W-Closed Submodule and Related Concepts

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Abstract

Let R be a commutative ring with identity, and M be a left untial module. In this paper we introduce and study the concept w-closed submodules, that is stronger form of the concept of closed submodules, where asubmodule K of a module M is called w-closed in M, "if it has no proper weak essential extension in M", that is if there exists a submodule L of M with K is weak essential submodule of L then K=L. Some basic properties, examples of w-closed submodules are investigated, and some relationships between w-closed submodules and other related modules are studied. Furthermore, modules with chain condition on w-closed submodules are studied.

Keywords: Closed submodules, Weak essential submodules, W-closed submodules, completely essential modules, y-closed submodules, Minimal semi-prime submodules.

Introduction

In this note, we shall assume that all rings are commutative with unity and all modules are unital left modules, and all R-modules under study contains semi-prime submodules. "A submodule L of a module M is called closed in M provided that L has no proper essential extension in M [1]" ," where a non-zero submodule N of M is called essential if N ∩ E $\neq (0)$ for all non-zero submodule E of M [1]", "and a non-zero submodule N of M is called weak essential if N ∩ S $\neq (0)$ ∀ non zero semi-prime submodule S of M [2]". "Equivalently, a submodule N of a module M is called weak essential if whenever N ∩ S $\neq (0)$, then S=(o) for every semi-prime submodule S of M [3]","where a submodule S of a module M is called semi-prime if for each $r \in R\ and\ y \in M\ with\ r^k y \in S, k \in Z^+\ then\ ry \in S$ [4]"."Equivalently if $r^2 y \in S, then\ ry \in S$ [5]".In this proper, "we introduce the concept of w-closed submodule "which is stronger than the concept of closed submodule" ,where a submodule K of an R-module M is called w-closed "if K has no proper weak essential extension in M". That is if K is weak essential in L , where L is a submodule of M, then K=L . A module M is called chaine if for each submodules E and D of M either $E \subseteq D\ or\ D \subseteq E$ [6]. An R-module M is called fully semi-prime, if every proper submodule of M is semi-prime submodule [3].A semi-prime radical of a module M denoted by Srad(M), and it is the intersection of all semi-prime submodule of M [3]. A submodule N of a module M is called y-closed submodule in M, if $\frac{M}{N}$ is a non-singular module [1],"where an R-module M is called non-singular if $Z(M) = \{x \in M: ann(x)\ is\ essential\ ideal\ in\ R\}$ =(0) [3]". A module M is called multiplication module, if every submodule N of M is equal IM. i.e N=IM for some ideal I of R [7].

Basic Properties of W-Closed Submodules

"In this section, we introduce the definition of" w-closed submodule, and we will give basic properties, examples of w-closed submodule.

Definition (2.1)

Asubmodule K of a module M is called w-closed in M ,"if K has no proper weak essential extension in M". That is if there exists asubmodule L of M with K "is a weak essential submodule of L", then K=L . An ideal J of R is called w-closed, if it is w-closed R-submodule.

Remark (2.2)

Every w-closed submodule in a module M is a closed submodule in M,but the converse is not true in general.

proof

Let K be a w-closed submodule in M and L is a submodule in M with K is essential in L, then by [2] K is weak essential in L. But K is w-closed in M, thus K=L. Hence K is closed submodule in M. For the converse, we give the following example:

Example(2.3)

Let M=Z_{24} as a Z-module, and $K = \langle \bar{3} \rangle$ is closed submodule in Z_{24}, since K is a direct summand of the Z-module Z_{24}, but K is not w-closed submodule in Z_{24} because K is weak essential submodule in Z_{24}.

Proposition (2.4)

If M is a module, and E is a submodule of M such that E is weak essential and w-closed in M,then E=M.

Proof

Follows from definition of w-closed submodule.

Remark (2.5)

(1) Every module M is a w-closed submodule in itself.

(2) The trivial submodule <0> may not be w-closed submodule of an R-module M, for example : $M = Z_2$ as a Z-module, $K = \langle\bar{0}\rangle$ is not w-closed submodule in M.

Proposition(2.6)

If M is a module, and let U be a non-zero submodule of M, then ∃ a w-closed submodule T in M with U is weak essential in T.

proof

Let $\mathcal{A}$ ={ Q : Q "is a submodule of M such that" U is weak essential in Q }. clearly $\mathcal{A}$ is a non-empty. $\mathcal{A}$ has maximal element say T "by Zorn's lemma". "To prove that" T is a w-closed submodule in M. Assume that there exists a submodule L of M with T weak essential in L. Since U is weak essential in T and T is weak essential in L so by [3, prop (1.4)]. U is weak essential in L. But this is a contradicts the maximality of T.Thus T=L. Hence T is w-closed submodule in M, with U is weak essential in T.

The following remark shows that w-closed property is not hereditary property.

Remark(2.7)

If Q_1 and Q_2 are submodules of an R-module M with Q_1 is a submodule of Q_2, and Q_2 is a w-closed submodule in M then Q_1 need not to be w-closed submodule in M. For example: M=Z the Z-module, M is a w-closed submodule of M, and 2Z is a submodule of M is not w-closed submodule in M, since 2Z has a proper weak essential extension.

The converse of remark (2.7) is not true. That is if Q_1 is w-closed in M, then Q_2 need not to be w-closed in M. As the next example explain:

Example(2.8)

Take the Z-module Z and N_1= <0>, N_2 = 2Z are Z-submodules of Z we notes that N_1 is w-closed submodule in Z. But N_2 is not w-closed submodule in Z.

The following propositions show that the transitive property for w-closed submodule hold under certain conditions.

Proposition (2.9)

If E and D are submodules of a module M, provided that D contained in any weak essential extensions of E, and E is a w-closed submodule in D and D is a w-closed submodule in M, then E is a w-closwed submodule in M.

Proof

Assume that K is a submodule of M such that E is weak essential in K. By hypothesis D is a submodule of K. Since E "is weak essential in K and E is a submodule of D" then by [2, Rem(1.5)(2)] we get D is weak essential in K. But D is w-closed submodule in M, then D=K. That is E weak essential in D. But E is w-closed submodule in D, so E=D. Hence E is a w-closed submodule in M.

Proposition(2.10)

If N_1 and N_2 are submodules of a module M, provided that N_2 is containing any weak essential extensions of N_1, and N_1 is a w-closed submodule in N_2 and N_2 is a w-closed submodule in M, then N_1 is a w-closed submodule in M.

Proof

Assume that $U \leq M$ with N_1 is weak essential submodule in U, then by hybothesis we get U is a submodule in N_2. Since N_1 is a w-closed in N_2, then N_1=U. Thus N_1 is a w-closed submodule in M.

Proposition(2.11)

If M is a chained module, and E, D are submodules of M with $E \leq D$, and $E \leq_W D$ and $D \leq_W M$, then $E \leq_W M$.

Proof

Let K be a submodule of M with E is weak essential in K. Since M is chained module, then either K is a submodule in D or D is a submodule in K. If K is a submodule in D, and since E is a w-closed submodule in D,then E=K.Hence E is a w-closed submodule in M. If D is a submodule in K, and since E is weak essential in K, then by [2, Rem(1.5)(2)] D is a weak essential submodule in K. But D is a w-closed submodule in M, hence D=K.Thus, E is a weak essential submodule in D. But E is a w-closed submodule in D, then E=D.Hence E is a w-closed submodule in M.

Before we give the next proposition, we introduce the following denifition.

"Definition(2.12)

A module M is called completly essential if every non zero weak essential submodule of M is an essential submodule of M".

Completely essential in [3] is called fully essential.

The following proposition show that closed submodules and w-closed submodules are equivalents under certain conditions.

Proposition(2.13)

"If M is a module, and E be a non zero submodule of M" such that every weak essential extensions of E is a completly essential, then E is a closed submodule in M if and only if E is a w-closed submodule in M.

Proof

Let E be a non zero closed submodule in M, and U be a submodule of M such that E is a weak essential in U. By hypothesis U is a completely essential, therefore E is an essential submodule in U. But E is a closed submodule in M, then E=U.That is E is a w-closed submodule.

The converse is direct.

Proposition(2.14)

If M is a fully semi-prime module, and E be a non zero submodule of M, then E is a closed submodule in M if and only if E is a w-closed submodule in M.

Proof

Assume that E is a non zero closed submodule in M, and U is a submodule of M such that E is a weak essential submodule in U. Then by [3, Cor(2.5)] E is an essential submodule in U. But E is a non-zero closed submodule in M, hence E=U. That is E is a w-closed submodule in M.

The converse is direct.

Corllary (2.15)

If M is a uniform module, and E be a non zero submodule of M, then E is a closed submodule in M if and only if E is a w-closed submodule in M.

Proof

Assume that E is a closed submodule in M and let E a weak essential in U where U is a submodule of M, then U is a uniform. Hence by [3,prop(2.7)] U is a completely essential. Thus E is an essential in U. But E is a closed, then E=U. Thus E is a w-closed in M.

The converse is direct.

The following propositions show that the transitive property for w-closed submodules hold under conditions fully semi-prime and completely essential.

Proposition(2.16)

Let M be a module, and E, D are non-zero submodules of M such that $E \leq D$ and every weak essential extensions of E is a completely essential submodule of M. If $E \leq_W D$ and $D \leq_W M$, then $E \leq_W M$.

Proof

Since $E \leq_W D$ and $D \leq_W M$. Then by remark(2.2), we get E is a closed submodule in D and D is a closed submodule in M. Then by [1,prop(1.5),P.18] "we get E is a closed submodule in M", then by prop(2.13), $E \leq_W M$.

Proposition (2.17)

Let M be a fully semi-prime module, and let E be a non-zero w-closed submodule in D and D is a w-closed submodule in M. Then E is a w-closed submodule in M.

Proof

Since E is a w-closed submodule in D and D is a w-closed submodule in M, then by remark(2.2), E is a closed submodule in D and D is a closed submodule in M. Hence by [1,prop(1.5), P.18] we get E is a closed submodule in M. Thus by prop(2.14), E is a w-closed submodule in M.

Remark (2.18)

The intersection of two w-closed submodule need not to be w-closed submodule as the following example shows:

In the Z-module $Z_8 \oplus Z_2$, the submodules $N = \langle(\bar{0},\bar{1})\rangle$ *and* $K = \langle(\bar{4},\bar{1})\rangle$ are w-closed submodule in $Z_8 \oplus Z_2$, but $N \cap K = (\bar{0},\bar{0})$ is not w-closed submodule in $Z_8 \oplus Z_2$.

The following results give more basic properties of w-closed submodules.

Proposition (2.19)

If every submodule of a module M is w-closed, then every submodule of M is a direct summand. Provided that M is a semi simple.

Proof

Since every submodule of M is w-closed, then every submodule of M is a closed. Hence by [8, Exc(6-c), P.139] "every submodule of M is a direct summand of M".

The following corollary is a direct consequence of proposition(2.19).

Corollary (2.20)

If every submodule of a module M is a w-closed, then M is a semi-simple.

Proposition(2.21)

If E and D are submodules of a module M with $E \leq D$, and $E \leq_W M$, then $E \leq_W D$.

Proof

Let $F \leq D$, then $F \leq M$, and E is a weak essential submodule of F. But $E \leq_W M$, then E=F. Hence $E \leq D$.

As a direct application of proposition(2.21) we get the following results.

Corollary (2.22)

If E and D are submodules of a module M with $E \cap D$ is a w-closed submodule in M , then $E \cap D$ is a w-closed submodule in E and D.

Corollary (2.23)

If M is a module, and E , U are w-closed submodules in M,then E and U are w-closed submodules in E + U.

Corollary(2.24)

If M is an R-module , and E is a w-closed submodule in M, then E is a w-closed submodule in $\sqrt{E}$.

Proof

Since $E \leq \sqrt{E} \leq M$, and E is a w-closed submodule in M then by proposition(2.21), E is a w-closed submodule in $\sqrt{E}$.

Remark (2.25)

A direct summand of a module M is not necessary w-closed submodule in M, as the following example show:

Let M=Z_{24}as a Z-module , where $Z_{24} = \langle\bar{3}\rangle \oplus \langle\bar{8}\rangle$, the direct summand $\langle\bar{3}\rangle$ is not w-closed submodule in Z_{24}. Since $\langle\bar{3}\rangle$ is a weak essential in Z_{24}.

Proposition(2.26)

Let $X = X_1 \oplus X_2$ be a module, where X_1 and X_2 are submodules of X, and let E be a non zero w-closed submodule in X_1 and D is a non zero w-closed submodule in X_2 such that ann X_1 + ann X_2=R, and all weak-essential extensions of $E \oplus D$ are completely essential submodule of $X_1 \oplus X_2$. Then $E \oplus D$ is a w-closed submodule in $X_1 \oplus X_2$.

Proof

Let $S \leq X$ with $E \oplus D$ "is a weak essential submodule in S". Since S is a submodule of X and ann X_1 + ann X_2=R, then by [9, prop(4.2)], $S = S_1 \oplus S_2$, where S_1 is a submodule of X_1 and S_2 is a submodule of X_2. Thus $E \oplus D$ is a weak essential submodule in $S_1 \oplus S_2$. But by hybothesis S is a completely essential , therefore $E \oplus D$ is an essential submodule in $S = S_1 \oplus S_2$, thus by [10, prop(5.20)] we are, "E is an essential submodule in S_1 and D is an essential submodule in S_2". Since both E and D are w-closed, it is a clear that E and D are closed submodules in S_1 and S_2 respectively. Then E=S_1 and D=S_2, thus $E \oplus D$=$S_1 \oplus S_2$. That is $E \oplus D$ is a w-closed submodule in X.

Proposition(2.27)

Let $X = X_1 \oplus X_2$ be a module, where X_1 and X_2 are submodules of X such that ann X_1 + ann X_2=R and all submodules of X are completely essential submodule of X. If E and D are non zero submodules of X_1 and X_2 respectively, then $E \oplus D$ is a w-closed submodule in X if and only if E is a w-closed submodule in X_1 and D is a w-closed submodule in X_2.

Proof

($\Leftarrow$) Suppose that $E \oplus D$ "is weak essential submodule of K", "where K is a submodule of M". Hence by [1, prop(4.2)] $K = K_1 \oplus K_2$ where K_1 is a submodule of X_1 andK_2 is a submodule of X_2. Thus $E \oplus D$ is weak essential submodule in $K_1 \oplus K_2$. But $K_1 \oplus K_2$ is a completely "essential submodule of " X, then $E \oplus D$ "is an essential submodule of " $K_1 \oplus K_2$. Hence by [10, prop(5.20), P.15] we get "E is an essential submodule in K_1 and D is an essential submodule in K_2". But by [2] every essential submodule is a weak essential. Hence E "is a weak essential submodule in K_1" and D is a weak essential submodule in K_2. But E and D are w-closed submodules of X, then E=K_1 and D=K_2. Thus $E \oplus D$=$K_1 \oplus K_2$. That is $E \oplus D$ is a w-closed submodule in X.

($\Rightarrow$) Assume that E "is a weak essential submodule in L" where L is a submodule of X, we have D is a weak essential submodule in D. But by hypothesis all submodules of X are completely essential, then E is an essential submodule in L and D is an essential submodule in D. Hence by [10, prop(5.20), P.15], we have. $E \oplus D$ is an essential submodule in $L \oplus D$, which implies that $E \oplus D$ is a weak essential submodule in $L \oplus D$. Hence $E \oplus D$= $L \oplus D$. That is E=L, implies that E is a w-closed submodule inX_1.

In similar way we can prove that D is w-closed submodule inX_2.

It is well-known that a fully semi-prime module is a completely essential [3, cor(2.6)]. So we have the following result.

Corollary(2.28)

If $X = X_1 \oplus X_2$ is a module, where X_1 and X_2 are submodules of X with ann X_1 + ann X_2=R and all submodules of X are fully-semi-prime. If E, D are submodules of X_1 and X_2 respectively, then $E \oplus D$ is a w-closed submodule in X if and only if E is a w-closed submodule in X_1 and D is a w-closed submodule in X_2.

The following remark shows that w-closed property is not algebrice property.

Remark(2.29)

If M is a module, and X is a w-closed submodule of M, and Y is asubmodule of M such that X≅Y , then it is not necessary that Y is a w-closed submodule in M, as the following example shows:- The Z-module Z is a w-closed in itself and Z ≅ 3Z, but 3Z as a Z-module is not a w-closed submodule in Z, since 3Z "is a weak-essential submodule of Z".

We introduce the following lemma, before we give the next proposition.

Lemma(2.30)

Let $f \in Hom(M_1, M_2)$ be module an epimorphism with $Ker\, f \leq Srad(M_1)$, if $E \leq_{weak} M_2$. Then $f^{-1}(E) \leq_{weak} M_1$.

Proof

Assume that $E \leq_{weak} M_2$, and $f^{-1}(E) \cap S = (0)$ where S is a semi-prime submodule of M_1. But $Ker\, f \leq Srad(M_1) \leq S$ for all semi-prime submodule S of M_1, hence by [5, prop(2.1)(A)] $f(S)$ is a semi-prime submodule of M_2. That is $E \cap f(S) = (0)$, but E "is a weak essential submodule of M_2", then $f(S) = (0)$. Implies that $S \leq Kerf \leq f^{-1}(E)$, and hence $f^{-1}(E) \cap S = (0)$ implies that $S = (0)$. Then $f^{-1}(E) \leq_{weak} M_1$.

Proposition(2.31)

Let $g: M_1 \rightarrow M_2$ be a module epimorphism, and let E be a submodule of M_1 such that $\ker g \leq Srad(M_1) \cap E$.If E is a w-closed submodule inM_1 then $g(E)$ is a w-closed submodule in M_2.

Proof

Suppose that E is a w-closed submodule in M_1, and let $g(E)$ "is a weak essential submodule of L", where L is a submodule of M_2. Since $\ker g \leq Srad(M_1) \cap E$. Hence by lemma(2.30), we get $g^{-1}(g(E))$ is a weak essential submodule in $g^{-1}(L)$, where $g^{-1}(L)$ is a submodule of M_1, but $Kerg \leq E$, then $g^{-1}g(E) = E$, i.e E is a weak essential in $g^{-1}(L)$. But E is a w-closed submodule in M_1, then E=$g^{-1}(L)$, and since g is an epimorphism so , $g(E) = L$. Hence $g(E)$ is a w-closed submodule in M_2.

As a direct consequence of proposition(2.31) we get the following corollary.

Corollary(2.32) : If E and D are submodules of a module M with $E \leq srad(M) \cap D$. If D is a w-closed submodule in M, then $\frac{D}{E}$ is a w-closed submodule in $\frac{M}{E}$.

The following proposition gives a relation between y-closed submodule and w-closed submodule in the class of a fully semi-prime module.

Proposition (2.33)

Let M be a fully semi-prime module. Then every non zero y-closed submodule is a w-closed submodule.

Proof

Let E be a non zero y-closed submodule in M, then by [11], every y-closed submodule is a closed. Hence E is a closed, then by proposition(2.14), E is a w-closed submodule in M.

"The following proposition shows that in the class of non-singular modules", the class of w-closed submodules is contained in the class of y-closed submodules.

Proposition (2.34)

If M is a non singular module and E is a w-closed submodule of M, then E is a y-closed submodule of M.

Proof

Let E be a w-closed submodule in M then E "is a closed submodule in M", but M is a non-singular R-module, then by [11, prop(2.1)(2)] E is a y-closed submodule in M.

The following proposition shows that in the class of non-singular and fully semi-prime R-module, w-closed submodule , y-closed submodule and closed submodule are equivalent:

Proposition (2.35)

Let M be a fully semi-prime and non-singular module, "and E be a non zero submodule of M. Then the following statements are equivalent" :

1- E is a y-closed submodule .

2- E is a closed submodule .

3- E is a w-closed submodule.

Proof

$(\mathbf{1}) \implies (\mathbf{2})$ Follows by [11].

Follows by proposition(2.14).$(\mathbf{2}) \implies (\mathbf{3})$

Follows by proposition(2.34).$(\mathbf{3}) \implies (\mathbf{1})$

3. W-closed submodule in multiplication modules

In this section, we establishe some relationships between w-closed submodule and multiplication modules.

"First we introduce the following definition".

Definition(3.1)

A non-zero semi-prime submodule E of a module M is called minimal semi-prime submodule of M, if whenever S "is a non zero semi-prime submodule of M such that" $S \leq E$, then S=E. That is by minimal semi-prime submodule E of M we mean a semi-prime submodule which is a minimal in the collection of semi-prime submodules of M. If A is a proper ideal of R, then a semi-prime ideal B is called a minimal semi-prime ideal of A provided that $A \leq B$ and $\frac{B}{A}$ is minimal semi-prime ideal of a ring $\frac{R}{A}$.

Remark(3.2)

In multiplication module since $ann(M) \neq R$ it follows that by [12, Th(2.5)], there exists a minimal ideal P of R such that $ann(M) \leq P, and\ M \neq PM$. But by [13, prop(2.5), P.36] PM is a semi-prime submodule of M.

Then from definition(3.1) we get the following facts:

(a) E is a minimal semi-prime submodule of M if and only if there exists a minimal semi-prime ideal A, with $ann(M) \leq A$ such that $E = AM \neq M$.

(b) Eveery semi-prime submodule of M contains a minimal semi-prime submodule.

Lemma(3.3)

If M is a faithful and multiplication module, and E be a non zero semi-prime submodule of M. If E is not minimal semi-prime, then E "is a weak-essential submodule of M".

Proof

Since M is a multiplication, and E is a semi-prime submodule of M, then by [13,prop(2.5), P.36] ∃ a "semi-prime ideal K of R" with $(0) = annM \leq K$ and E=KM. "Let S be a non-zero semi-prime submodule of M" such that $E \cap S = (0)$.But E is not minimal semi-prime, then by remark(3.2)(b) every semi-prime submodule of M contain a minimal semi-prime submodule say $E_1 \leq E$. Hence by remark(3.2)(a), there exists a minimal semi-prime ideal K_1 of R such that $ann(M) \leq K_1$ and $E_1 = K_1M \neq M$, $(K \cap [S:M])\, M = KM \cap [S:M]\, M = E \cap S = (0) = (0)$. But M is faithful, then $K \cap [S:M] = (0)$, which implies that $K \cap [S:M] \leq K_1$, that is either $K \leq K_1 or\ [S:M] \leq K_1$. If $K \leq K_1$, then $KM \leq K_1M$, implies that $E \leq E_1$ which is a contradiction. Thus, $[S:M] \leq K_1$. That is $[S:M]\, M \leq K_1M$, implies that $S \leq E_1 \leq E$ which is contradict the minimality of E_1. Thus $E \cap S = (0)$ is not true. Thus $E \cap S \neq (0)$, which implies that E is a weak essential submodule of M.

Proposition (3.4)

If M is a faithful and multiplication module, and E be a non-zero semi-prime submodule and w-closed submodule of M, then E is a minimal semi-prime submodule of M.

Proof

Suppose that E is not minimal semi-prime submodule of M, then by lemma(3.3), E "is a weak essential submodule of M". But E is a w-closed submodule in M, then E=M. On the other hand E is a semi-prime submodule of M, that E must be a proper submodule of M, so we get contradiction. Hence E must be a minimal "semi-prime submodule of M".

Proposition (3.5)

Let M be a non zero multiplication module with only one non zero maximal submodule E. Then E can not be w-closed submodule in M.

Proof

Assume that E is a w-closed submodule in M, then by [3, prop(2.20)] E "is a weak essential submodule of M". Hence E=M. "But this contradict the maximality of E". Therefore E is not W-closed submodule in M.

"Recal that for any module M and any ideals I and J of R if I is a semi-prime ideal of J then IM is a semi-prime submodule of JM this is called condition$(*)$ in [3]".

Proposition(3.6)

Let M be a faithful and multiplication module such that M satisfies condition$(*)$, if L is a w-closed ideal in K then LM is a w-closed submodule in KM.

Proof

Suppose that L is a w-closed ideal in K, and LM is a weak essential submodule of T where T is a submodule of KM, we have to show that LM=T. Since M is a multiplication module, then T=PM for some ideal P of R with $P \leq K$. That is LM "is a weak essential submodule of PM", and since M is faithful and satisfies condition$(*)$ then by [3,prop(2.17)], we have L is a weak essential ideal in P and $P \leq K$. But L is a w-closed ideal in K, then L=P. That is LM=PM=T. Hence LM is a w-closed submodule in KM.

The following proposition gives the converse of proposition(3.6).

Proposition (3.7)

If M is a finitely generated,faithful and multiplication module, and LM is a w-closed submodule in KM, then L is a w-closed ideal in K.

Proof

Suppose that LM is a w-closed submodules in KM, where L and K are ideals in R, and let L is a weak essential ideal in U where U is an ideal of K. "Since M is finitely generated faithful and multiplication", then by [3, prop(2.18)] we have LM is a weak essential in UM which is a submodule of KM. But LM is a w-closed submodule in KM, then LM=UM. Hence by [12, Th,(3.1)], L=U. Then L is a w-closed ideal in K.

From proposition (3.6) and proposition(3.7) we get the following corollary.

Corollary(3.8)

"If M is a finitely generated faithful and multiplication module which satisfies condition($*$)", then L is a w-closed ideal in K if and only if LM is a w-closed submodule in KM.

Theorem(3.9)

If M is a finitely generated faithful and multiplication module, and let E be a submodule of M, such that M satisfies condition($*$), "then the following statements are equivalent" :

1- E is a w-closed submodule in M.

2-$[E \underset{R}{:} M]$ is a w-closed ideal in R.

3- E=PM for some w-closed ideal P in R.

Proof

$(1) \Longrightarrow (2)$ Suppose that E is a w-closed submodule in M. Since M is a multiplication, then by [7] $E = [E \underset{R}{:} M]\, M$. Put $[E \underset{R}{:} M] = P$, then we have PM=E is a w-closed submodule in M. Hence by cor(3.8), P is a w-closed ideal in R. That is$[E \underset{R}{:} M]$ is a w-closed ideal in R.

$(2) \Longrightarrow (3)$: Suppose that $[E \underset{R}{:} M]$ is a w-closed ideal in R. Then $E = [E \underset{R}{:} M]\, M$ since M is multiplication,i.e E=PM where $P = [E \underset{R}{:} M]$ is a w-closed ideal in R.

: Suppose that E=PM for some w-closed submodule P in R. Then by cor(3.8), $(3) \Longrightarrow (1)$ PM=E is a w-closed submodule in RM=M.

4- Chain conditions on w-closed submodules

We start this section by introducing the definitions of a modules that have ascending (descending) chain condition on w-closed submodules.

Definition(4.1)

A module M is said to have the ascending chain condition on w-closed submodule(briefly acc on w-closed submodules), if every ascending chain $E_1 \subseteq E_2 \subseteq \ldots$ of w-closed submodule in M is finite. That is $\exists\, m \in Z_+$ such that $E_n = E_m$ for all $n \geq m$.

Definition(4.2)

A module M is said to have the descending chain condition on w-closed submodule(briefly dcc on w-closed submodules), if every descending chain $E_1 \supseteq E_2 \supseteq \ldots$ of w-closed submodule in M is finite. That is $\exists\, m \in Z_+$ such that $E_n = E_m$ for all $n \geq m$.

Remarks (4.3)

1- Zp^∞ as a Z-module satisfies dcc on w-closed submodules, but Zp^∞ as a Z-modules does not satisfies acc on w-closed submodules because Zp^∞ is an artinian but not noetherian.

2- Z as Z-module satisfies (acc) on w-closed submodules, but does not satisfies (dcc) on w-closed submodules because Z as a Z-module is a noetherian but not artinian.

Proposition (4.4)

If M is a module and satisfies (dcc) on closed submodules, then M satisfies (dcc) on w-closed submodules.

Proof

Let $E_1 \supseteq E_2 \supseteq \ldots$ "be a descending chain" of w-closed submodules of M. But by remark(2.2) every w-closed submodule is closed, then E_i is a closed submodule for each i=1,2,. . . . Since M satisfies (dcc) on closed submodule, then $\exists\, m \in Z_+$ such that $E_n = E_m$ for each $n \geq m$. Thus, M satisfies (dcc) on w-closed submodules.

The proof of the following proposition is similar to the proof of proposition (4.4) and hence is omited.

Proposition (4.5)

If M is a module and satisfies (acc) on closed submodules, then M satisfies (acc) on w-closed submodules.

Since w-closed submodules and closed submodules are equivalent in the class of fully semi-prime modules by proposition (2.14), "we get the following results".

Proposition (4.6)

If M is a fully semi-prime module, then M satisfies (acc) on w-closed submodules if and only if M satisfies (acc) on closed submodules.

Proof

$(\Longrightarrow)$ Let $E_1 \subseteq E_2 \subseteq \ldots$ "be ascending chain of closed submodules". Then by prop(2.14), E_i is a w-closed submodule for each i=1,2, ... But M satisfies (acc) on w-closed submodules, so $\exists\, m \in Z_+$ such that $E_n = E_m$ for all $n \geq m$. Thus M satisfies (acc) on closed submodules.

$(\Longleftarrow)$ By proposition (4.5).

The proof of the following proposition is similar to proof of proposition (4.6).

Proposition (4.7)

Let M be a fully semi-prime module. "Then M satisfies (dcc) on closed submodules if and only if M satisfies (dcc)" on w-closed submodules.

Proposition(4.8)

If M is a module , and $E_1 \subseteq E_2 \subseteq \ldots$ "be ascending chain of submodules such that" each weak essential extension of E_i is a completely essential for each i=1,2, . . . , then M satisfies (acc) on w-closed submodules if and only if M satisfies (acc) on closed submodules.

Proof

($\Longrightarrow$) Let $E_1 \subseteq E_2 \subseteq \ldots$ "be ascending chain" of closed submodules. Then by prop(2.13), E_i is a w-closed submodules for each i=1,2, ... But M satisfies (acc) on w-closed submodules, then there exists a non zero integer m such that $E_n = E_m$ for all $n \geq m$. Hence M satisfies (acc) on closed submodules.

Follows by proposition (4.5).($\Longleftarrow$)

The proof the following proposition is similar to proof of proposition (4.8).

Proposition(4.9)

If M is a module, and $E_1 \supseteq E_2 \supseteq \ldots$ be a descending chain of submodules such that each weak essential extension of E_i is a completely essential for each i=1,2, ... Then M satisfies (dcc) on w-closed submodules if and only if M satisfies (dcc) on closed submodules.

Proposition(4.10)

If M is a module, and D be a submodule of M such that $D \leq Srad(M) \cap K$, where K is any w-closed submodule in M. If $\frac{M}{D}$ satisfies (dcc) on w-closed submodules, then M satisfies (dcc) on w-closed submodules.

Proof

Let $E_1 \supseteq E_2 \supseteq \ldots$ be a descending chain of w-closed submodules in M. Since E_i is a w-closed submodule in M for each i=1,2, . . . , and $D \leq Srad(M) \cap E_i$ then by Corollary(2.32), we have $\frac{E_i}{D}$ is a w-closed submodule in $\frac{M}{D}$ for each i=1,2, ... Hence $\frac{E_1}{D} \supseteq \frac{E_2}{D} \supseteq \ldots$, is a descending chain of w-closed submodules in $\frac{M}{D}$. But $\frac{M}{D}$ satisfies (dcc) on w-closed submodules, so there exists a positive integer m such that $\frac{E_n}{D} = \frac{E_m}{D}$ for each $n \geq m$. So, that $E_n = E_m$ for each $n \geq m$. Thus M satisfies (dcc) on w-closed submodules.

Proposition(4.11)

If M is a module, and D be a submodule of M such that $D \leq Srad(M) \cap K$, where K is any w-closed submodule in M. If $\frac{M}{D}$ satisfies (acc) on w-closed submodules, then M satisfies (acc) on w-closed submodules.

Proof

Similar to proof of proposition (4.10).

Proposition(4.12)

If $X = X_1 \oplus X_2$ is a module, where X_1 and X_2 are submodules of X, provided that ann X_1 + ann X_2=R, and all weak essential extensions of $E_i \oplus X_2$ (or $X_1 \oplus E_i$

) are completely essential modules where E_i is a non zero w-closed submodule in X_1 (or X_2) for each i=1,2, ... If X satisfies (dcc) on w-closed submodules, then X_1 $(or\ X_2)$ satisfies (dcc) on non- zero w-closed submodules.

Proof

Let $E_1 \supseteq E_2 \supseteq . . .$ "be a descending chain" of a non-zero w-closed submodules of X_1. If X_2 is equal to zero, then X=X_1 and this, implies that X_1 satisfies (dcc) on non-zero w-closed submodules. Otherwise, since E_i is a non-zero w-closed submodule in X_1, and X_2 is a w-closed in X_2, so by proposition(2.26), $E_i \oplus X_2$ is a w-closed submodule in X for each i=1,2, . . . , $E_1 \oplus X_2 \supseteq E_2 \oplus X_2 \supseteq . . .$, "is a descending chain" of w-closed submodule in X. But X satisfies (dcc) on w-closed submodules, then there exists a positive integer m such that $E_n \oplus X_2 = E_m \oplus X_2$ for all $n \geq m$. Thus $E_n = E_m$ for all $n \geq m$. Thus X_1 satisfies (dcc) on w-closed submodule.

Similarly we can prove that X_2 satisfies (dcc) on w-closed submodule.

Proposition(4.13)

If $X = X_1 \oplus X_2$ is a module, where X_1 and X_2 are submodules of X, provided that ann X_1 + ann X_2=R, and all weak essential extensions of $E_i \oplus X_2$ (or $X_1 \oplus E_i$) are completely essential modules where E_i is a non zero w-closed submodule in X_1 (or X_2) for each i=1,2, ... If X satisfies (acc) on w-closed submodules, then X_1 $(or\ X_2)$ satisfies (acc) on non-zero w-closed submodules.

Proof

Similarly as in proposition (4.12).

We end this section by the following propositions.

Proposition(4.14)

"If M is a finitely generated,faithful and multiplication module, and M satisfies condition($*$)", then M satisfies (dcc) on w-closed submodules, if and only if R satisfies (dcc) on w-closed ideals.

Proof

($\Longrightarrow$) Let $L_1 \supseteq L_2 \supseteq . . .$, "be a descending chain" of w-closed ideals in R. Since L_i is a w-closed ideal in R for each i=1,2, ... Then by cor(3.8) $L_i M$ is a w-closed submodule in M for each i=1,2, . . . , then $L_1 M \supseteq L_2 M \supseteq . . .$, be a "descending chain" of w-closed submodules in M. But M satisfies (dcc) on w-closed submodules, "so there exists a positive integer m such that" $L_n M = L_m M$ for each $n \geq m$. But M is a finitely generated faithful and multiplication R-module, then by [12, Th(3.1)], $L_n = L_m$ foe each $n \geq m$. Therefore R satisfies (dcc) on w-closed ideals.

($\Longleftarrow$) Let $E_1 \supseteq E_2 \supseteq . . .$, be a descending chain of w-closed submodules in M. Since M is multiplication module, then $E_i = L_i M$ for some ideal L_i of R ∀ i=1,2, . . . , then $L_1 M \supseteq L_2 M \supseteq . . .$. Since E_i is a w-closed submodule in M for each i=1,2, . . . , so by cor(3.8), L_i is a w-closed ideal in R for each i=1,2, . . . , . But M is a finitely generated, faithful and multiplication module, then by [12, Th(3.1)] we have $L_1 \supseteq L_2 \supseteq . . .$, is a "descending chain" of w-closed ideals in R. But R satisfies (dcc) on w-closed ideals, therefore, there exists a positive integer m such that $L_n M = L_m M$ for each $n \geq m$, thus $E_n = E_m$ for each $n \geq m$.

The proof the following proposition is similar to the proof of prop(4.14), hence we omited.

Proposition(4.15)

"If M is a finitely generated,faithful and multiplication module, and M satisfies condition(∗)", then M satisfies (acc) on w-closed submodules, if and only if R satisfies (acc) on w-closed ideals.

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