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Using Approximation Non-Bayesian Computation with Fuzzy Data to Estimation Inverse Weibull Parameters and Reliability Function

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Abstract

In real situations all observations and measurements are not exact numbers but more or less non-exact, also called fuzzy. So, in this paper, we use approximate non-Bayesian computational methods to estimate inverse Weibull parameters and reliability function with fuzzy data. The maximum likelihood and moment estimations are obtained as non-Bayesian estimation. The maximum likelihood estimators have been derived numerically based on two iterative techniques namely “Newton-Raphson” and the “Expectation-Maximization” techniques. In addition, we provide compared numerically through Monte-Carlo simulation study to obtained estimates of the parameters and reliability function in terms of their mean squared error values and integrated mean squared error values respectively.

Keywords: Non-Bayesian Methods; Inverse Weibull distribution; Expectation Maximization algorithm; Newton-Raphson algorithm; Moment method; Fuzzy data.

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1.Introduction

Statistical analysis of inverse Weibull distribution is based on exact data. However, in real situations all observations and measurements are not exact numbers but more or less non-exact, also called fuzzy. Thus, in this paper we present a non-Bayesian approach to estimate the parameters and reliability function of inverse Weibull distribution with fuzzy data. Keller et al. (1985) [4] introduced inverse Weibull distribution as a suitable model to describe degradation phenomena of mechanical components of diesel engines. Other names for this distribution are complementary Weibull distribution, reciprocal Weibull distribution and reverse Weibull distribution [7].
A random variable X is said to have a two-parameter inverse Weibull distribution if it has the following probability density function (PDF)[8],

$$f_X(x;\beta,\lambda) = \beta\lambda x^{-(\beta+1)} e^{-\lambda x^{-\beta}}; x \geq 0, \beta, \lambda > 0 \qquad \text{... (1)}$$

The cumulative distribution function(CDF), reliability function is given respectively by:

$$F_X(x;\beta,\lambda) = e^{-\lambda x^{-\beta}}; x \geq 0, \beta, \lambda > 0 \qquad \text{... (2)}$$

$$R(x) = 1 - e^{-\lambda x^{-\beta}}; x \geq 0, \beta, \lambda > 0 \qquad \text{... (3)}$$

where β is the shape parameter and λ is the scale parameter.

Definition [9]: let $(\mathbb{R}^n, \mathcal{A}, \mathrm{P})$ be a probability space in which $\mathcal{A}$ is the $\sigma-$ field of Borel sets in $\mathbb{R}^n$ and P is a probability measure over $\mathbb{R}^n$. Then, the probability of a fuzzy event $\widetilde{\mathrm{A}}$ in $\mathbb{R}^n$ is defined by:

$$P(\tilde{A}) = \int \mu_{\tilde{A}}(x)\, dP \qquad ; for\ all\ x \in \mathbb{R}^n \qquad \text{... (4)}$$

In particular, suppose that P be the probability distribution of a continuous random variable X with PDF $g(x)$.The conditional density of X given $\tilde{A}$ is given by:

$$g(x|\tilde{A}) = \frac{\mu_{\tilde{A}}(x)\, g(x)}{\int \mu_{\tilde{A}}(u) g(u) du} \qquad \text{... (5)}$$

Fuzzy Data and the Likelihood Function

Let $\underline{\mathrm{x}} = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n)$ be an (i.i.d.) random vector of a random sample of size n from inverse Weibull distribution with PDF given by (1). If an observations of $\underline{x}$ was known exactly, then the complete-data likelihood function is:

$$L(\beta,\lambda;\underline{x}) = \prod_{i=1}^{n} f_x(x_i,\beta,\lambda) = \prod_{i=1}^{n} \beta\, \lambda\, x_i^{-(\beta+1)}\, exp\left(-\lambda x_i^{-\beta}\right)$$

$$\Rightarrow L(\beta,\lambda;\underline{x}) = \beta^n \lambda^n \prod_{i=1}^{n} x_i^{-(\beta+1)} \ exp\left(-\lambda \sum_{i=1}^{n} x_i^{-\beta}\right) \qquad \dots(6)$$

Now consider the problem where $\underline{x}$ is not observed precisely and only partial information about $\underline{x}$ is available in the form of a fuzzy subset $\underline{\tilde{x}}$ with the Borel measurable membership function $\mu_{\tilde{x}}(x)$. The observed-data natural log-likelihood function can be obtained, using Zadeh's definition of the probability of a fuzzy event in the expression (5) as:

$$\ell(\beta,\lambda;\underline{\tilde{x}}) = \ ln\, L(\beta,\lambda;\underline{\tilde{x}}) = ln \prod_{i=1}^{n} \int \beta\lambda\, x^{-(\beta+1)}\, exp(-\lambda x^{-\beta})\, \mu_{\tilde{x}_i}(x) dx$$

$$\Rightarrow \ell(\beta,\lambda;\underline{\tilde{x}}) = n\, ln\, \beta + n\, ln\, \lambda + \sum_{i=1}^{n} ln \int x^{-(\beta+1)}\, exp(-\lambda x^{-\beta})\, \mu_{\tilde{x}_i}(x) dx \qquad \dots(7)$$

Maximum likelihood Estimations

Differentiating the natural log-likelihood function $\ell(\beta,\lambda;\underline{\tilde{x}})$, given by (7), partially with respect to β and λ and then equating to zero we have:

$$\frac{\partial \ell(\beta,\lambda;\underline{\tilde{x}})}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{\int [\frac{\lambda}{x^{2\beta+1}} - \frac{1}{x^{\beta+1}}]\, ln\, x \ \ e^{-\lambda x^{-\beta}}\, \mu_{\tilde{x}_i}(x) dx}{\int \frac{1}{x^{\beta+1}} \ \ e^{(-\lambda x^{-\beta})}\, \mu_{\tilde{x}_i}(x) dx} = 0 \qquad \dots(8)$$

$$\frac{\partial \ell(\beta,\lambda;\underline{\tilde{x}})}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \frac{\int \frac{1}{x^{2\beta+1}} \ \ e^{-\lambda x^{-\beta}} \mu_{\tilde{x}_i}(x) dx}{\int \frac{1}{x^{\beta+1}} e^{-\lambda x^{-\beta}} \mu_{\tilde{x}_i}(x) dx} = 0 \qquad \dots(9)$$

The maximum likelihood estimates (MLEs) of β and λ are the solution of the likelihood equations (8) and (9). Since there is no closed form of the solutions, an iterative approximation technique can be used to obtain the MLEs. In the following, we consider two iterative approximation techniques namely the expectation-maximization (EM) algorithm and Newton Raphson (NR) algorithm.

Expectation-Maximization Algorithm

Dempster et al. (1977) [2] presented a general approach to iterative computation of maximum-likelihood estimates when the observations can be viewed as incomplete data. Now, since the observed fuzzy data $\tilde{x}$ can be seen as an incomplete specification of a complete data vector $\underline{x}$, the EM algorithm is appropriate to find the MLEs of the unknown parameters [3][5].

Form equation (6), the natural log-likelihood function for $\underline{x}$ becomes:

$$\ell(\beta,\lambda;\underline{x}) = ln\, L(\beta,\lambda;\underline{x})$$

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$$\Rightarrow \ell(\beta,\lambda;\underline{x}) = n\ln\beta + n\ln\lambda - (\beta+1)\sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}{x_i}^{-\beta} \qquad \ldots(10)$$

The EM algorithm is given by the following iterative process:

1. Set $h = 0$ and initial values of β and λ, say $\beta^{(0)}$ and $\lambda^{(0)}$.
2. At iteration $(h+1)$, compute the E and M steps.
3. Repeat step (2) until the convergence occurs, i.e. the absolute difference between two successive iterations is less than ε for some pre-fixed $\varepsilon > 0$. when the convergence occurs then the present $\beta^{(h+1)}$ and $\lambda^{(h+1)}$ be the MLEs of β and λ via EM algorithm which we referred to as $(\hat{\beta}_{EM}, \hat{\lambda}_{EM})$.

E-step: The E-step of the EM algorithm at iteration $(h+1)$ requires to compute the following conditional expectations using the expression (5),

$$\begin{aligned} &E_{\beta^{(h)},\lambda^{(h)}}\left[\ell(\beta,\lambda;\underline{x}|\underline{\tilde{x}})\right] \\ &\qquad = n\ln\beta + n\ln\lambda - (\beta+1)\sum_{i=1}^{n}E_{\beta^{(h)},\lambda^{(h)}}[\ln x_i|\tilde{x}_i] \\ &\qquad - \lambda\sum_{i=1}^{n}E_{\beta^{(h)},\lambda^{(h)}}\left[{x_i}^{-\beta}|\tilde{x}_i\right] \end{aligned} \qquad \ldots(11)$$

where the conditional expectations of X given $\tilde{x}$ is computed as:

$$E_{\beta^{(h)},\lambda^{(h)}}(\ln X\,|\tilde{x}_i) = \frac{\int x^{-(\beta^{(h)}+1)}\ \ln(x)\ e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\ \mu_{\tilde{x}_i}(x)dx}{\int x^{-(\beta^{(h)}+1)}\ e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\ \mu_{\tilde{x}_i}(x)dx} \qquad \ldots(12)$$

$$E_{\beta^{(h)},\lambda^{(h)}}(X^{-\beta}|\tilde{x}_i) = \frac{\int x^{-(2\beta^{(h)}+1)}\ e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\ \mu_{\tilde{x}_i}(x)dx}{\int x^{-(\beta^{(h)}+1)}\ e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\ \mu_{\tilde{x}_i}(x)dx} \qquad \ldots(13)$$

M-step: The M-step of EM algorithm involves maximizing equation (11) with respect to β and λ. This is easily achieved by differentiating equation (11) partially with respect to β and λ and then equating to zero which implies that:

$$\hat{\beta}^{(h+1)} = \frac{n}{\left[\sum_{i=1}^{n}E_{\beta^{(h)},\lambda^{(h)}}(\ln X\,|\tilde{x}_i) - \lambda^{(h+1)}\sum_{i=1}^{n}E_{\beta^{(h)},\lambda^{(h)}}(X^{-\beta}\ln X\,|\tilde{x}_i)\right]} \qquad \ldots(14)$$

$$\hat{\lambda}^{(h+1)} = \frac{n}{\sum_{i=1}^{n}E_{\beta^{(h)},\lambda^{(h)}}(X^{-\beta}|\tilde{x}_i)} \qquad \ldots(15)$$

where $E_{\beta^{(h)},\lambda^{(h)}}(X^{-\beta}\ln X\,|\tilde{x}_i)$ computed as:

$$E_{\beta^{(h)},\lambda^{(h)}}(X^{-\beta}\ln X\,|\tilde{x}_i) = \frac{\int x^{-(2\beta^{(h)}+1)}\ln x\ e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\ \mu_{\tilde{x}_i}(x)dx}{\int x^{-(\beta^{(h)}+1)}\ e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\ \mu_{\tilde{x}_i}(x)dx} \qquad \ldots(16)$$

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$E_{\beta^{(h)},\lambda^{(h)}}(ln\, X\,|\tilde{x}_i)$ and $E_{\beta^{(h)},\lambda^{(h)}}\left(X^{-\beta}|\tilde{x}_i\right)$ as in (12) and (13). We can rewrite $\hat{\beta}^{(h+1)}$ and $\hat{\lambda}^{(h+1)}$ in (14) and (15) as:

$$\hat{\beta}^{(h+1)} = \left\{\frac{1}{n}\left[\sum_{i=1}^{n} E_{1i} - \lambda^{(h+1)}\sum_{i=1}^{n} E_{3i}\right]\right\}^{-1} \qquad \dots (17)$$

$$\hat{\lambda}^{(h+1)} = \frac{n}{\sum_{i=1}^{n} E_{2i}} \qquad \dots (18)$$

where:

$$E_{1i} = E_{\beta^{(h)},\lambda^{(h)}}(ln\, X\,|\tilde{x}_i)\,,\; E_{2i} = E_{\beta^{(h)},\lambda^{(h)}}\left(X^{-\beta}|\tilde{x}_i\right)\; and\; E_{3i} = E_{\beta^{(h)},\lambda^{(h)}}\left(X^{-\beta}\, ln\, X\,|\tilde{x}_i\right)$$

Newton-Raphson Algorithm

The steps of the NR algorithm are [5][7]:

1. Set $h = 0$ and initial values of β and λ, say $\beta^{(0)}$ and $\lambda^{(0)}$.
2. At iteration $(h+1)$, estimate the new value of β and λ, as:

$$\begin{bmatrix}\hat{\beta}^{(h+1)}\\ \hat{\lambda}^{(h+1)}\end{bmatrix} = \begin{bmatrix}\hat{\beta}^{(h)}\\ \hat{\lambda}^{(h)}\end{bmatrix} - \begin{bmatrix}\frac{\partial^2\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\beta^2} & \frac{\partial^2\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\beta\partial\lambda}\\ \frac{\partial^2\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\beta\partial\lambda} & \frac{\partial^2\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\lambda^2}\end{bmatrix}^{-1}_{\substack{\beta=\hat{\beta}^{(h)}\\ \lambda=\hat{\lambda}^{(h)}}} \begin{bmatrix}\frac{\partial\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\beta}\\ \frac{\partial\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\lambda}\end{bmatrix}_{\substack{\beta=\hat{\beta}^{(h)}\\ \lambda=\hat{\lambda}^{(h)}}} \qquad \dots (19)$$

where the first–order derivatives of the natural log-likelihood with respect to β and λ , required for proceeding with the NR algorithm, are obtained as in (8) and (9) and the second–order derivatives are obtained as follows.

$$\begin{aligned}\frac{\partial^2\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\beta^2} = &-\frac{n}{\beta^2}\\ &+\sum_{i=1}^{n}\left[\frac{\int\left(\frac{\lambda^2}{x^{3\beta+1}}-\frac{\lambda}{x^{2\beta+1}}\right)e^{-\lambda x^{-\beta}}\left(ln(x)\right)^2\mu_{\tilde{x}_i}(x)dx}{\int\frac{1}{x^{\beta+1}}\,e^{-\lambda x^{-\beta}}\,\mu_{\tilde{x}_i}(x)dx}\right.\\ &\left.+\frac{\int\left(\frac{1}{x^{\beta+1}}-\frac{2\lambda}{x^{2\beta+1}}\right)\left(ln(x)\right)^2\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}{\int\frac{1}{x^{\beta+1}}\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}\right]\\ &-\sum_{i=1}^{n}\left[\frac{\int\left(\frac{\lambda}{x^{2\beta+1}}-\frac{1}{x^{\beta+1}}\right)ln(x)\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}{\int\frac{1}{x^{\beta+1}}\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}\right]^2\end{aligned} \qquad \dots (20)$$

$$\begin{aligned}\frac{\partial^2\ell(\beta,\lambda;\underset{\sim}{\tilde{x}})}{\partial\lambda^2} = &-\frac{n}{\lambda^2}+\sum_{i=1}^{n}\frac{\int\frac{1}{x^{3\beta+1}}\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}{\int\frac{1}{x^{\beta+1}}\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}\\ &-\sum_{i=1}^{n}\left[\frac{\int\frac{1}{x^{2\beta+1}}\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}{\int\frac{1}{x^{\beta+1}}\,e^{-\lambda x^{-\beta}}\mu_{\tilde{x}_i}(x)dx}\right]^2\end{aligned} \qquad \dots (21)$$

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$$\frac{\partial^2 \ell(\beta,\lambda;\underline{\tilde{x}})}{\partial\beta\partial\lambda} = -\sum_{i=1}^{n} \frac{\int \left(\frac{\lambda}{x^{3\beta+1}} - \frac{2}{x^{2\beta+1}}\right) ln(x)\ e^{-\lambda x^{-\beta}} \mu_{\tilde{x}_i}(x)dx}{\int \frac{1}{x^{\beta+1}}\ e^{-\lambda x^{-\beta}} \mu_{\tilde{x}_i}(x)dx}$$

$$-\sum_{i=1}^{n} \frac{\int \left(1 - \frac{\lambda}{x^{\beta}}\right) \frac{1}{x^{\beta+1}}\ ln(x)\, e^{-\lambda x^{-\beta}} \mu_{\tilde{x}_i}(x)dx\ \int \frac{1}{x^{2\beta+1}}\ e^{-\lambda x^{-\beta}} \mu_{\tilde{x}_i}(x)dx}{\left(\int \frac{1}{x^{\beta+1}}\ e^{-\lambda x^{-\beta}} \mu_{\tilde{x}_i}(x)dx\right)^2} \quad \ldots (22)$$

3. Repeat step (2) until the convergence occurs. When the convergence occurs then the present $\hat{\beta}^{(h+1)}$ and $\hat{\lambda}^{(h+1)}$ be the MLEs of β and λ via NR algorithm which we referred to as $(\hat{\beta}_{NR}, \hat{\lambda}_{NR})$.

Now, depending on the invariant property of MLE, the MLE of the reliability function of inverse Weibull distribution via EM and NR algorithms, denoted by $\hat{R}(t)_{EM}$ and $\hat{R}(t)_{NR}$ respectively, can be obtained by replacing β and λ in (3) by their MLE estimates as:

$$\hat{R}(t)_{EM} = 1 - e^{-\hat{\lambda}_{EM}\, t^{-\hat{\beta}_{EM}}}\ ;\ t \geq 0 \quad \ldots (23)$$

$$\hat{R}(t)_{NR} = 1 - e^{-\hat{\lambda}_{NR}\, t^{-\hat{\beta}_{NR}}}\ ;\ t \geq 0 \quad \ldots (24)$$

Moment Estimations

The method of moments is one the oldest method for deriving point estimators. The moment estimates for β and λ of inverse Weibull distribution can be found by the following two equations which are obtained by equating the first and the second population moments to the corresponding sample moments, that is:

$$\lambda^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\beta,\lambda}\,(X|\tilde{x}_i) \quad \ldots (25)$$

$$\lambda^{\frac{2}{\beta}} \Gamma\left(1 - \frac{2}{\beta}\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\beta,\lambda}\,(X^2|\tilde{x}_i) \quad \ldots (26)$$

Note that, the direct form of the solutions to equations (25) and (26) could not be obtained. However, by using an iterative numerical process, we can obtain the parameter estimates as described below:

Step (1) Set $h = 0$ and initial values of β and λ, say $\beta^{(0)}$ and $\lambda^{(0)}$.

Step (2) At $(h+1)^{th}$ iteration, using the expression (5) to compute the following conditional expectation,

$$E_{\beta^{(h)},\lambda^{(h)}}(X^r|\tilde{x}_i) = \frac{\int x^{-\beta^{(h)}+r-1}\, e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\mu_{\tilde{x}_i}(x)\, dx}{\int x^{-\beta^{(h)}-1}\, e^{-\lambda^{(h)}x^{-\beta^{(h)}}}\mu_{\tilde{x}_i}(x)\, dx}\ ;\ r = 1,2$$

Step (3) Solve the following equation for β, based on equations (25) and (26), to obtain the solution as $\beta^{(h+1)}$,

$$\frac{\left[\Gamma\left(1-\frac{1}{\beta}\right)\right]^2}{\Gamma\left(1-\frac{2}{\beta}\right)} = \frac{\left[\sum_{i=1}^{n} E_{\beta^{(h)},\lambda^{(h)}}(X|\tilde{x}_i)\right]^2}{n\ \sum_{i=1}^{n} E_{\beta^{(h)},\lambda^{(h)}}(X^2|\tilde{x}_i)}$$

Step (4) Obtain the solution for λ, say $\lambda^{(h+1)}$, through the following equation,

$$\lambda^{(h+1)} = \left[\frac{\sum_{i=1}^{n} E_{\beta^{(h)},\lambda^{(h)}}(X|\tilde{x}_i)}{n\,\Gamma\left(1-\frac{1}{\beta^{(h+1)}}\right)}\right]^{\beta^{(h+1)}}$$

Step (5) Setting $h = h + 1$, repeat step (2) to step (4) until convergence occurs. When the convergence occurs then the present $\hat{\beta}^{(h+1)}$ and $\hat{\lambda}^{(h+1)}$ be the moment estimates of β and λ which we referred to as $(\hat{\beta}_{MO}, \hat{\lambda}_{MO})$.

Now, depending on the moment estimates of the shape and scale parameters, the approximated moment estimate of the reliability function of inverse Weibull distribution at mission time t, denoted by $\hat{R}_{MO}(t)$, can be obtained by replacing β and λ in equation (3) by their moment estimates as:

$$\hat{R}_{MO}(t) = 1 - e^{-\hat{\lambda}_{MO}\, t^{-\hat{\beta}_{MO}}}\ ;\ t \geq 0 \qquad \ldots(27)$$

Simulation Study

In trying to illustrate and compare the algorithms as described above, a Monte-Carlo simulation study was perform to generate an independent identical distributed random samples, say $\underline{x}$, according to inverse Weibull distribution through the adoption of inverse transformation method with size n = 20, 30 and 90 to take care of small, medium and large data sets. The number of sample replicated chosen to be (100). The shape parameter was chosen to be 3, 2.1, 1 and 0.5 and the scale parameter 0.5, 1, 3. Then, each observation of $\underline{x}$ was made fuzzy based on an appropriate selected membership function among the following eight membership functions in the FIS shown in figure (1). The simulation program has been written by using MATLAB (R2010b) program. The results of Monte-Carlo simulation have been summarized in the tables (1)…(3).

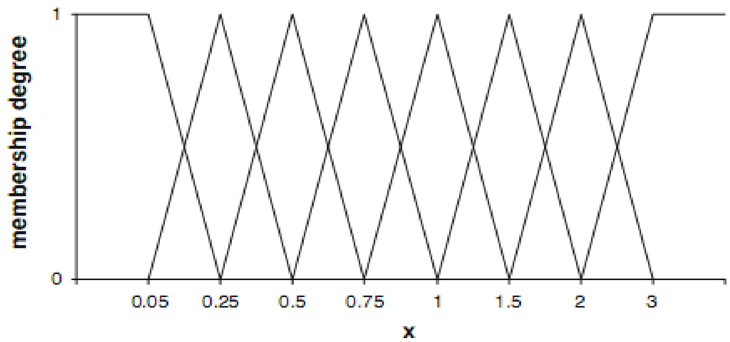

Figure (1): FIS used to Encode the Simulated Data [5 & 6]

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & ; \quad x \le 0.05 \\ \dfrac{0.25 - x}{0.2} & ; \quad 0.05 \le x \le 0.25 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \dfrac{x - 0.05}{0.2} & ; \quad 0.05 \le x \le 0.25 \\ \dfrac{0.5 - x}{0.25} & ; \quad 0.25 \le x \le 0.5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \dfrac{x - 0.25}{0.25} & ; \quad 0.25 \le x \le 0.5 \\ \dfrac{0.75 - x}{0.25} & ; \quad 0.5 \le x \le 0.75 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \dfrac{x - 0.5}{0.25} & ; \quad 0.5 \le x \le 0.75 \\ \dfrac{1 - x}{0.25} & ; \quad 0.75 \le x \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \dfrac{x - 0.75}{0.25} & ; \quad 0.75 \le x \le 1 \\ \dfrac{1.5 - x}{0.5} & ; \quad 1 \le x \le 1.5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_6}(x) = \begin{cases} \dfrac{x - 1}{0.5} & ; \quad 1 \le x \le 1.5 \\ \dfrac{2 - x}{0.5} & ; \quad 1.5 \le x \le 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \dfrac{x - 1.5}{0.5} & ; \quad 1.5 \le x \le 2 \\ 3 - x & ; \quad 2 \le x \le 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_8}(x) = \begin{cases} x - 2 & ; \quad 2 \le x \le 3 \\ 1 & ; \quad x \ge 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The initial values required for proceeding with the Expectation-Maximization , Newton-Raphson algorithms and moment method chosen to be the symmetrical rank regression estimators and the iterative process stops when the absolute difference between two successive iterations becomes less than $\varepsilon = 0.0001$. The comparisons between parameter estimates were based on values from Mean Square Error (MSE) while it were based on values from Integrated Mean Square Error (IMSE) for the estimates of the reliability function [1], where:

$$MSE(\hat{\beta}) = \frac{\sum_{j=1}^{L}(\hat{\beta}_j - \beta)^2}{L} \qquad \ldots (28)$$

$$MSE(\hat{\lambda}) = \frac{\sum_{j=1}^{L}(\hat{\lambda}_j - \lambda)^2}{L} \quad \ldots (29)$$

$$IMSE(\hat{R}(t)) = \frac{1}{L}\sum_{j=1}^{L}\left(\frac{1}{n_t}\sum_{i=1}^{n_t}(\hat{R}_j(t_i) - R(t_i))^2\right) \quad \ldots (30)$$

$\hat{\beta}_j, \hat{\lambda}_j$: is the estimate of β and λ respectively at the j th replicate (run).

L: is the number of sample replicated.

n_t : is the number of times chosen to be (4) where $(t = 1,2,3,4)$.

$\hat{R}_j(t_i)$: is the estimates of $R(t)$ at the j^{th} replicate (run) and i^{th} time.

2. Conclusions and Recommendations

Approximate maximum likelihood and moment estimations as non-Bayesian estimations have been used to estimate the parameters and reliability function of inverse Weibull distribution according to fuzzy data. The maximum likelihood estimators have been derived numerically based on two iterative techniques namely "Newton-Raphson" and the "Expectation-Maximization" techniques. In addition, we provide compared numerically through Monte-Carlo simulation study to obtained estimates of the parameters and reliability function in terms of their mean squared error (MSE) values and integrated mean squared error (IMSE) values respectively.
The most important conclusions of Monte-Carlo simulation results are:

Table (1): The maximum likelihood estimates based on Newton-Raphson algorithm introduced the best perform compared with other estimates with different values of the shape and scale parameters for all sample sizes except for small sample size with parameters, $\beta = 3, \lambda = 0.5$ and $\beta = 3, \lambda = 3$.

Table (2): The maximum likelihood estimates based on Newton-Raphson algorithm introduced the best perform compared with other estimates for all sample sizes except for large sample size with parameters $\beta = 2.1, \lambda = 1$, where Moment estimate is the best.

Table (3): The maximum likelihood estimates based on NR introduced the best perform compared with other estimates with different values of the shape and scale parameters for all sample sizes with $\beta = 2.1, 3$ and $\lambda = 0.5, 1$ as well as for moderate sample size with $\beta = 2.1$ and $\lambda = 3$. On the other hand, the maximum likelihood estimates based on EM

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introduced the best perform compared with other non-Bayes estimates for all sample sizes with $\beta=3$ and $\lambda=3$ as well as for small and large sample sizes with $\beta = 2.1$ and $\lambda = 3$.

Tables 1, 2 and 3 clearly show that as the sample size increases, the MSE and IMSE values of the estimates decrease.

Based on this, we recommend,

Using the maximum likelihood estimates based on Newton-Raphson algorithm for estimating the shape parameter of the inverse Weibull distribution especially with moderate and large sample sizes and we have to be careful in choosing the approximation techniques for estimating the shape parameter of this distribution when dealing with small sample size.

Using the maximum likelihood estimates based on Newton-Raphson algorithm for estimating the scale parameter of the inverse Weibull distribution especially with small and moderate sizes and we have to be careful in choosing the approximation techniques for estimating the scale parameter of this distribution when dealing with large sample size.

Using the maximum likelihood estimates based on Newton-Raphson algorithm for estimating the reliability function of the inverse Weibull distribution with $\beta = 2.1, 3$ and $\lambda = 0.5, 1$ and using the maximum likelihood estimates based on Expectation-Maximization algorithm with $\beta = 3$ and $\lambda = 3$.

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Table (1): MSE Values for Non-Bayes Estimates of the Shape Parameter (β) of Inverse Weibull Distribution with Different Cases

n	MO	ML		Best Estimate
		EM	NR	
$\beta = 2.1, \lambda = 0.5$				
20	0.5925534	0.3276789	0.3210075	ML-NR
30	0.5668387	0.1731912	0.1720976	ML-NR
90	0.5026410	0.0807156	0.0802433	ML-NR
$\beta = 3, \lambda = 0.5$				
20	0.5945321	0.8529363	0.7876280	MO
30	0.4313349	0.3780456	0.3622704	ML-NR
90	0.4008172	0.0871851	0.0861431	ML-NR
$\beta = 2.1, \lambda = 1$				
20	1.0234089	0.4793374	0.4697234	ML-NR
30	0.8987515	0.3765942	0.3700855	ML-NR
90	0.8584988	0.1780456	0.1764240	ML-NR
$\beta = 3, \lambda = 1$				
20	0.8207644	0.7696865	0.7355563	ML-NR
30	0.6615921	0.3721112	0.3652719	ML-NR
90	0.5914806	0.1426038	0.1412609	ML-NR
$\beta = 2.1, \lambda = 3$				
20	2.4264343	1.4173415	1.3050008	ML-NR
30	2.1658246	0.9206690	0.8674874	ML-NR
90	2.1064719	0.5787366	0.5542025	ML-NR
$\beta = 3, \lambda = 3$				
20	1.2356461	0.8079456	0.8175703	ML-EM
30	1.2346853	0.7552212	0.7170664	ML-NR
90	1.1146329	0.2692730	0.2583540	ML-NR

n: Sample Size
MO: Moment Estimate
ML: Maximum Likelihood Estimate
EM: Expectation-Maximization Algorithm
NR: Newton-Raphson Algorithm

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Table (2): MSE Values for Non-Bayes Estimates of the Scale Parameter (λ) of Inverse Weibull Distribution with Different Cases

n	MO	ML		Best Estimate
		EM	NR	
$\beta = 2.1, \lambda = 0.5$				
20	0.0261692	0.0212376	0.0211103	ML-NR
30	0.0185599	0.0146520	0.0145815	ML-NR
90	0.0096941	0.0066130	0.0065777	ML-NR
$\beta = 3, \lambda = 0.5$				
20	0.0258200	0.0221413	0.0216447	ML-NR
30	0.0187999	0.0153414	0.0151493	ML-NR
90	0.0061082	0.0049155	0.0048935	ML-NR
$\beta = 2.1, \lambda = 1$				
20	0.1178850	0.0874576	0.0864178	ML-NR
30	0.0539672	0.0495897	0.0493294	ML-NR
90	0.0171838	0.0222780	0.0220901	MO
$\beta = 3, \lambda = 1$				
20	0.1266382	0.0949276	0.0932050	ML-NR
30	0.0689319	0.0553591	0.0549835	ML-NR
90	0.0431480	0.0176048	0.0172905	ML-NR
$\beta = 2.1, \lambda = 3$				
20	10.1555872	5.9423094	5.0189589	ML-NR
30	6.6359697	3.0022532	2.5016068	ML-NR
90	3.9878570	0.5756619	0.5583213	ML-NR
$\beta = 3, \lambda = 3$				
20	4.2045048	5.2462299	3.5490963	ML-NR
30	3.8376384	5.1730105	3.0378761	ML-NR
90	2.1648816	0.3457172	0.3411649	ML-NR

n: Sample Size
MO: Moment Estimate
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EM: Expectation-Maximization Algorithm
NR: Newton-Raphson Algorithm

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Table (3): IMSE Values for Non-Bayes Estimates of the Reliability Function of Inverse Weibull Distribution with Different Cases

n	MO	ML		Best Estimate
		EM	NR	
$\beta = 2.1, \lambda = 0.5$				
20	0.0038869	0.0030015	0.0029777	ML-NR
30	0.0028681	0.0020454	0.0020271	ML-NR
90	0.0019454	0.0010003	0.0009953	ML-NR
$\beta = 3, \lambda = 0.5$				
20	0.0027426	0.0025390	0.0024743	ML-NR
30	0.0018299	0.0015330	0.0015252	ML-NR
90	0.0007605	0.0005639	0.0005614	ML-NR
$\beta = 2.1, \lambda = 1$				
20	0.0068683	0.0050789	0.0050384	ML-NR
30	0.0062469	0.0045062	0.0044766	ML-NR
90	0.0041694	0.0024271	0.0024047	ML-NR
$\beta = 3, \lambda = 1$				
20	0.0037806	0.0033036	0.0032750	ML-NR
30	0.0024769	0.0022311	0.0022310	ML-NR
90	0.0016173	0.0008200	0.0008185	ML-NR
$\beta = 2.1, \lambda = 3$				
20	0.0196349	0.0138542	0.0140972	ML-EM
30	0.0189882	0.0120847	0.0120259	ML-NR
90	0.0182735	0.0106072	0.0106827	ML-EM
$\beta = 3, \lambda = 3$				
20	0.0040808	0.0031302	0.0032523	ML-EM
30	0.0035055	0.0024299	0.0025396	ML-EM
90	0.0032125	0.0015585	0.0015589	ML-EM

n: Sample Size
MO: Moment Estimate
ML: Maximum Likelihood Estimate
EM: Expectation-Maximization Algorithm
NR: Newton-Raphson Algorithm

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