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The Comparison Between the Bayes Estimator and the Maximum Likelihood Estimator of the Reliability Function for Negative Exponential Distribution

Hazim Mansour Gorgees
College of Education for Pure Science /(Ibn Al-Haitham)University of Baghdad
Bushra Abdualrasool Ali
Ministry of Education
Raghad Ibrahim Kathum
agoldenfish@yahoo.com
College of Al-Yarmouk University

Abstract

In this paper, the maximum likelihood estimator and the Bayes estimator of the reliability function for negative exponential distribution has been derived, then a Monte – Carlo simulation technique was employed to compare the performance of such estimators. The integral mean square error (IMSE) was used as a criterion for this comparison. The simulation results displayed that the Bayes estimator performed better than the maximum likelihood estimator for different samples sizes.

Keywords: -Negative Exponential Distribution, Maximum likelihood estimator, Bayes estimator, integral men square error.

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1. Introduction

There are many situations in which one would expect negative exponential distribution to give a useful description of observed variation. One of the most widely quoted is that of events recurring at random in time [1]. The mathematics associated with the negative exponential distribution is often of a simple nature. It is often possible to obtain explicit formulas in terms of elementary functions. For these reasons models constructed from exponential variables are sometimes used as an approximate representation of other models. The exponential distribution is the first and most popular model for failure times. [5]

This paper is organized as follows:

In section 2 the purpose of the research is given. In section 3 the theoretical part of the negative exponential distribution is presented. The experimental part including the description of the simulation experiment steps is discussed in section 4. Finally, the conclusions and recommendations are presented in section 5.

2. Purpose of research

The main aim of this paper is to derive the Bayes estimator and the maximum likelihood estimator for the reliability functions of negative exponential distribution and then compare them by employing the Monte Carlo simulation procedures in order to obtain the best method for estimating this function.

2.1 Theoretical part

The random variable X has a negative exponential (or just exponential) distribution if it has a probability density function $f(x,\theta,\sigma) = \sigma^{2} \exp[-(x-\theta] \; l \;\; \sigma], x > \theta . \sigma > 0$ ….(1)

The reliability function of this distribution is [6]

$$R(t) = pr \; (x > t) = \int_{t}^{\infty} f \; (x,\theta,\sigma) \; dx$$

$$\frac{1}{\sigma}\int_{t}^{\infty} e^{-\frac{(x-\theta)}{\sigma}} dx = [e^{-\frac{(t-\theta)}{\sigma}}] \qquad \text{…..(2)}$$

1-Maximum likelihood estimator (MLE)

The Maximum likelihood method is one of the classical methods for estimation which depends upon the assumption that the parameter to be estimated is fixed quantity. This method has many good features, especially, the invariant property. [3]

The MLE can be defined as those values of parameters that maximize the likelihood function of observation.

Let $x_1, x_2 \; x_3 .. x_n$ be a random sample of size n from population having negative exponential distribution with two parameters

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(θ,σ), then the likelihood function is given as:

$$L\ (X_1, X_2 \ldots\ldots\ldots X_2, \theta, \sigma) = \prod_{i-1}^{n} \frac{1}{\sigma} e^{\frac{-(x_i-\theta)}{\sigma}} = \frac{1}{\sigma^n}\ e^{\frac{-\Sigma\ (x_i-\theta)}{\sigma}}$$

$$LnL\ = -n\ \ln\sigma - \Sigma\,\frac{(x_i-\theta)}{\sigma} \qquad \ldots.\ (3)$$

The maximum likelihood estimator of θ is the smallest order statistic of observations $x_{(1)}$ say, that is:

$$\hat{\theta}_{MLE} = Min\ \ (X_1, X_2 \ldots\ldots\ldots X_n) = \ x_{(1)} \qquad \ldots(4)$$

Differentiating equation (3) partially with respect to σ we get

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\sum (x_i - x_{(1)})}{\sigma^2} \qquad \ldots.\ (5)$$

By equations the derivative in equation (5) to zero and solving for σ we get:

$$\hat{\sigma}_{MLE} = \frac{\Sigma(x_i - x_{(1)})}{n} = \bar{x} - x_{(1)} \qquad \ldots..(6)$$

Since the maximum likelihood estimator has invariant property then

$$\hat{R}\ (t)_{ML} = [\ e^{\frac{-\ (t-\hat{\theta}_{ML})}{\hat{\sigma}_{ML}}}\ ] \qquad ..\ (7)$$

2-Bayes estimator

In this case, assuming that the two parameters θ and σ are random variables each has a prior distribution, moreover, assuming that the quadratic loss function is employed, we have to determine an estimator for the reliability function which minimizes the expected loss.

According to Jeffry's approach, the prior p.d.f for θ and σ are: [2]

$$I_1(\theta)\ \alpha \frac{1}{\theta}, \quad I_2(\sigma)\ \alpha \frac{1}{\sigma} \qquad ..\ \ (8)$$

Hence, the joint prior p.d.f for the two random independent parameters is:

$$I\ (\theta, \sigma)\ \ \alpha\ \ \frac{1}{\theta\ \ \sigma}, \qquad \ldots\ \ (9)$$

By using the Bayes formula, the joint posterior p.d.f for θ and σ is given as

$$h\ \ (\theta, \sigma \setminus x_1, x_2 \ldots.. x_n)\ \ \alpha\ \frac{1}{\theta\ \ \sigma^{n+1}}\ \ e^{-\ \Sigma \frac{[x_i - \theta]}{\sigma}}$$

$$h\ \ (\theta, \sigma \setminus x_1, x_2 \ldots.. x_n) = \ c\ \frac{1}{\theta\ \ \sigma^{n+1}}\ \ e^{-\Sigma \frac{[x_i - \theta]}{\sigma}}$$

$$h\ (\theta,\sigma \setminus x_{1,} x_{2} .....x_{n}) = c\frac{1}{\theta\ \sigma^{n+1}}\ e^{-\frac{1}{\sigma}\sum x_{i} + \frac{\theta n}{\sigma}} \qquad ...(10)$$

Where c is the proportionality constant which represents the reciprocal of the marginal density function f $((X_1, X_2 ....X_n)$ that is

$$C^{-1} = \int_0^{\infty} \int_0^{x(1)} \frac{1}{\theta\ \sigma^{n+1}} e^{-\frac{1}{\sigma}\sum x_i + \frac{\theta n}{\sigma}}\ d\theta d\sigma$$

Since the quadratic loss function is employed, then the Bayes estimator for the reliability function is the posterior mean of this function [2], that is

$$\hat{R}_B(t) = E[R(t) \setminus x_1, x_2 ......x_n]$$

$$= \int_0^{\infty} \int_0^{x(1)} R(t)\ h\ (\theta,\sigma \setminus x_1, x_2 ......x_n)\ d\theta\ d\sigma \qquad ....(11)$$

3. Experimental part

In this section, we describe the steps of simulation experiments [4]

Step1: choosing the assumed values for θ, σ. For example, let θ =1, 1.5 and σ =1, 2.7, hence there will be four simulation experiments. Also at this step we assume the sample sizes to be n= 10, 30, 50,100 (say) and the number of replications for each experiment is L =1000

Step2: Generating the data according to the negative exponential distribution by using the cumulative distribution function F(x) where

$$F(x) = 1 - [R(X)] = 1 - e^{\frac{-(x-\theta)}{\sigma}}$$

Let the random variable u has a uniform distribution on the interval (0, 1) then

$$U = 1 - e^{\frac{-(x-\theta)}{\sigma}}$$

$$\Rightarrow e^{\frac{-(x-\theta)}{\sigma}} = 1-u \Rightarrow \frac{-(x-\theta)}{\sigma} = \ln(1-u)$$

$$\Rightarrow \frac{(x-\theta)}{\sigma} = -\ln(1-u) \Rightarrow x = -\sigma \ln(1-u) + \theta \qquad ....(12)$$

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Step 3: Estimating the reliability function for the negative exponential distribution with two parameters θ, σ by employing the formula's (7), (11)

Step 4: comparing the two estimators by using the integral mean square error (IMSE) where

$$\text{IMSE}\ [\hat{R}(t)] = \frac{1}{L}\sum_{i=1}^{L}\left[\frac{1}{n_i}\sum_{j=1}^{n}[\hat{R}(t_j) - R(t_j)]^2\right] \quad \dots (13)$$

4. Conclusion and Recommendation

According to the simulation results, it is obvious that the Bayes estimator of the reliability function for the negative experimental distribution performs better than the maximum likelihood estimator for all sample sizes in the sense of IMSE as it is shown table (1). However, for the purpose of function works one can use other classical methods of estimation such as the moments method, median-first order statistics method and the ordinary least squares method. Moreover, in addition to IMSE, many other criteria may be used to measure the performance of the studied estimators such as mean Absolute Error (MAE) and mean absolute percentage error (MAPE).

Table (1): The integrated mean square error (IMSE) of reliability estimator for all experiments

Model	N	ML	Bays	Best
$\theta = 1$ $\sigma = 1$	10	0.0014	0.000355	Bays
	30	0.000481	0.000118	Bays
	50	0.000291	0.000071	Bays
	100	0.000147	0.000035	Bays
$\theta = 1$ $\sigma = 2.7$	10	0.000697	0.000033	Bays
	30	0.000207	0.000011	Bays
	50	0.000125	0.000007	Bays
	100	0.000064	0.000003	Bays
$\theta = 1.5$ $\sigma = 1$	10	0.0048	0.0026	Bays
	30	0.0016	0.00085	Bays
	50	0.00097	0.00051	Bays
	100	0.00049	0.000255	Bays
$\theta = 1.5$ $\sigma = 2.7$	10	0.0010	0.00017	Bays
	30	0.000345	0.000057	Bays
	50	0.000209	0.000034	Bays
	100	0.000106	0.000017	Bays

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