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Fuzzy Internal Direct Product

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Abstract

The purpose of this research is to show a constructive method for using known fuzzy groups as building blocks to form more fuzzy subgroups. As we shall describe employing this procedure with the fuzzy generating subgroups will give us a large class of fuzzy subgroup of abelian groups which include all fuzzy subgroup of abelian groups of finite order.

Some definitious are given below:

Definition

Let A be a fuzzy subgroup of a group (G,.) and $N_1, N_2, \ldots, N_n$ fuzzy normal subgroups in A such that

1- $A = N_1 \times N_2 \times \ldots \times N_n$

2- Let $x_t \subseteq A$, $x \in G$, $t \in [0,1]$, then

$$x_t(y) = \begin{cases} \sup\{\min\{n_1(y_1), n_2(y_2), \ldots, n_n(y_n)\}\} & \\ \qquad y = y_1 \cdot y_2 \cdots y_n \text{ and } y_1, y_2, \ldots, y_n \in G & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

$n_i \subseteq N_i$ in unique way.

We then say that A is the fuzzy internal direct product of $N_1, N_2, \ldots, N_n$.

Now the following theorem is introduced:

Theorem

If a fuzzy subgroup A of a group (G,.) is the fuzzy internal direct product of $N_1, N_2, \ldots, N_n$ then for $i \neq j$, $N_i \cap N_j = \{e_t\}$, and if $n_i \subseteq N_i$, $n_j \subseteq N_j$ then $n_i n_j = n_j n_i$.

Proof:

Suppose that $n_s \subseteq N_i \cap N_j$ then we can write n_s as

$$n_s(y) = \begin{cases} \sup\{\min\{e_1,\ldots,e_{i-1},n_s(y),e_{i+1}\ldots,e_j,\ldots,e_n\}\} & \text{if } y = n, n_i \subseteq N_i \ \ i = 1,\ldots,n \\ 0 & \text{other wise} \end{cases}$$

Viewing n_s as an fuzzy singleton in N_i . similarly, we can write n_s as

$$n_s(y) = \begin{cases} \sup\{\min\{e_1,\ldots,e_i,\ldots,e_{j-1},n_s(y),e_{j+1},\ldots,e_n\}\} & \text{if } y = n, n_i \subseteq N_i \ \ i = 1,\ldots n \\ 0 & \text{other wise} \end{cases}$$

Since the two decompositions in this form for n_s must coincide, the entry from N_i in each must be equal. In our first decomposition this entry is n_s, in the other it is e_t; hence $n_s = e_t$. Thus, $N_i \cap N_j = \{e_t\}$ for $i \neq j$.

Suppose $n_i \subseteq N_i$, $n_j \subseteq N_j$, and $i \neq j$ then $n_i n_j n_i^{-1} \in N_j$ since N_j is normal fuzzy; thus, $n_i n_j n_i^{-1} n_j^{-1} \subseteq N_j$. Similarly, since $n_i^{-1} \subseteq N_i$, $n_j n_i^{-1} n_j^{-1} \subseteq N_i$, whence $n_i n_j n_i^{-1} n_j^{-1} \subseteq N_i$ then $n_i n_j n_i^{-1} n_j^{-1} \subseteq N_i \cap N_j = \{e_t\}$.

Thus $n_i n_j n_i^{-1} n_j^{-1} = e_t$; this gives the desired result $n_i n_j = n_j n_i$.

One should point out that if $k_1,\ldots,k_n$ are normal fuzzy subgroups of A such that $A = k_1 \times k_2 \times \ldots \times k_n$ and $k_i \cap k_j = \{e_t\}$ for $i \neq j$ it need not be true that A is the fuzzy internal direct product of $k_1,\ldots,k_n$. Amone stringent condition is needed.

Clearly from the above theorem the following corollary can be obtained:

Corollary

A fuzzy subgroup A of a group (G, •) is the fuzzy internal direct product of the normal fuzzy subgroups $N_1,\ldots,N_n$ if and only if

(1) $A = N_1 \times N_2 \times \ldots \times N_n$

(2) $Ni \cap (N_1 \times N_2 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_n) = \{e\}$ for $i = 1,\ldots,n$

Definition

Let H and k fuzzy subgroups of a group (G, •). The join $H \vee k$ of H and k is the intersection of all fuzzy subgroups of G containing H•k.

$$H.k = \begin{cases} \sup\{\min\{H(x_1),K(x_2)\} x = x_1 \cdot x_2, x_1, x_2 \in G\} & \text{if } x \in \operatorname{Im}(\cdot) \\ 0 & \text{other wise} \end{cases}$$

Clearly, this intersection will be the smallest possible fuzzy subgroup of G containing H•k, and if elements in H and k commute, in particular, if G is abelian, we have $H \vee k = H \bullet k$.

Note that since $H(x_1) = \sup\{\min \{H(x_1), k(e)\}\ x_1 \in G\}$ and $K(x_2) = \sup \{\min \{H(e), k(x_2)\}\ x_2 \in G\}$, then

$H \subseteq H \bullet k$ and $k \subseteq H \bullet k$, so $H \leq H \vee k$ and $k \leq H \vee k$.

But clearly, $H \vee k$ would be contained in any fuzzy subgroup containing both Hand k.

Thus it is seen that $H \vee k$ is the smallest fuzzy subgroup of G containing both H and k.

Theorem

A fuzzy subgroup A of a group (G, •) is the fuzzy internal direct product of fuzzy subgroups H and k if and only if

1) $A = H \vee k$

2) $x_t \bullet y_s = y_s \bullet x_t$ for all $x_t \subseteq H$ and $y_s \subseteq k$, t, s ∈ [0,1], x, y ∈ G.

3) $H \cap k = \{e\}$

Proof

Let A be the fuzzy internal direct product of H and k. It is claimed that (1), (2) and (3) are obvious if one will regard A as isomorphic to the fuzzy internal direct product of H and k under the map ϕ, with $\phi(x_t, y_s) = x_t \bullet y_s = (x \bullet y)_r$ where $r = \min \{t, s\}$, x, y ∈ G, r, s ∈ [0, 1]. Under this map. $\overline{H}$ and $\overline{k}$

Corresponds to H, and k respectiviy.

$$\overline{H} = \{(x_t, e) \mid x_t \subseteq H\}$$

$$\overline{k} = \{(e, y_s) \mid y_s \subseteq k\}$$

Then (1), (2) and (3) follow immediately from the corresponding assertions regarding $\overline{H}$ and $\overline{K}$ in $H \times k$, which are obvious.

Conversely, let (1), (2) and (3) hold. It must be shown that the map ϕ of the fuzzy internal direct product $H \times k$ in to A, given by

$\phi(x_t, y_s) = x_t \bullet y_s = (x \bullet y)_r$ where $r = \min\{t, s\}$, x, y ∈ G, r, s ∈ [0,1], is an isomorphism. The map ϕ has already been defined suppose $\phi(x_{t1}, y_{s1}) = \phi(x_{t2}, y_{s2})$.

Then $x_{t1} \bullet y_{s1} = x_{t2} \bullet y_{s2}$; consequently $x_{t2}^{-1} \cdot x_{t1} = y_{s2} \cdot y_{s1}^{-1}$

But $x^{-1}_{t2} \bullet x_{t1} \subseteq H$ and $y_{s2} \bullet y^{-1}_{s1} \subseteq k$ and they are the same element and thus in $H \cap k = \{e\}$ by (3). Therefore, $x^{-1}_{t2} \bullet x_{t1} = e$ and $x_{t1} = x_{t2}$.

Likewise, $y_{s1} = y_{s2}$, so $(x_{t1}, y_{s1}) = (x_{t2}, y_{s2})$. This shows that ϕ is one to one. The fact that $x_t \bullet y_s = y_s \bullet x_t$ by (2) for all $x_t \subseteq H$ and $y_s \subseteq k$ means that

$H\bullet k(x) = \{\sup\{\min\{H(x_1), k(x_2)\}x = x_1\bullet x_2 , x_1, x_2 \in G\}$ is a fuzzy subgroup, for we have seen that this is the case if fuzzy singletons of H commute with those of k.

Thus by (1), $H\bullet k = H \vee k = A$, so ϕ is on to A. Finally,

$\phi[(x_{t1} , y_{s1}) (x_{t2} , y_{s2})] = \phi(x_{t1} \bullet x_{t2} , y_{s1} \bullet y_{s2}) = x_{t1} \bullet x_{t2} \bullet y_{s1} \bullet y_{s2}$, while

$[\phi(x_{t1} , y_{s1})][\phi(x_{t2} , y_{s2})] = x_{t1} \bullet y_{s1} \bullet x_{t2} \bullet y_{s2}$.But by (2) we have $y_{s1}\, x_{t2} = x_{t2} \bullet y_{s1}$. Thus

$\phi[(x_{t1} , y_{s1})(x_{t2} , y_{s2})] = [\phi(x_{t1} , y_{s1})][\phi(x_{t2} , y_{s2})]$.

Remark

Not every fuzzy subgroup of abelian group is the fuzzy internal direct product of two proper fuzzy subgroups.

The following corollary are immediate consequences of the above theorem.

Corollary

Let A be fuzzy subgroup of a group $(G, \bullet)$. Let x_t and y_s be fuzzy singletons of A which commute and are of relatively prime orders r and s and $\langle x_t\rangle$, $\langle y_s\rangle$ are fuzzy subgroups of $\langle x_t\rangle \vee \langle y_s\rangle$. Then $x_t \bullet y_s$ is of order r.s.

Thus for fuzzy invariants of fuzzy subgroup, we have the following definition and Lemma.

Definition

Let A be fuzzy subgroup of a belian group $(G, \bullet)$ of order p^n, p a prime, and

$A = A_1 \times A_2 \times \ldots\times A_k$ where each A_i is fuzzy generating set of order p^{ni} with $n_1 \geq n_2 \geq \ldots \geq n_k > 0$, then the integers $n_1, n_2, \ldots, n_k$ are called the fuzzy invariants of A just because we called the integers above the fuzzy invariants of A does not mean that they are really the fuzzy invariants of A. That is, it is possible that we can assign different sets of fuzzy invariants to A. We shall soon show that the fuzzy invariants of A are indeed unique and completely describe A. Note one other thing about the fuzzy invariants of A. If $A = A_1 \times \ldots \times A_k$, where A_i is fuzzy generating set of order P^{ni}, $n_1 \geq n_2 \geq \ldots \geq n_k > 0$, then $0(A) = 0(A_1)0(A_2)\ldots\ 0(A_k)$, hence $P^n = P^{n1}P^{n2} \ldots P^{nk} = P^{n1+n2+\ldots+nk}$, whence $n = n_1 + n_2 + \ldots + n_k$.

In other words, $n_1, n_2, \ldots, n_k$ give us a partition of n.

Before discussing the uniqueness of the fuzzy invariants of A, one thing should be made absolutely clear the singleton fuzzy $a_1,\ldots, a_k$ and the fuzzy subgroups $A_1, \ldots, A_k$ which they generate, which a rose above to give the decomposition of A in to a fuzzy internal direct product of fuzzy generating subgroups, are not unique. Let's see this in a very simple example.

Let G = {e,a,b,a.b} be an abelian group of order 4 where $a^2 = b^2 = e$, ab = ba and A(e) = A(a) = 1, A(b) = A(a.b) = 3/4.

Then $A = H \times k$ where $H = \langle a_t \rangle$, $k = \langle b_t \rangle$ are fuzzy generating subgroups of order 2. But we have another decomposition of A as a fuzzy internal direct product, namely $A = N \times k$ where $N = \langle ab \rangle$ and $k = \langle b \rangle$. So, even in this fuzzy subgroup of very small order, we can get distinct decompositions of the fuzzy subgroup as the internal direct product of fuzzy generating subgroups.

Lemma –1-

Let A be fuzzy subgroup of abelian group (G, •) of order p^n, p a prime. Suppose that $A = A_1 \times A_2 \times \ldots \times A_k$, where each $A_i = \langle a_{si} \rangle$ is fuzzy generating of order p^{ni}, and $n_1 \geq n_2 \geq \ldots \geq n_k > 0$. If m is an integer such that $n_t > m \geq n_{t+1}$ then

$A(p^m) = B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$ where B_i is fuzzy generating of order p^m, generated by $a_{si}^{P^{ni-m}}$, for $i \leq t$. The order of $A(p^m)$ is p^u, where

$$u = mt + \sum_{i=t+1}^{k} n_i$$

Proof:

First of all, we claim that $A_{t+1}, \ldots, A_k$ are all in $A(p^m)$, since $m \geq n_{t+1} \geq \ldots \geq n_k > 0$, if $j \geq t+1$,

$$a_j^{p^m} = (a_j^{p^{nj}})^{p^{m-nj}} = e$$

Hence Aj, for $j \geq t+1$ lies in $A(p^m)$.

Secondly, if $i \leq t$ then $n_i > m$ and $(a_i^{p^{ni-m}}) = a_i^{p^{ni}} = e$ whence each such $a_i^{p^{ni}}$ is in $A(p^m)$ and so the fuzzy subgroup it generates, B_i, is also in $A(p^m)$.

Since $B_1, \ldots, B_t, A_{t+1}, \ldots, A_k$ all in $A(p^m)$, their product (which is fuzzy direct, since the product $A_1 \times A_2 \times \ldots \times A_k$ is fuzzy direct) is in $A(p^m)$.

Hence $A(p^m) \supset B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$.

On the other hand, if $y_r = a_1^{\lambda 1}.a_2^{\lambda 2}....a_k^{\lambda k}$ is in $A(p^m)$, since it then satisfies $y_r^{p^m} = e$, we set

$$e = y_r^{p^m} = a_1^{\lambda 1 p^m}....a_k^{\lambda k p^m}$$

However, the product of the fuzzy subgroups $A_1, \ldots, A_k$ is fuzzy direct, so we get

$$a_1^{\lambda 1 p^m} = e, ..., a_k^{\lambda k p^m} = e$$

Thus, the order of a_i, that is, p^{ni} must divide $\lambda_i p^m$ for $i = 1, 2, \ldots, k$.
If $i \geq t+1$ this is automatically true whatever be the choice of $\lambda_{t+1}, \ldots, \lambda_k$ since $m \geq n_{t+1} \geq \ldots \geq n_k$,
Hence $p^{ni} | p^m$, $i \geq t+1$. However, for $i \leq t$, we get from $p^{ni} | \lambda_i p^m$ that $p^{ni-m} | \lambda_i$. therefore $\lambda_i = v_i P^{ni-m}$ for some integer v_i.
Putting all this in formation in to the values of the λ_i's in the expression for y_r as

$y_r = a_1^{\lambda 1}...a_k^{\lambda k}$ We see that $y_r = a_1^{v1 p^{n1-m}}...a_t^{vt p^{nt-m}} a_{t+1}^{\lambda t+1}....a_k^{\lambda k}$

This says that $y_r \in B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$.
Now since each B_i is of order p^m and since $o(A_i) = p^{ni}$ and since $A = B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$,
$o(A) = o(B_1)\ o\ (B_2) \ldots o(B_t)\ o\ (A_{t+1}) \ldots o\ (A_k) = p^m p^m \ldots p^m\ p^{nt+1} \ldots p^{nk}$
Thus, if we write $o\ (A) = p^u$, then

$u = mt + \sum_{i=t+1}^{k} n_i$ The lemma is proved.

Corollary

If A is a fuzzy subgroup of a group in lemma –1-, then $o(A(p)) = p^k$.
Proof
Apply the above lemma to the case m = 1.
Then t = k, hence u = 1. k = k and so $o\ (A) = p^k$.

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الخلاصة

يتضمن البحث تقديم تعريف الجداء المباشر الداخلي الضبابي وبعض الخواص والمبرهنات المتعلقة به.