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Weak Essential Fuzzy Submodules Of Fuzzy Modules

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Abstract

Throughout this paper, we introduce the notion of weak essential F-submodules of F-modules as a generalization of weak essential submodules. Also, we study the homomorphic image and inverse image of weak essential F-submodules.

Keywords: Semi-prime F-submodules, essential F-submodules.

1.Introduction

Let S $\neq \emptyset$. Zadeh [1] defined F-subset X of S as a mapping X: S $\longrightarrow$[0,1]. Negoita and Ralescu [2] introduced the concept of F-modules. Mashinchi and Zahedi [3] introduced the notion of F-submodules.

Mona [4] introduced and studied the concept of weak essential submodules, where a submodule H of $\mathcal{M}$ is called a weak essential, if H $\cap$ L $\neq$ (0), for each non-zero semiprime submodule L of $\mathcal{M}$. In this paper, we introduce the notion weak essential F- submodule of F-module. We investigate some basic results about weak essential submodules.

Next, throughout this paper $\mathcal{R}$ is a commutative ring with identity, $\mathcal{M}$ is an $\mathcal{R}$-module and X is a F-module of an $\mathcal{R}$-module $\mathcal{M}$.

Finally, (shortly fuzzy set, fuzzy submodule and fuzzy module is F-set, F-submodule and F- module).

S.1 Preliminaries

In this section, we shall give the concepts of F-sets and operations on F-sets, with some important properties of them, which are used in this paper.

Definition 1.1 [1]:

Let S be a non-empty set and let I be a closed interval [0,1] of the real line (real number). ***A F-set X*** in S (a fuzzy subset X of S) is characterized by a membership function $X : S \longrightarrow I$,

Definition 1.2 [2]

Let $x_t : S \longrightarrow I$, be a F-set in S, where x ∈ S , t ∈ I, defined by:

$$x_t = \begin{cases} 1 & if \quad x = y \\ 0 & if \quad x \neq y \end{cases}$$

Then x_t a said *F- **singleton.***

If x = 0 and t = 1 then :

$$0_1(y) = \begin{cases} 1 & if \quad y = 0 \\ 0 & if \quad y \neq 0 \end{cases}$$

We shall call such F-singleton the ***F-zero singleton.***

Proposition 1.3 [3]:

Let a_t, b_k be two F-singletons of a set S. If $a_t = b_k$, then a = b and t = k, where t, k ∈ I.

Definition 1.4 [5]:

Let A_1, A_2 are F-sets in S, then :

1. $A_1 = A_2$ if and only if $A_1(x) = A_2(x)$, ∀ x ∈ S.
2. $A_1 \subseteq A_2$ if and only if $A_1(x) \leq A_2(x)$, ∀ x ∈ S.

If $A_1 \subset A_2$ and there exists x ∈ S such that $A_1(x) < A_2(x)$, then A_1 is called a proper F-subset of A_2.

3. $x_t \subseteq A$ if and only $x_t(y) \leq A(y)$, ∀ y ∈ S *and* if $t > 0$ *then* $A(x) \geq t$. Thus $x_t \subseteq A$ (x ∈ A_t), (that is x ∈ A_t if and only if $x_t \subseteq A$)

Definition 1.5 [5]:

Let A_1, A_2 are F-sets in S, then:

1.$(A_1 \cup A_2)(x) = \max\{A_1(x), A_2(x)\}$, ∀ x ∈ S.
2.$(A_1 \cap A_2)(x) = \min\{A_1(x), A_2(x)\}$, ∀ x ∈ S.

$A_1 \cup A_2$ *and* $A_1 \cap A_2$ are F-sets in S.

In general if $\{A_\alpha, \alpha \in \Lambda\}$, is a family of F-sets in S, then:

$$\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)(x) = inf\{A_\alpha(x), \alpha \in \Lambda\}, for\ all\ x \in S.$$

$$\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)(x) = \sup\{A_\alpha(x), \alpha \in \Lambda\}, for\ all\ x \in S.$$

Now, we give the definition of level subset, which is a set between F-set and ordinary set.

Definition 1.6 [6]:

Let A be a F-set in S. For t ∈ I, the set $A_t = \{x \in S,\ A(x) \geq t\}$ is called ***level subset of* X**."

The following are some properties of the level subset:

Remark 1.7 [1]:

Let A, B are F-subsets of S, t ∈ I, then:

1. $(A \cap B)_t = A_t \cap B_t$.
2. $(A \cup B)_t = A_t \cup B_t$.
3. A = B if and only if $A_t = B_t$, for all t [0,1].

Definition1.8 [7]:

Let f be a mapping from a set $\mathcal{M}_1$ into a set $\mathcal{M}_2$, let A be a F-set in $\mathcal{M}_1$ and B be a F-set in $\mathcal{M}_2$. The image of A denoted by f(A) is the F-set in $\mathcal{M}_2$ defined by:

$$f(A)(y) = \begin{cases} sup\{A(z) \mid z \in f^{-1}(y)\} & if \quad f^{-1}(y) \neq \emptyset, \; for\ each\ y \in \mathcal{M}_2 \\ 0 & o.w \end{cases}$$

where $f^{-1}(y) = \{ x : f(x) = y\}$

And the inverse of B(x), denoted by $f^{-1}(B)$ is the F-set in $\mathcal{M}_1$defined by: $f^{-1}(B) = B\big(f(x)\big)$, for all x $\in \mathcal{M}_1$.

Definition 1.9 [8]:

Let f be a function from a set $\mathcal{M}_1$ into a set $\mathcal{M}_2$. A F-subset A of $\mathcal{M}_1$ is a said ***f*-*invariant*** if A(x) = A(y), whenever f(x) = f(y), where x, y $\in \mathcal{M}_1$.

Proposition 1.10 [8]:

If f is a function defined on a set $\mathcal{M}$, A_1 $and A_2$ are F-subsets of $\mathcal{M}$, B_1 and B_2 are F-subset of $f(\mathcal{M})$. The followings are true:

1. $A_1 \subseteq f^{-1}\big(f(A_1)\big)$.
2. $A_1 = f^{-1}\big(f(A_1)\big)$, whenever A_1 is f-invariant.
3. $f\big(f^{-1}(B_1)\big) = B_1$.
4. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_{2)}$.
5. If $B_1 \subseteq B_2$,then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
6. Let f be a function from a set $\mathcal{M}$ into N. If B_1 and B_2 are F-subsets of N, then $f^{-1}(B_1 \cap B_2)$ =$f^{-1}(B_1) \cap f^{-1}(B_2)$ [9] .

Definition 1.11 [2]:

A said F-set X is F-module of an $\mathcal{R}$-module $\mathcal{M}$ if:

1. $X(\nu - \mu) \geq \min\{X(\nu), X(\mu)\}$, $\forall\, \nu, \mu \in \mathcal{M}$.
2. $X(r\nu) \geq X(\nu)$, $\forall\, \nu \in \mathcal{M}$ and r $\in \mathcal{R}$.
3. X(0) = 1 (0 is the zero element of $\mathcal{M}$).

Definition 1.12 [3]:

Let X_1, X_2 are F-modules of an $\mathcal{R}$-module $\mathcal{M}$. X_2 is a said F-submodule of X_1 if $X_2 \subseteq X_1$."

Proposition 1.13 [10]:

Let X_1, X_2 be two F-modules of an $\mathcal{R}$-module $\mathcal{M}_1$ and $\mathcal{M}_2$ resp. Let $f : X_1 \longrightarrow X_2$ be F-homomorphism.

If A_1 and A_2 are two F-submodules of X_1 and X_2 resp., then:

1. $f(A_1)$ is a F-submodule of X_2 .
2. $f^{-1}(A_2)$ is a F-submodule of X_1.

Proposition 1.14 [11]:

Let A be a F-set of an $\mathcal{R}$-module $\mathcal{M}$. Then, the level subset A_t, t $\in$ I, is a submodule of $\mathcal{M}$ iff A is F-submodule of X.

Definition 1.15 [3]:

Let A be a F-module in $\mathcal{M}$, then we define:

1. $A^* = \{x \in \mathcal{M} : A(x) > 0\}$ is called support of A, also
   $A^* = \cup A_t$, t $\in (0,1]$.
2. $A_* = \{x \in \mathcal{M} : A(x) = 1 = A(0_{\mathcal{M}})\}$.

Definition1.16 [12]:

A F-submodule A of a F-module X is called an essential (briefly $A \leq_e X$), if $A \cap B \neq 0_1$, for any non-trivial F-submodule B of X.

2. Weak Essential Fuzzy Submodules

Mona in [4] introduced the concept of weak essential submodule, where a submodule H of $\mathcal{M}$ is a said weak essential, if $H \cap L \neq (0)$, for each non-zero semiprime submodule L of $\mathcal{M}$, where a submodule N of an $\mathcal{R}$-module $\mathcal{M}$ is called semiprime if for each $r \in \mathcal{R}$ and $m \in \mathcal{M}$, if $r^2x \in N$, then $rx \in N$ [13]. We shall fuzzify this concept.

Definition 2.1 [14]:

Let A be F-submodule of F-module X is a said a semiprime F-submodule if ${r_t}^k a_s \subseteq A$, for F-singleton r_t of $\mathcal{R}$, $a_s \subseteq X$, $k \in Z_+$, then $r_t a_s \subseteq A$. Equivalently, A is semiprime F-submodule if ${r_t}^2 a_s \subseteq A$ for $a_s \subseteq X$ and r_t a F-singleton of $\mathcal{R}$, then $r_t a_s \subseteq A$."

Definition 2.2:

Let A_1 be F-submodule of F-module X. A_1 is a said weak essential F-submodule if $A_1 \cap S \neq 0_1$, for each non-trivial semiprime F-submodules of X. Equivalently F-submodule A of a F-module X is called weak essential F-submodule if $A \cap S = 0_1$, then $S = 0_1$, for every semiprime F-submodule of X .

Next, proposition is a characterization of a weak essential F-submodule.

Proposition 2.3:

Let X be a F-module and A a non-trivial F-submodule of X is a weak essential F-submodule if and only if for each non-trivial semiprime F-submodule S of X, there exists $x_s \subseteq S$ and r_t of $\mathcal{R}$, such that $x_s r_t \subseteq A$, $\forall\, t \in (0,1]$.

Proof:

Suppose that non-trivial semiprime F-submodule S of X, there exists $x_s \subseteq S$ and r_t of $\mathcal{R}$ such that $0_1 \neq x_s r_t \subseteq A$. Note that $x_s r_t \subseteq S$.

$0_1 \neq x_s r_t \subseteq A \cap B$. Thus $A \cap B \neq 0_1$, that is A is weak essential F-submodule.

Conversely, A is weak essential F-submodule, then $A \cap S \neq 0_1$, for each non-trivial semiprime F-submodule S of X. Thus, there exists $0_1 \neq x_t \subseteq A \cap S$, implying that $x_t \subseteq A$ and hence $0_1 \neq x_s r_t \subseteq A$, $\forall\, t \in (0,1]$.

Now, we give the following Lemma, which we will need in proving the next result.

Lemma 2.4:

Let A be a F-submodule of a F-module X if A_t weak essential submodule of X_t, $\forall\, t \in I$. Then A is weak essential F-submodule in X.

Proof:

Assume B a semiprime F-submodule of X such that $B \neq 0_1$, since B semiprime F-submodule of X, hence B_t semiprime submodule of X_t, $\forall\, t \in (0,1]$, see [14, Theorem(2.4)], which implies $A_t \cap B_t \neq (0)$, since A_t is weak essential submodule and $A_t \cap B_t = (A \cap B)_t \neq (0)$, hence $A \cap B \neq 0_1$ by Remark (1.7)(3). Thus, A is a weak essential F-submodule of X.

Remark 2.5:

Every essential F-submodule is weak essential F-submodule. But the converse is not true in general, for example:

Example:

Let $\mathcal{M} = Z_{36}$ as Z-module. Define $X : \mathcal{M} \longrightarrow I$, by:

X(a) = 1, for all $a \in Z_{36}$

Let A : $\mathcal{M} \longrightarrow$ I, define by: A(x) = $\begin{cases} 1 & if \quad x = 0 \\ 1/2 & if \quad x \in (\overline{9}) - (0) \\ 0 & otherwise \end{cases}$

It is clear that A F-submodule of X, $A_{\frac{1}{2}} = (\overline{9})$ is weak essential by [4, Remarks(1.5)], then A is weak essential F-submodule by Lemma(2.4). Let

B : $\mathcal{M} \longrightarrow$ I, as defined by: B(x) = $\begin{cases} 1 & if \ x = 0 \\ 1/2 & if \ x \in (\overline{4}) - (0) \\ 0 & otherwise \end{cases}$

It is clear that B F-submodule of X. A is not essential, since

$$A \cap B(x) = \begin{cases} 1 & if \ x = 0 \\ 0 & otherwise \end{cases}$$

$A \cap B = 0_1$ and B $\neq 0_1$; therefore A is not essential F-submodule .

Remark 2.6*:*

The converse of Lemma (2.4) is not true in general.

Example 2.7:

Let $\mathcal{M}$ = Z_6 as Z-module. Define X : $\mathcal{M} \longrightarrow$ I, A : $\mathcal{M} \longrightarrow$ I by: X(a) = $\begin{cases} 1 & if \ a = 0 \\ 1/2 & if \ a = 2,4 \\ 0 & otherwise \end{cases}$

, A(a) = $\begin{cases} 1 & if \ a = 0 \\ 1/3 & if \ a = 2,4 \\ 0 & otherwise \end{cases}$

A is an essential F-submodule, then A is weak essential by Remark (2.5), but $A_{\frac{1}{2}} = (0)$ is not essential see [15, Remark (2.1)]. Also $A_{\frac{1}{2}}$ is not weak essential, since $A_{\frac{1}{2}} \cap S = (0)$, where S any semiprime submodule. Therefore $A_t$is not weak essential of X_t.

Proposition 2.8:

Let A be a F-submodule of a F-module X, then A is weak essential in X iff A_* is weak essential submodule in X_*.

Proof:

Let A_* is a weak essential submodule in X_*. To show A is weak essential F-submodule in X .

Assume that S is semiprime F-submodule of X and $A \cap S = 0_1$,then $(A \cap S)_* = (0)$, implies that $A_* \cap S_* = (0)$. But S is semiprime F-submodule, then S_t is semiprime see [14, Theorem (2.4)], so S_* is semiprime, hence $S_* = (0)$, so S = 0_1. Thus, A is weak essential F-submodule in X.

Conversely, let A is a weak essential F-submodule in X, we have to show that A_* is weak essential submodule in X_*.

Let N is semiprime submodule of X_* and $A_* \cap N = (0)$, we must prove N = (0).

Define B : $\mathcal{M} \longrightarrow$ I by: B(x) = $\begin{cases} 1 & if \ x \in N \\ 0 & otherwise \end{cases}$

It is clear that B F-submodule of X, $B_* = N$, so $A_* \cap B_* = (0)$, then $(A \cap B)_* = (0)$, hence by Remark(1.7)(3), A$\cap B = 0_1$ and B = 0_1, since A is weak essential F-submodule in X, so $B_* = (0)$; therefore

N = (0). Thus A_* is weak essential submodule in X_*.

Remarks 2.9:

1. Let A, B are F-submodules of X such that $A \subseteq B$ and B is weak essential F-submodule of X, then A need not be weak essential F-submodule for example:

Let $\mathcal{M}$ be as Z-module Z_{36}. Let X : $\mathcal{M} \longrightarrow$ I, define by :

X(a) = 1, for all $a \in Z_{36}$.

Define A: $\mathcal{M} \longrightarrow$ I , B : $\mathcal{M} \longrightarrow$ I by:

$A(x) = \begin{cases} 1 & if \ \ x \in (\overline{18}) \\ 0 & otherwise \end{cases}$, $B(x) = \begin{cases} 1 & if \ x \in (\overline{2}) \\ 0 & otherwise \end{cases}$

It is clear that $X_t = Z_{36}$ and A, B are F-submodules of X .

B_t a weak essential submodule in X_t see [4, Remarks(1.5)]. Thus B is weak essential F-submodule of X by Lemma (2.4). Let C : $\mathcal{M} \longrightarrow$ I, as defined by:

$C(x) = \begin{cases} 1 & if \ x \in (\overline{12}) \\ 0 & otherwise \end{cases}$, where C semiprime F-submodule

$C_t = (\overline{12})$, is semiprime submodule of X_t ($\forall\, t > 0$). But $A \cap C = 0_1$, therefore A is not weak essential F-submodule of X .

2. Let A, B are F-submodule such that $A \subseteq B$. If A is weak essential F-submodule in X implying B is a weak essential F-submodule of X.

Proof:

Assume that $B \cap S = 0_1$, for some semi-prime F-submodule S of X, then $A \cap S = 0_1$. But A is weak essential F-submodule, hence S = 0_1. That is B is weak essential F-submodule of X .

3. Let A, B be are F-submodules of F-module X if $A \cap B$ a weak essential F-submodule of X, then both of A and B are weak essential F-submodules of X.

Proof:

It is clear by (2).

Note that, the converse is not true in general, for example:

Example:

Let $\mathcal{M}$ be Z_{36} as Z-module. Define X : $\mathcal{M} \longrightarrow$ I by:

X(a) = 1, for all $a \in Z_{36}$.

Let A : $\mathcal{M} \longrightarrow$ I, B : $\mathcal{M} \longrightarrow$ I, define by:

$A(x) = \begin{cases} 1 & if \ \ x \in (\overline{12}) \\ 0 & otherwise \end{cases}$, $B(x) = \begin{cases} 1 & if \ x \in (\overline{18}) \\ 0 & otherwise \end{cases}$

Clearly A, B are F-submodules of X, $A_t = (\overline{12})$,

$B_t = (\overline{18})$, $\forall\, t \in (0,1]$ are weak essential submodules of X_t by [4, Remark(1.5)]. Hence A, B are weak essential F-submodules of X; see Lemma(2.4). But $A \cap B = 0_1$; that is A ∩ B is not weak essential F-submodule of X .

Under some conditions the converse (3) will be true as in the following proposition.

Proposition 2.10:

Let A, B are F-submodules of F-module X such that A is an essential F-submodule, B weak essential F-submodule, then $A \cap B$ is a weak essential F-submodule of X.

Proof:

Suppose S is a non-trivial semiprime F-submodule of X, but B is weak essential F-submodule of X, hence B ∩ $S \neq 0_1$. So A is an essential F-submodule of X and we have A ∩ $(B \cap S) = (A \cap B) \cap S \neq 0_1$,

Hence, A ∩ B is weak essential F-submodule of X.

Lemma 2.11:

If S is a semiprime F-submodule of F-module X, B be a F-submodule of X such that B ⊈ S, then S ∩ B is semiprime F-submodule in B.

Proof:

Let S be a semiprime F-submodule of X, then by [14,Theorem(2.4)], S_t semiprime submodule and B_t submodule of X_t; see Proposition (1.14) such that $B_t \nsubseteq X_t$, then by [13 ,Proposition(1.11)], $S_t \cap B_t = (S \cap B)_t$; see Proposition (1.7)(1) is a semiprime submodule in B_t, therefore S ∩ B is a semiprime F-submodule in B; see [14, Theorem(2.4)].

In the following proposition, we prove the transitive property for non-trivial F-submodule.

Proposition 2.12:

Let A, B be a non-trivial F-submodules of F-module X such that A ⊆ B. If A is a weak essential F-submodule in B and B is a weak essential F-submodule in X implying A is a weak essential F-submodule in X.

Proof:

Assume that S is a semiprime F-submodule in X, such that A ∩ S $=0_1$. Note that 0_1 =A ∩ S = (A ∩ S) ∩ B = A ∩ (S ∩ B). But S is a semi-prime F-submodule of X, so we have two cases. If B ⊆ S, then 0_1 = A ∩ (S ∩ B) = A ∩ B. Hence, A ∩ B = 0_1, but A ⊆ B so A ∩ B = A implies A = 0_1 which is a contradiction with our assumption. Thus B ⊈ S and by Lemma (2.11), S ∩ B is a semiprime F-submodule in B. Since A is a weak essential F-submodule in B, therefore S ∩ B = 0_1 and since B is a weak essential F-submodule in X, then S = 0_1, then A is a weak essential F-submodule in X.

Now, we study a homomorphic image of a weak essential F-submodule.

Proposition 2.13:

Let X_1, X_2 be F-modules of an $\mathcal{R}$-module $\mathcal{M}_1$and $\mathcal{M}_2$ resp. and $f : X_1 \longrightarrow X_2$ be F-epimorphism. If A_1 is a weak essential F-submodule of X_1 such that A_1 is f-invariant, then $f(A_1)$ is a weak essential F-submodule of X_2.

Proof:

To show $f(A_1)$ is a weak essential F-submodule of X_2, since A_1 is a F-submodule of X_1, then $f(A_1)$ is a F-submodule of X_2 by Proposition (1.13)(1).Now suppose that S semiprime F-submodule of X_2 such that $f(A_1) \cap S = 0_1$; therefore $f^{-1}(f(A_1) \cap S) = f^{-1}(0_1)$, then $f^{-1}(f(A_1)) \cap f^{-1}(S) = 0_1$, see Proposition (1.10)(2). But A_1 is f-invariant implying that $A_1 \cap f^{-1}(S) = 0_1$, and $f^{-1}(S) = 0_1$, since A_1 is weak essential F-submodule and $f^{-1}(S)$ F-submodule of X_1 by Proposition (1.13)(2). $f(f^{-1}(S)) = f(0_1)$, then S = 0_1, by Proposition (1.10)(3). That is $f(A_1)$ is a weak essential F-submodule.

Now, we consider the inverse image of a weak F-submodule.

Proposition 2.14:

Let X_1, X_2 are F-modules of an $\mathcal{R}$-module $\mathcal{M}_1$and $\mathcal{M}_2$ resp. and $f : X_1 \longrightarrow X_2$ be F-epimorphism. If A_2 is weak essential F-submodule of X_2, then f^{-1} (A_2) is a weak essential F-submodule of X_1.

Proof:

Since A_2 F-submodule of X_2, then $f^{-1}(A_2)$ is F-submodule of X see Proposition(1.13)(2).Now suppose S is semiprime F-submodule of X_1,such that $f^{-1}(A_2) \cap S = 0_1$, hence $f\ (f^{-1}(A_2) \cap S) = f(0_1)$, implies that $f\ (f^{-1}(A_2)) \cap f(S) = f(0_1)$ see Proposition (1.10)(6). $A_2 \cap f(S) = 0_1$(since A_2 is f-invariant and f is epimorphism), then $f^{-1}\big(f(S)\big) = f^{-1}(0_1)$, implies that S = 0_1, since every F-submodule of X_1 is f-invariant, implies $f^{-1}(A_2)$ is weak essential F-submodule of X_1.

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