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New Properties of Anti Fuzzy Ideals of Regular Semigroups

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Abstract

In this article, we study some properties of anti-fuzzy sub-semigroup, anti fuzzy left (right, two sided) ideal, anti fuzzy ideal, anti fuzzy generalized bi-ideal, anti fuzzy interior ideals and anti fuzzy two sided ideal of regular semigroup. Also, we characterized regular LA-semigroup in terms of their anti fuzzy ideal.

Keywords: Fuzzy ideal, regular, anti fuzzy interior ideal, anti fuzzy ideal.

1. Introduction and Basic Concept

Fuzzy sub-semigroup and fuzzy interior ideal in semigroup was introduced by Hong, et al., in [1]. And the concept of fuzzy ideal and fuzzy bi-ideals in semigroups was studied by Nobuaki Kuroki in (1981)", [2]. The concept of the product of two fuzzy subset and anti product of two fuzzy subset was introduced by Shabir and Nawaz [3]. The concept of characterizations of semigroups by their anti fuzzy ideals was studied by Khan and Asif in [4]. The concept of intra-regular (left almost semigroup denoted by LA-semigroups) characterized by their anti fuzzy ideals by Khan and Faisal in [5]. Many other authors interested studied of fuzzy ideal, for example see [6-9]. Through out of this paper we are denoted of a regular semigroup by$\aleph_r$.

Definition 1 [1].

A fuzzy subset ζ in a semigroup ℵ is said to be a fuzzy sub-semigroup of ℵ if ζ(wz)≥ min{ζ(w), ζ(z)}, whenever w, z ∈ ℵ.

Definition 2 [1].

A fuzzy sub-semigroup ζ of a semigroup ℵ is said to be a fuzzy interior ideal of ℵ if ζ(swr)≥ ζ(w), whenever s, w, r ∈ ℵ.

Definition 3 [2].

A fuzzy function ζ of a semigroup ℵ is said to be a fuzzy ideal if ζ(swr)≤max{ζ(s), ζ(r)}={ζ(s)∨ ζ(r)}, whenever s, w, r ∈ ℵ.

Definition 4 [2].

A fuzzy sub-semigroup ζ of a semigroup ℵ is said to be a fuzzy bi ideal in ℵ if ζ(swr)≥min{ζ(s), ζ(r)}, whenever s, w, r ∈ ℵ.

Definition 5 [3].

Let ζ and φ be any fuzzy subsets of a semigroup ℵ then the product ζ ∘ φ is defined by

$$(\zeta \circ \varphi)(w) = \begin{cases} \bigvee_{w=sr}\{\zeta(s) \wedge \varphi(r)\}, \exists\ s, r \in \aleph \text{ s.t } w = sr \\ 0; \quad \text{other wise} \end{cases}$$

Definition 6 [3].

Let ζ and φ be any fuzzy subsets of a semigroup ℵ then the anti product ζ ∗ φ is defined by

$$(\zeta * \varphi)(w) = \begin{cases} \bigwedge_{w=s\,r}\{\zeta(s) \vee \varphi(r)\}, \exists\ s, r \in \aleph \text{ s.t } w = s\,r \\ 1; \quad \text{other wise} \end{cases}$$

Definition 7 [4].

A fuzzy subset ζ of a semigroup ℵ is said to be anti fuzzy sub-semigroup of ℵ if ζ(sr) ≤ ζ(s) ∨ ζ(r), whenever s, r ∈ ℵ.

Definition 8 [4].

A fuzzy subset ζ of a semigroup ℵ is said to be anti fuzzy left (right) ideal of ℵ if ζ(sr) ≤ ζ(r), (ζ(sr) ≤ ζ(s)), whenever s, r ∈ ℵ.

Definition 9 [4].

A fuzzy subset ζ of a semigroup ℵ is said to be anti fuzzy ideal of ℵ if it is both anti fuzzy left ideal and anti fuzzy right ideal.

Definition 10 [4].

A fuzzy subset ζ of a semigroup ℵ is said to be anti fuzzy interior ideal of ℵ if ζ(swr) ≤ ζ(w), whenever s, w, r ∈ ℵ.

Definition 11 [4].

A fuzzy subset ζ of a semigroup ℵ is said to be anti fuzzy generalized bi-ideal of ℵ if ζ(swr) ≤ ζ(s) ∨ ζ(r), whenever s, w, r ∈ ℵ.

Definition 12 [4].

A fuzzy sub-semigroup ζ is said to be anti fuzzy bi-ideal of ℵ if ζ(swr) ≤ ζ(s) ∨ ζ(r) whenever s, w, r ∈ ℵ.

Definition 13 [5].

A fuzzy subset ζ of a LA-semigroup ℵ is said to be a fuzzy LA-sub-semigroup if ζ(sr) ≥ ζ(s) ∧ ζ(r), whenever s, r ∈ ℵ.

Definition 14 [5].

A fuzzy subset ζ of a LA-semigroup ℵ is said to be a fuzzy left(right)ideal of ℵ if ζ(sr) ≥ ζ(r), (ζ(sr)≥ ζ(s)), whenever s, r ∈ ℵ.

Definition 15 [5].

A fuzzy LA-sub-semigroup ζ of a LA-semigroup ℵ is said to be a fuzzy bi-ideal if $\zeta((sr)t) \geq \zeta(s) \wedge \zeta(t)$, whenever s, r, t ∈ ℵ.

Definition 16 [5].

A fuzzy LA-sub-semigroup ζ of a LA-semigroup ℵ is said to be fuzzy interior ideal if $\zeta((sr)t) \geq \zeta(r)$, whenever s, r, t ∈ ℵ.

2. New Properties of Anti Fuzzy Ideals of a Regular Semigroup

In this section we introduce some properties anti fuzzy ideal

Definition 17

ℵ is said to be a regular semigroup if w=wzw, whenever w, z ∈ ℵ or equivalently w ∈ wℵw.

Theorem 18

Every fuzzy interior ideal in $\aleph_r$ is idempotent.

Proof

Suppose that ζ is a fuzzy interior ideal of a semigroup ℵ, then clearly $\zeta \circ \zeta \subseteq \zeta$,
Let w ∈ ℵ then ∃ z ∈ ℵ s.t $w = wzw \Longrightarrow$

w=wzw =(wz)w(z)w=(wz)w(z)wzw=((wz)w(z))(wzwzw)

$(\zeta \circ \zeta)_{(w)} = \vee_{w=((wz)w(z))(wzwzw)}\{\zeta(wz)w(z) \wedge \zeta(wz)w(zw)\}$

$\geq \zeta(wz)w(z) \wedge \zeta(wz)w(zw)$

$\geq \zeta(w) \wedge \zeta(w) = \zeta(w)$

This is implies that $\zeta \circ \zeta \supseteq \zeta$, hence $\zeta \circ \zeta = \zeta$. Then ζ is idempotent.

Theorem 19

Let ζ be a fuzzy subset in $\aleph_r$ then it is an anti fuzzy two sided ideal of ℵ iff it is an anti fuzzy interior ideal of ℵ.

Proof

⟹Since ζ be anti fuzzy two sided ideal of ℵ, then obviously, ζ is an anti fuzzy interior ideal of ℵ.

⟸ Suppose that ζ is an anti fuzzy interior ideal of ℵ. Let w, z ∈ ℵ, by by hypotheses so ∃ s, r ∈ ℵ, s.t w=wsw and z=zrz

$\zeta(wz)=\zeta((wsw)z)=\zeta((ws)wswz))=\zeta((ws)w(swz)) \leq \zeta(w)$, and

Also $\zeta(wz)= \zeta(w(zrz)) = \zeta(wzrzrz) = \zeta((wzr)z(rz)) \leq \zeta(z)$,

Hence, ζ is an anti fuzzy two sided ideal of ℵ.

Example 20

Let ℵ ={s, r, t, v} be a set with operation as follows:

.	*s*	*r*	*t*	*v*
s	*s*	*s*	*s*	*s*
r	*s*	*s*	*s*	*s*
s	*s*	*s*	*r*	*s*
v	*s*	*s*	*r*	*r*

Then we can easily see that (ℵ,.) is not a regular semigroup.
Define the fuzzy subset ζ of ℵ as
$\zeta(s) = 0.3,\ \zeta(r) = 0.9, \zeta(t) = 0.5, \zeta(v) = 0.7.$
Then clearly, ζ is anti fuzzy interior ideal of ℵ but it is not an anti fuzzy two sided ideal of ℵ, since {s, r} is not a two sided ideal of ℵ.

Proposition 21

In regular semigroup ℵ, then

i- Every anti fuzzy right ideal is idempotent.

ii- Every anti fuzzy interior ideal is idempotent.

Proof

i- Suppose that ζ is an anti fuzzy right ideal of semigroup ℵ, then clearly $\zeta \subseteq \zeta * \zeta$.
Since ℵ is a regular so whenever w ∈ ℵ, ∃ z ∈ ℵ, s.t w=wzw, so

$(\zeta * \zeta)_{(w)} = \wedge_{w=wzw=wzwzw}\{\zeta(wz) \vee \zeta(wzw)\}$

$= \wedge_{w=(wz)(wzw)}\{\zeta(wz) \vee \zeta(wt)\}$ where t=zw

$\leq \zeta(wz) \vee \zeta(wt) \leq \zeta(w) \vee \zeta(w) = \zeta(w)$

This implies that $\zeta * \zeta \subseteq \zeta$.
Hence $\zeta * \zeta = \zeta$.

ii- Suppose that ζ is an anti fuzzy interior ideal of semigroup ℵ, then clearly $\zeta \subseteq \zeta * \zeta$.
Since ℵ is a regular so whenever w ∈ ℵ, ∃ z∈ ℵ, s.t w=wzw,
so w=wzw=wzwzw=((wz)w(z)) ((wz)w (z w))

$(\zeta * \zeta)_{(w)} = \wedge_{w=((wz)w(z))\,((wz)w(zw))}\{\zeta((wz)w(z)) \vee \zeta((wz)w(zw))\}$

$\leq \zeta((wz)w(z)) \vee \zeta((wz)w(zw)) \leq \zeta(w) \vee \zeta(w) = \zeta(w).$

This implies that $\zeta * \zeta \subseteq \zeta$.
Hence $\zeta * \zeta = \zeta$.

Proposition 22 [3].

Let ζ be an anti fuzzy right ideal and μ an anti fuzzy left ideal of a semigroup ℵ.
Then $\zeta * \mu \supseteq \zeta \cup \mu$.
It is clear that from Proposition 22. $\zeta * \mu \supseteq \zeta \cup \mu$, but the converse needs not at all be true. Consider the following example,

Example 23

Consider the semigroup ℵ= {s, r, t , v} with the operation as follows:

.	s	r	t	v
s	s	s	s	s
r	s	s	s	s
t	s	s	r	s
v	s	s	r	r

The ideals of ℵ are {s}, {s, r}, {s, r, t} and {s, r, t, v} Let us define two fuzzy subsets ζ and μ of ℵ as follows

ζ(s)=0.5, ζ(r)=0.6, ζ(t)=0.7, ζ(v)=0.8.

μ(s)=0.6, μ(r)=0.7, μ(t)=0.8, μ(v)=0.9.

Then ζ and μ are an anti fuzzy ideal of ℵ, and we note that:

$(\zeta * \mu)_{(r)} = \wedge_{r = xy}\{\zeta(x) \vee \mu(y)\} = \wedge\{0.8, 0.8, 0.9\} = 0.8 \geq (\zeta \cup \mu)_{(r)} = 0.7$.

To consider the converse of proposition 22, we need to strengthen the condition of semigroupℵ.

Theorem 24

If ζ, μ are any anti fuzzy two sided ideals of $\aleph_r$, then $\zeta * \mu = \zeta \cup \mu$.

Proof

Let ζ and μ be any anti fuzzy two sided ideals of ℵ, then obviously $\zeta * \mu \supseteq \zeta \cup \mu$.

since ℵ is a regular so whenever element w ∈ ℵ, ∃ z ∈ ℵ, s.t w=wzw, so

$(\zeta * \mu)_{(w)} = \wedge_{w=wzw=wzwzw}\{\zeta(wz) \vee \mu(wzw)\}$

$\leq \zeta(wz) \vee \mu(wzw) \leq \zeta(w) \vee \mu(w) = (\zeta \cup \mu)(w)$

Then $(\zeta * \mu) \subseteq \zeta \cup \mu$. Hence, $\zeta * \mu = \zeta \cup \mu$.

Example 25

Let ℵ ={s, r, t} be a semigroup with the following table:

.	s	r	t
s	s	r	t
r	r	r	t
t	t	t	t

Define a fuzzy subset ζ of ℵ by ζ(s)=0.6, ζ(r)=0.5, ζ(t)=0.4. By routine calculation, we can check that ζ is an anti fuzzy ideal, anti fuzzy interior ideal and anti fuzzy bi-ideal of $\aleph_r$.

Now, we give other fuzzy characterizations of a regular semigroup.

Proposition 26

A fuzzy subset ζ of $\aleph_r$, then ζ is anti fuzzy bi-ideal of ℵ iff it is an anti fuzzy generalized bi-ideal of ℵ.

Proof

⟹ Suppose that ζ be any anti fuzzy bi-ideal of ℵ, the obviously, ζ is an anti fuzzy generalized bi-ideal of ℵ.

⟸ Suppose that ζ be any anti fuzzy generalized bi-ideal of ℵ, since ℵ is a regular of a semigroup, so whenever w ∈ ℵ, ∃ z ∈ ℵ s.t w=w z w.

we have

$\zeta(wr)=\zeta(wzwr)=\zeta(w\,t\,r) \leq \zeta(w) \vee \zeta(r)$ where $t=zw$.

Therefore, ζ is an anti fuzzy sub-semigroup of ℵ.

Hence, ζ is an anti fuzzy generalized bi-ideal of ℵ.

Theorem 27

For anti fuzzy generalized bi-ideal ζ and anti fuzzy right ideal μ of $\aleph_r$, then ζ ∗ μ ⊆ ζ ∪ μ.

Proof

Let ζ and μ are any anti fuzzy generalized bi-ideal and anti fuzzy right ideal of ℵ, respectively, then whenever w ∈ ℵ, ∃ z ∈ ℵ s.t w=wzw.

Then $(\zeta * \mu)_{(w)} = \wedge_{w = bc}\{\zeta(b) \vee \mu(c)\}$

$\leq \zeta(wzw) \vee \mu(zw) \leq \zeta(w) \vee \mu(w)=(\zeta \vee \mu)(w)$

And so we have ζ ∗ μ ⊆ ζ ∪ μ.

Theorem 28

If ζ and μ are any anti fuzzy interior ideals of $\aleph_r$, then (ζ ∗ μ) ∪ (μ ∗ ζ) ⊆ ζ ∨ μ.

Proof

Let ζ, μ be any anti fuzzy interior ideals of ℵ, and w ∈ ℵ. Then since ℵ is regular semigroup then, ∃ z ∈ ℵ s.t *w*= wzw=((wz)w(z)) (w(zw))=((wz)w(z)) ((wz)w(zw)).

Hence $(\zeta * \mu)_{(w)} = \wedge_{w = bc.}\{\zeta(b) \vee \mu(c)\}$

$\leq \zeta((wz)w(z)) \vee \mu((w z)w(zw)) \leq \zeta(w) \vee \mu(w)=(\zeta \vee \mu)(w)$

And so we have ζ ∗ μ ⊆ ζ ∪ μ. Similarly, we have (μ ∗ ζ) ⊆ ζ ∪ μ

Therefore (ζ ∗ μ) ∪ (μ ∗ ζ) ⊆ ζ ∪ μ.

Theorem 29

For every anti fuzzy left ideal α, every anti fuzzy generalized bi-ideal μ, and every anti fuzzy interior ideal ζ of $\aleph_r$, then μ ∗ α ∗ ζ ⊆ μ ∪ α ∪ ζ.

Proof

Let α, μ and ζ be any anti fuzzy left ideal, any anti fuzzy generalized bi-ideal and anti fuzzy interior ideal of $\aleph_r$, respectively, whenever w ∈ ℵ, ∃ z ∈ ℵ. Because ℵ is a regular, s.t w=wzw=wzwzw=(wzw) (zw)zw=((wzw) [(zw) ((z)w(zw)]).

Then we have:

$(\mu * \alpha * \zeta)_{(w)}=\wedge_{w=((wzw)[(zw))((z)w(zw))}\{\mu((wzw)) \vee (\alpha * \zeta)((zw)((z)w(zw))\}$

$\leq \mu(w) \vee \{\wedge_{((zw)(z)w(zw))}\{\alpha(zw) \vee \zeta((z)w(zw))\}$

$\leq \mu(w) \vee \alpha(w) \vee \zeta(w)$

$= (\mu \cup \alpha \cup \zeta)(w)$

And so we have μ ∗ α ∗ ζ ⊆ μ ∪ α ∪ ζ .

Now, we characterized regular **(**left almost-semigroup for short LA-semigroup) by the properties of their fuzzy left (right, two sided) ideal.

Let ℵ be a gropoid. Then

1. ℵ is called LA-semigroup if (wr) j=(jr) w; whenever w, r, j ∈ ℵ.
2. Medial law of a LA-semigroup means (wr) (jv) = (wj) (rv); whenever w, r, j, v ∈ ℵ.
3. In additional if ℵ has a left identity(necessary unique) the paramedical law mean
4. (wr) (jv)=(vr) (jw); whenever w, r, j, v ∈ ℵ.
5. An LA-semigroup with right identity becomes a commutative semigroup with identity. if an LA-semigroup contains left identity, the following law holds w (r j) = r (w j); whenever w, r, j ∈ ℵ.

Proposition 30

A fuzzy subset ζ of $\aleph_r$ is a fuzzy right ideal iff it is a fuzzy left ideal.

Proof

⟹ Suppose that ζ is a fuzzy right ideal of ℵ, since ℵ is a regular so whenever w∈ ℵ, ∃ z ∈ ℵ, s.t w=wzw, so by using (1)

$\zeta(wb) = \zeta((wzw)\, b)$

$=\zeta((wzw)(zw)b))$

$=\zeta(b\,(zw)(wzw))$

$\geq \zeta(b(zw)) \geq \zeta(b)$

⟸ Suppose that ζ is a fuzzy left ideal of $\aleph_r$, then using (1)

$\zeta(wr) = \zeta((wzw)r)=\zeta((wzw)(zw)r)$

$=\zeta(r(zw)(wzw))\geq \zeta(wzw)$

$= \zeta((wz)w)\geq \zeta((w)w) \geq \zeta(w^2) \geq \zeta(w).$

Theorem 31

Every fuzzy two sided ideal of a regular LA-semigroup ℵ, with left identity is idempotent.

Proof

Suppose that ζ is a fuzzy two sided ideal of ℵ, then clearly $\zeta \circ \zeta \subseteq \zeta \circ \aleph \subseteq \zeta$.

Since ℵ is a regular so whenever w ∈ ℵ, ∃ z ∈ ℵ, s.t w=wzw so by using (1)

w=wzw=w(zw)(zw)=(zwzw)w,

$(\zeta \circ \zeta)_{(w)}=\bigvee_{w=(zwzw)w} \zeta(zwzw) \wedge \zeta(w)$

$\geq \zeta(zwzw) \wedge \zeta(w)$

$\geq \zeta(w) \wedge \zeta(w)=\zeta(w).$

And this implies that $\zeta \circ \zeta \supseteq \zeta$, hence $\zeta \circ \zeta = \zeta$.

Theorem 32

For a fuzzy subset ζ of a regular LA-semigroup ℵ, with left identity then ζ is a fuzzy two sided ideal of ℵ iff it is a fuzzy interior ideal of ℵ.

Proof

⟹ Suppose that ζ be a fuzzy two sided ideal of ℵ, then obviously, ζ is a fuzzy interior ideal of ℵ.

⟸ Suppose that ζ be a fuzzy interior ideal of ℵ, and w, r∈ ℵ, then since ℵ is a regular of AL-semigroup, so ∃ z, y ∈ ℵ s.t w=wzw, r=ryr, then

$\zeta(wr) = \zeta((w\, z\, w)r)$ using (1)

$= \zeta(r(z\, w))(w\, z\, w)\,)$ using (2)

$= \zeta(rw)((zw)(zw)) = \zeta(rw)t)$ where t= ((z w)(z w))

$\geq \zeta(w),$

Also $\zeta(wr) =\zeta(w(ryr))= \zeta(w(ryry)r))$ using (4)

$=\zeta((ryry)(wr)\,)= \zeta((ry)r(ywr)\,)=\zeta(jrt)$

Where j= ry and t= y w r and

$\geq \zeta(r),$

Hence, ζ is a fuzzy two sided ideal.

2. Conclusion

From the research / the evidence we conclude that

1. Let ζ be a fuzzy subset in $\aleph_r$ then it is an anti fuzzy two sided ideal of ℵ iff is an anti fuzzy interior ideal of ℵ.
2. In a regular semigroup ℵ, then the following are satisfy the following
   i) Every anti fuzzy right ideal is idempotent.
   ii) Every anti fuzzy interior ideal is idempotent.
3. If ζ , μ are an anti fuzzy two sided ideals of $\aleph_r$, then $\zeta * \mu = \zeta \cup \mu$.
4. For anti fuzzy generalized bi-ideal ζ and anti fuzzy right ideal μ of $\aleph_r$,
5. Then $\zeta * \mu \leq \zeta \vee \mu$.
6. For every anti fuzzy left ideal α, every anti fuzzy generalized bi-ideal μ, and every anti fuzzy interior ideal ζ of $\aleph_r$, then $\mu * \alpha * \zeta \subseteq \mu \cup \alpha \cup \zeta$.
7. For a fuzzy subset ζ of a regular LA-semigroup ℵ, with left identity then ζ is a fuzzy two sided ideal of ℵ iff it is a fuzzy interior ideal of ℵ.

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