Ibn Al-Haitham Jour. for Pure & Appl. Sci. 33 (4) 2020

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Approximaitly Quasi-primary Submodules

Ali Sh. Ajeel
Directorate General of education Salahaddin, the ministry of education, Tikrit, Iraq.
Ali.shebl@st.tu.edu.iq

Omar A. Abdulla
Directorate General of education Salahaddin, the ministry of education, Tikrit, Iraq.
omar.aldoori87@gmail.com

Haibat K. Mohammadali
Department of Mathematics, College of Computer Sciences and Mathematics, Tikrit University, Tikrit, Iraq.
H.mohammadali@tu.edu.iq

Article history: Received 7 January 2020, Accepted 12 February 2020, Published in October 2020

Doi: 10.30526/33.4.2513

Abstract

In this paper, we introduce and study the notation of approximaitly quasi-primary submodules of a unitary left R-module Q over a commutative ring R with identity. This concept is a generalization of prime and primary submodules, where a proper submodule E of an R-module Q is called an approximaitly quasi-primary (for short App-qp) submodule of Q, if $rq \in E$, for $r \in R$, $q \in Q$, implies that either $q \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$, for some $n \in Z^+$. Many basic properties, examples and characterizations of this concept are introduced.

Keywords: Prime submodules, Primary submodules, Socle of modules, Radical of submodules, Multiplication modules, Nonsingular modules.

1. Introduction

In this article all rings are commutative with identity, and all modules are left unitary R-modules. Dauns, J. in 1978 introduced and studied the concept of prime submodule, where a proper submodule E of an R- module Q was prime if $rq \in E$, for $r \in R, q \in Q$, implying that either $q \in E$ or $rQ \subseteq E$ [1]. Recently many generalizations of prime submodule have been introduced for example, see [2-5]. Primary submodules as a generalization of prime submodules was first introduced in [6], where a proper submodule E of Q was called primary submodule if whenever $rq \in E$, for $r \in R, q \in Q$, implying that either $q \in E$ or $r^n Q \subseteq E$, for some $n \in Z^+$. The concept of quasi-primary ideal which was introduced and studied by Fuchs, L. [7], where a proper ideal I of a ring R was called quasi-primary ideal if $rs \in I$, for $r, s \in R$, implying that $r \in \sqrt{I}$ or $s \in \sqrt{I}$, where $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in Z^+\}$. In particular I is quasi-primary ideal of R if and only if $\sqrt{I}$ is a prime ideal of R [7, p. 176]. In 2016 Hosein, F. et. Extended the notation of quasi-primary ideal to submodules, where a proper submodule E of an R-module Q was called quasi-primary if $rq \in E$, for $r \in R$, $q \in Q$, implying that either $q \in rad_Q(E)$ or $r \in \sqrt{[E:_R Q]}$, "where $rad_Q(E)$ define the intersection of all prime submodules of Q contining E [8]". Those two concepts led us to introduce the notation of approximaitly quasi-primary submodule as generalization of prime and primary submodules, where a proper submodule E of an R-module Q is called an approximaitly quasi-primary (for short App-qp) submodule of Q, if $rq \in E$, for $r \in R$, $q \in Q$, implies that either $q \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$, for some $n \in Z^+$. The socle of a module Q denoted by $soc(Q)$ is the intersection of all essential submodules of Q [9]. Several results of approximaitly quasi-primary are introduced.

2. Approximaitly Quasi-primary Submodules

In this part of the paper, we introduce the definition of approximaitly quasi-primary submodule and give it some basic properties and characterizations.

Definition (1)

A proper submodule E of an R-module Q is called an approximaitly quasi-primary (for short App-qp) submodule of Q, if $rq \in E$, for $r \in R$, $q \in Q$, implies that either $q \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$, for some $n \in Z^+$. And an ideal A of a ring R is called App-qp ideal of R if A is an App-qp submodule of an R-module R.

Remarks and examples (2)

1) It is clear that every primary submodule is an App-qp, but not conversely. The following example explains that:

Consider the Z-module Z_{12}, the submodule $E = \langle\bar{0}\rangle$ is not primary submodule of Z-module Z_{12}, since $4.\bar{3} \in \langle\bar{0}\rangle$, for $4 \in Z, \bar{3} \in Z_{12}$, but $\bar{3} \notin \langle\bar{0}\rangle$ and $4 \notin \sqrt{[\langle\bar{0}\rangle:_Z Z_{12}]} = \sqrt{12Z} = 6Z$. But $E = \langle\bar{0}\rangle$ is an App-qp submodule of the Z-module Z_{12} , since for all $r \in R$, $q \in Z_{12}$ such that $rq \in E$, implies that either $q \in rad_{Z_{12}}(\langle\bar{0}\rangle) + soc(Z_{12}) = \langle\bar{6}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $r \in \sqrt{[\langle\bar{0}\rangle + soc(Z_{12}):_Z Z_{12}]} = \sqrt{[\langle\bar{2}\rangle:_Z Z_{12}]} = \sqrt{2Z} = 2Z$. That is if $4.\bar{3} \in E$, for $4 \in Z$, $\bar{3} \in Z_{12}$, and $\bar{3} \notin rad_{Z_{12}}(\langle\bar{0}\rangle) + soc(Z_{12}) = \langle\bar{2}\rangle$ but $4 \in \sqrt{[\langle\bar{0}\rangle + soc(Z_{12}):_Z Z_{12}]} = 2Z$.

2) It is clear that every prime submodule is an App-qp submodule, but not conversely. The following example shows that:

Consider the Z-module Z_4 , the submodule $E = \langle\bar{0}\rangle$ is not prime submodule of the Z-module Z_4 , since $2.\bar{2} \in E$, for $2 \in Z$, $\bar{2} \in Z_4$, but $\bar{2} \notin E$ and $2 \notin [\langle\bar{0}\rangle:_Z Z_4] = 4Z$. While E is an App-qp submodule of the Z-module Z_4 , since $soc(Z_4) = \langle\bar{2}\rangle$ and for all $r \in Z$, $q \in Z_4$ such that $rq \in E$, implies that either $q \in rad_{Z_4}(\langle\bar{0}\rangle) + soc(Z_4) = \langle\bar{2}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $r \in \sqrt{[\langle\bar{0}\rangle + soc(Z_4):_Z Z_4]} = \sqrt{2Z} = 2Z$. That is if $2.\bar{2} \in E$, for $2 \in Z$, $\bar{2} \in Z_4$ implies that $\bar{2} \in rad_{Z_4}(\langle\bar{0}\rangle) + soc(Z_4) = \langle\bar{2}\rangle$ and $2 \in \sqrt{[\langle\bar{0}\rangle + soc(Z_4):_Z Z_4]} = 2Z$.

3) It is clear that every quasi-prime submodule is an App-qp submodule, but not conversely, where a proper submodule E of Q is called quasi-prime if $rsq \in E$. For $r, s \in R$, $q \in Q$, implies that either $rq \in E$ or $sq \in E$ [10]. The following example explains that:

Consider the Z-module Z, and the submodule $4Z$ is not quasi-prime submodule of Z, since $2.2.1 = 4 \in 4Z$,, but $2.1 \notin 4Z$. While $4Z$ is an App-qp submodule of the Z-module Z, since for all $r \in Z$, $q \in Z$ such that $rq \in 4Z$, implies that either $q \in rad_Z(4Z) + soc(Z) = \langle \bar{2} \rangle + (0) = \langle \bar{2} \rangle$ or $r \in \sqrt{[4Z + soc(Z):_Z Z]} = \sqrt{4Z} = 2Z$. That is, if $2.2 \in 4Z$, implies that $2 \in rad_Z(4Z) + soc(Z) = \langle \bar{2} \rangle$ and $2 \in \sqrt{[4Z + soc(Z):_Z Z]} = 2Z$.

The following results are characterizations of App-qp submodules.

Proposition (3)

Let Q be an R-module, and E be a proper submodule of Q. Then E is an App-qp submodule of Q if and only if $IF \subseteq E$, for I is an ideal of R and F is a submodule of Q, implies that either $F \subseteq rad_Q(E) + soc(Q)$ or $I^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$.

Proof

($\Longrightarrow$) Suppose $IF \subseteq E$, for I is an ideal of R and F is a submodule of Q with $F \nsubseteq rad_Q(E) + soc(Q)$, then there exists $k \in F$ such that $k \notin rad_Q(E) + soc(Q)$. Now we have $IF \subseteq E$, then for any $a \in I$, $ak \in E$. Since E is an App-qp submodule of Q and $k \notin rad_Q(E) + soc(Q)$, it follows that $a^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$, that is $I^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$.

($\Longleftarrow$) Assume that $rq \in E$, for $r \in R$, $q \in Q$, then $rq = \langle r \rangle \langle q \rangle$, that is $IF \subseteq E$ where $I = \langle r \rangle, F = \langle q \rangle$, then by hypothesis, either $F \subseteq rad_Q(E) + soc(Q)$ or $I^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$. Hence either$q \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$. Thus E is an App-qp submodule of Q.

The following Corollary is a direct consequence Proposition (3).

Corollary (4)

Let Q be an R-module, and E be a proper submodule of Q.Then, E is an App-qp submodule of Q if and only if for every submodule F of Q and every $r \in R$ with $rF \subseteq E$, implies that either $F \subseteq rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$.

Proposition (5)

A zero submodule of a non-zero R-module Q is an App-qp submodule of Q if and only if $ann_R(F) \subseteq \sqrt{[soc(Q):_R Q]}$ for all non-zero submodule F of Q, with $F \nsubseteq rad_Q(0) + soc(Q)$.

Proof

($\Longrightarrow$) Let F be a non-zero submodule of Q, such that $F \nsubseteq rad_Q(0) + soc(Q)$, and let $x \in ann_R(F)$, implies that $xF = (0)$ but (0) is an App-qp submodule of Q and $F \nsubseteq rad_Q(0) + soc(Q)$, it follows by Corollary (4) that $x^n Q \subseteq (0) + soc(Q)$ for some $n \in Z^+$, that is $x \in \sqrt{[soc(Q):_R Q]}$. Hence $ann_R(F) \subseteq \sqrt{[soc(Q):_R Q]}$.

($\Longleftarrow$) Suppose that $xF \subseteq (0)$, for $r \in R$ and F is a non-zero submodule of Q, with $F \nsubseteq rad_Q(0) + soc(Q)$. Since $xF \subseteq (0)$ it follows that $x \in ann_R(F)$, by hypothesis $x \in \sqrt{[soc(Q):_R Q]}$, that is $x \in \sqrt{[(0) + soc(Q):_R Q]}$. Hence $x^n Q \subseteq (0) + soc(Q)$ for some $n \in Z^+$. Thus by Corollary (4) a zero submodule of an R-module Q is an app-primary submodule of Q.

Proposition (6)

Let Q be an R-module, and E be a proper submodule of Q. Then, E is an App-qp submodule of Q if and only if for every $q \in Q$, $[E:_R q] \subseteq \sqrt{[E + soc(Q):_R Q]}$ with $q \notin rad_Q(E) + soc(Q)$.

Proof

($\Longrightarrow$) Suppose that E is an App-qp submodule of Q, and $r \in [E:_R q]$, implies that $rq \in E$. Since E is an App-qp submodule of Q. and $q \notin rad_Q(E) + soc(Q)$, then $r^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$, that is, $r \in \sqrt{[E + soc(Q):_R Q]}$. Thus $[E:_R q] \subseteq \sqrt{[E + soc(Q):_R Q]}$.

($\Longleftarrow$) Let $rq \in E$, for $r \in R$, $q \in Q$, and suppose that $q \notin rad_Q(E) + soc(Q)$. Since $rq \in E$ it follows that $r \in [E:_R q]$ by hypothesis $r \in \sqrt{[E + soc(Q):_R Q]}$. Hence, $r^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$. Thus E is an App-qp submodule of Q.

Proposition (7)

Let Q be an R-module, and E be a proper submodule of Q. Then, E is an App-qp submodule of Q if and only if $[E:_Q r] \subseteq [E + soc(Q):_Q r^n]$ for $r \in R, n \in Z^+$.

Proof

($\Longrightarrow$) Suppose that E is an App-qp submodule of Q, and let $q \in [E:_Q r]$, such that $q \notin rad_Q(E) + soc(Q)$. Since $q \in [E:_Q r]$ it follows that $rq \in E$. But E is an App-qp submodule of Q. and $q \notin rad_Q(E) + soc(Q)$, then $r^n Q \subseteq [E + soc(Q):_R Q]$ for some $n \in Z^+$. That is $r^n q \in E + soc(Q)$ for all $q \in Q$, it follows that $q \in [E + soc(Q):_Q r^n]$. Thus $[E:_Q r] \subseteq [E + soc(Q):_Q r^n]$.

($\Longleftarrow$) Let $rq \in E$, for $r \in R$, $q \in Q$, and suppose that $q \notin rad_Q(E) + soc(Q)$. Since $rq \in E$ it follows that $q \in [E:_Q r] \subseteq \left[E + soc(Q):_Q r^n\right]$, implies that $q \in \left[E + soc(Q):_Q r^n\right]$, that is $r^n q \in E + soc(Q)$ for all $q \in Q$, hence $r^n Q \subseteq E + soc(Q)$. Thus E is an App-qp submodule of Q.

Before we give the next result we need to recall the following Lemma.

Lemma (8) [11, Coro. (9.9)]

Let E be a submodule of an R-module Q, then $soc(E) = E \cap soc(Q)$.

Proposition (9)

Let E and F are proper submodules of an R-module Q with $E \subset F$ and $soc(Q) \subseteq F$. If E is an App-qp submodule of Q, then E is an App-qp submodule of F.

Proof

Let $rq \in E$, with $r \in R$, $q \in F \subseteq Q$. Since E is an App-qp submodule of Q, then either $q \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$, for some $n \in Z^+$. That is either $q \in \left(rad_Q(E) + soc(Q)\right) \cap F$ or $r^n Q \subseteq \left(E + soc(Q)\right) \cap F$. But since $soc(Q) \subseteq F$, then by modular law we have either $q \in \left(rad_Q(E) \cap F\right) + (soc(Q) \cap F)$ or $r^n Q \subseteq (E \cap F) + (soc(Q) \cap F)$. Now by Lemma (8) $soc(Q) \cap F = soc(F)$, so either $q \in \left(rad_Q(E) \cap F\right) + soc(F) \subseteq rad_Q(E) + soc(F)$ or $r^n Q \subseteq (E \cap F) + soc(F) \subseteq E + soc(F)$. Hence E is an App-qp submodule of F.

Remark (10)

If E is an App-qp submodule of an R-module Q, then $[E:_R Q]$ need not to be an App-qp ideal of R. The following example explains that:

Consider the Z-module Z_{12} , the submodule $E = \langle \bar{0} \rangle$ is an App-qp submodule of the Z-module Z_{12} [see Remarks and Examples (2) (1)]. But $[E:_Z Z_{12}] = 12Z$ is not App-qp ideal of Z because $4.3 \in 12Z$, for $4,3 \in Z$, but $3 \notin rad_Z(12Z) + soc(Z) = \langle \bar{6} \rangle + (0) = \langle \bar{6} \rangle$ and $4 \notin \sqrt{[12Z + soc(Z):_Z Z]} = \sqrt{12Z} = 6Z$.

Now before we offer under certain condition the residual of App-qp submodule is an App-qp ideal we need to revise the following Lemma:

Recall that an R-module Q is called multiplication if every submodule E of Q is of the form $E = IQ$ for some ideal I of Q [12].

Lemma (11) [12, Coro. 14(i)]

Let Q be a faithful multiplication R-module, then $soc(Q) = soc(R)Q$.

Proposition (12)

Let Q be a faithful multiplication R-module and E be a proper submodule of Q. Then E is an App-qp submodule of Q if and only if $[E:_R Q]$ is an App-qp ideal of R.

Proof

($\Longrightarrow$) Let $rs \in [E:_R Q]$, for $r, s \in R$, so $rsQ \subseteq E$. But E is an App-qp submodule of Q then by Corollary (4) either $(sQ) \subseteq rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$, for some $n \in Z^+$. Since Q is multiplication then $rad_Q(E) = \sqrt{[E:_R Q]}Q$, and since Q is faithful multiplication then by Lemma (11) $soc(R)Q = soc(Q)$, we get either $(sQ) \subseteq \sqrt{[E:_R Q]}Q + soc(R)Q$ or $r^n Q \subseteq [E:_R Q]Q + soc(R)Q$, that is either $s \in \sqrt{[E:_R Q]} + soc(R)$ or $r^n \subseteq [E:_R Q] + soc(R) \subseteq [[E:_R Q] + soc(R):_R R]$. Hence $[E:_R Q]$ is an App-qp ideal of R.

($\Longleftarrow$) Suppose that $[E:_R Q]$ is an App-qp ideal of R, and $IF \subseteq E$, for I is an ideal of R and F is a submodule of Q. Since Q is multiplication then $F = JQ$ for some ideal J of R, that is $IJQ \subseteq E$, implies that $IJ \subseteq [E:_R Q]$. But $[E:_R Q]$ is an App-qp ideal of R then either $J \subseteq \sqrt{[E:_R Q]} + soc(R)$ or $I^n \subseteq \left[[E:_R Q] + soc(R):_R R\right] = [E:_R Q] + soc(R)$ for some $n \in Z^+$. It follows that either $JQ \subseteq \sqrt{[E:_R Q]}Q + soc(R)Q$ or $I^n Q \subseteq [E:_R Q]Q + soc(R)Q$. Since Q is faithful multiplication then by Lemma (11) $soc(R)Q = soc(Q)$, and since Q is multiplication then $[E:_R Q]Q = E$ and $rad_Q(E) = \sqrt{[E:_R Q]}Q$. Hence either $JQ \subseteq rad_Q(E) + soc(Q)$ or $I^n Q \subseteq E + soc(Q)$, that is either $F \subseteq rad_Q(E) + soc(Q)$ or $I^n Q \subseteq E + soc(Q)$. Hence, by Proposition (3) E is an App-qp submodule of Q.

Recall that an R-module Q is called non-singular if $Z(Q) = Q$, where $Z(Q) = \{q \in Q: qJ = (0) \text{ for some essentail ideal } J \text{ of } R\}$ [9].

We need to recall the following Lemma:

Lemma (13) [9, Coro. (1.26)]

If Q is a non-singular R-module, then $soc(R)Q = soc(Q)$.

Proposition (14)

Let E be a propoer submodule of a non-singular multiplication R-module T. Then, E is an App-qp submodule of Q if and only if $[E:_R Q]$ is an App-qp ideal of R.

Proof

Follow as in Proposition (12) by using Lemma (13).

We need to recall the following Lemma:

Lemma (15) [13, Coro. of Theo. 9]

Let I and J are ideals of a ring R, and Q be a finitely generated multiplication R-module. Then $IQ \subseteq JQ$ if and only if $I \subseteq J + ann_R(Q)$.

Proposition (16)

Let Q be a faithful finitely generated multiplication R-module and I is an App-qp ideal of R. Then IQ is an App-qp submodule of Q.

Proof

Let $rF \subseteq IQ$ for $r \in R$, and F is a submodule of Q with $r^n Q \nsubseteq IQ + soc(Q)$ for some $n \in Z^+$. Since Q is faithful multiplication then by Lemma (11) $soc(Q) = soc(R)Q$, that is $r^n Q \nsubseteq IQ + soc(R)Q$ for some $n \in Z^+$, it follows that $r^n \notin I + soc(R) = [I + soc(R):_R R]$ implies that $r^n R \nsubseteq I + soc(R)$, Now, since $rF \subseteq IQ$ and Q is a multiplication then $F = JQ$ for some ideal J of R, thus $rJQ \subseteq IQ$. Hence by Lemma (15) $rJ \subseteq I + ann_R(Q)$, but Q is a faithful, then $rJ \subseteq I + (0) = I$. Since I is an App-qp ideal of R and $r^n R \nsubseteq I + soc(R)$ then by Corollary (4) either $\subseteq \sqrt{I} + soc(R)$, hence $JQ \subseteq \sqrt{I}Q + soc(R)Q$. It follows by Lemma (11) $JQ \subseteq rad_Q(IQ) + soc(Q)$. That is $F \subseteq rad_Q(IQ) + soc(Q)$. Hence by Corollary (4) IQ is an App-qp submodule of Q.

Proposition (17)

Let Q be a finitely generated multiplication non-singular R-module and I is an App-qp ideal of R with $ann_R(Q) \subseteq I$. Then IQ is an App-qp submodule of Q.

Proof

Follows similar as in Proposition (16) and using Lemma (13).

Proposition (18)

Let Q be a faithful finitely generated multiplication R-module and E be a proper submodule of Q. Then the following statements are equivalent.

1) E is an App-qp submodule of Q.

2) $[E:_R Q]$ is an App-qp ideal of R.

3) $E = IQ$ for some an App-qp ideal I of R.

Proof

(1) $\Longleftrightarrow$ **(2)** It follows by Proposition (12).

(2) $\Longrightarrow$ **(3)** It is clear.

(3) $\Longrightarrow$ **(2)** Suppose that $E = IQ$ for some App-qp ideal I of R. Since Q is a multiplication, then $E = [E:_R Q]Q = IQ$. But Q is faithful finitely generated multiplication, then $I = [E:_R Q]$, it follows that $[E:_R Q]$ an App-qp ideal of R.

Proposition (19)

Let Q be a finitely generated multiplication non-singular R-module and E be a proper submodule of Q. Then the following statements are equivalent.

1) E is an App-qp submodule of Q.

2) $[E:_R Q]$ is an App-qp ideal of R.

3) $E = JQ$ for some an App-qp ideal J of R with $ann_R(Q) \subseteq J$.

Proof

It follows similar as Proposition (18) by using Proposition (14) and Lemma (15).

We need the following Lemma.

Lemma (20) [14. Coro. (1.3)]

Let $f: Q \longrightarrow Q'$ be an R-epimorphism and E is a submodule of Q' with $ker\ (f) \subseteq E$, then $f\left(rad_Q(E)\right) = rad_{Q'}(f(E))$.

Proposition (21)

Let $f: Q \longrightarrow Q'$ be an R-epimorphism and E' is an App-qp submodule of Q'. Then $f^{-1}(E')$ is an App-qp submodule of Q.

Proof

It is clear that $f^{-1}(E')$ is a proper submodule of Q. Now, suppose that $rq \in f^{-1}(E')$, for $r \in R$, $q \in Q$, implies that $rf(q) \in E'$. But E' is an App-qp submodule of Q', it follows that either $f(q) \in rad_{Q'}(E') + soc(Q')$ or $r^n Q' \subseteq E' + soc(Q')$ for some $n \in Z^+$. It follows that by Lemma (20), either $q \in f^{-1}\left(rad_{Q'}(E')\right) + f^{-1}\big(soc(Q')\big) \subseteq rad_Q\big(f^{-1}(E')\big) + soc(Q)$ or $r^n f^{-1}(f(Q))) \subseteq f^{-1}(E') + f^{-1}(soc(Q')) \subseteq f^{-1}(E') + soc(Q)$.That is either $q \in rad_Q\big(f^{-1}(E')\big) + soc(Q)$ or $r^n Q \subseteq f^{-1}(E') + soc(Q)$. Hence $f^{-1}(E')$ be an App-qp submodule of Q.

Proposition (22)

Let $f: Q \longrightarrow Q'$ be an R-epimorphism and E is an App-qp submodule of Q with ker $(f) \subseteq E$. Then $f(E)$ is an App-qp submodule of Q'.

Proof

$f(E)$ is a proper submodule of Q'. If not, that is $f(E) = Q'$. Let $q \in Q$, then $f(q) \in Q' = f(E)$, so there exists $x \in E$ such that $f(q) = f(x)$, implies that $f(q - x) = 0$, that is $q - x \in Fer\ f \subseteq E$, it follows that $q \in E$. Thus, $E = Q$ contradiction. Now suppose that $rq' \in f(E)$, for $r \in R$, $q' \in Q'$, $f(q) = q'$ for some $q \in Q$ (since f is onto), that is $rq' = rf(q) = f(rq) \in f(E)$, it follows that there exists $e \in E$ such that $f(rq) = f(e)$, that is $f(e - rq) = 0$, so $e - rq \in ker\ (f) \subseteq E$, implies that $rq \in E$. But E is an App-qp submodule of Q, then either $q \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$. Hence, by using Lemma (20) either $q' = f(q) \in f\left(rad_Q(E)\right) + f(soc(Q)) \subseteq rad_{Q'}\big(f(E)\big) + soc(Q')$ or $r^n Q' = r^n f(Q) \subseteq f(E) + f(soc(Q)) \subseteq f(E) + soc(Q')$. Thus $f(E)$ is an App-qp submodule of Q'.

Remark (23)

The intersection of two App-qp submodules of an R-module Q need not to be an App-qp submodule of Q.The following example explains that:

Consider the Z-module Z and the submodules $2Z$, $3Z$ are App-qp submodules of Z-modules Z (because they are prime) but $2Z \cap 3Z = 6Z$ is not App-qp submodule of Z-module Z, since $2.3 \in 6Z$, but $3 \notin rad_Z(6Z) + soc(Z) = 6Z + (0) = 6Z$ and $2 \notin \sqrt{[6Z + soc(Z):_Z Z]} = \sqrt{[6Z:_Z Z]} = \sqrt{6Z} = 6Z$.

We need the following Lemma:

Lemma (24) [15, Theo. 15(3)]

Let Q be a multiplication R-module and E, F be a submodules of Q. Then $rad_Q(E \cap F) = rad_Q(E) \cap rad_Q(F)$.

Proposition (25)

Let E and F be a proper submodules of multiplication R-module Q with $soc(Q) \subseteq E$ or $soc(Q) \subseteq F$. If E and F are App-qp submodules of Q, then $E \cap F$ is an App-qp submodule of Q.

Proof

Suppose $rq \in E \cap F$ for $r, \in R$, $q \in Q$, then $rq \in E$ and $rq \in F$. But both E and F are App-qp submodules of Q, then either $q \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$ and either $q \in rad_Q(F) + soc(Q)$ or $r^n Q \subseteq F + soc(Q)$ for some $n \in Z^+$. Hence either $q \in \left(rad_Q(E) + soc(Q)\right) \cap \left(rad_Q(F) + soc(Q)\right)$ or $r^n Q \subseteq (E + soc(Q)) \cap (F + soc(Q))$. If $soc(Q) \subseteq F \subseteq rad_Q(E)$, then $F + soc(Q) = F$ and $rad_Q(F) + soc(Q) = rad_Q(F)$. Thus either $q \in (rad_Q(E) + soc(T)) \cap rad_Q(F)$ or $r^n Q \subseteq (E + soc(Q)) \cap F$. It follows that by modular law either $q \in \left(rad_Q(E) \cap rad_Q(F)\right) + soc(Q)$ or $r^n Q \subseteq (E \cap F) + soc(Q)$. Hence by Lemma (24) either $q \in rad_Q(E \cap F) + soc(Q)$ or $r^n Q \subseteq (E \cap F) + soc(Q)$ for some $n \in Z^+$. Thus $E \cap F$ is an App-qp submodule of Q. Similarly if $soc(Q) \subseteq E$, we got $E \cap F$ is an App-qp submodule of Q.

Proposition (26)

Let $Q = Q_1 \oplus Q_2$ be an R-module, where Q_1, Q_2 are R-modules, and $E = E_1 \oplus E_2$ be a submodule of Q, with E_1, E_2 are submodules of Q_1, Q_2 respectively with $rad_Q(E) \subseteq soc(Q)$. If E is an App-qp submodule of Q, then E_1 is an App-qp submodule of Q_1 and E_2 is an App-qp submodule of Q_2.

Proof

Let $rq_1 \in E_1$, for $r \in R$, $q_1 \in Q_1$, then $r(q_1, 0) \in E$. Since E is an App-qp submodule of Q, then $(q_1, 0) \in rad_Q(E) + soc(Q)$ or $r^n Q \subseteq E + soc(Q)$ for some $n \in Z^+$. But $rad_Q(E) \subseteq soc(Q)$, implies that $rad_Q(E) + soc(Q) = soc(Q)$, and $E + soc(Q) = soc(Q)$[since $E \subseteq rad_Q(E) \subseteq soc(Q)$]. It follows that either $(q_1, 0) \in soc(Q) = soc(Q) = soc(Q_1 \oplus Q_2)$ or $r^n(Q_1 \oplus Q_2) \subseteq soc(Q) = soc(Q_1 \oplus Q_2)$, that is either $(q_1, 0) \in soc(Q_1) \oplus soc(Q_2)$ or $r^n(Q_1 \oplus Q_2) \subseteq soc(Q_1) \oplus soc(Q_2)$, hence either $q_1 \in soc(Q_1) \subseteq rad_{Q_1}(E_1) + soc(Q_1)$ or $r^n Q_1 \subseteq soc(Q_1) \subseteq E_1 + soc(Q_1)$. Thus E_1 is an App-qp submodule of Q_1.Similarly we can prove that E_2 is an App-qp submodule of Q_2.

Proposition (27)

Let $Q = Q_1 \oplus Q_2$ be an R-module, where Q_1 and Q_2 are R-modules. Then, the following are held:

1) E_1 is an App-qp submodule of Q_1 such that $rad_{Q_1}(E_1) \subseteq soc(Q_1)$ and $soc(Q_2) = Q_2$ if and only if $E_1 \oplus Q_2$ is an App-qp submodule of Q.

2) E_2 is an App-qp submodule of Q_2 such that $rad_{Q_2}(E_2) \subseteq soc(2)$ and $soc(Q_1) = Q_1$ if and only if $Q_1 \oplus E_2$ is an App-qp submodule of Q.

Proof

1) ($\Longrightarrow$) Let $r(q_1, q_2) \in E_1 \oplus Q_2$, for $r \in R$, $(q_1, q_2) \in Q$, then $rq_1 \in E_1$. But E_1 is an App-qp submodule of Q_1 and $rad_{Q_1}(E_1) \subseteq soc(Q_1)$, then either $q_1 \in rad_{Q_1}(E_1) + soc(Q_1) = soc(Q_1)$ or $r^n Q_1 \subseteq E_1 + soc(Q_1) = soc(Q_1)$ for some $n \in Z^+$. Since $soc(Q_2) = Q_2$, then either $(q_1, q_2) \in soc(Q_1) \oplus soc(Q_2) = soc(Q_1 \oplus Q_2) \subseteq rad_Q(E_1 \oplus Q_2) + soc(Q_1 \oplus Q_2)$ or $r^n(Q_1 \oplus Q_2) \subseteq soc(Q_1) \oplus soc(Q_2) = soc(Q_1 \oplus Q_2) \subseteq E_1 \oplus Q_2 + soc(Q_1 \oplus Q_2)$. Thus $E_1 \oplus Q_2$ is an App-qp submodule of Q.

($\Longleftarrow$) Suppose $rq_1 \in E_1$, for $r \in R$, $q_1 \in Q_1$. Then for each $q_2 \in Q_2$, $(q_1, q_2) \in E_1 \oplus Q_2$, but $E_1 \oplus Q_2$ is an App-qp submodule of Q, implies that either $(q_1, q_2) \in rad_Q(E_1 \oplus Q_2) + soc(Q)$ or $r^n Q \subseteq E_1 \oplus Q_2 + soc(Q)$ for some $n \in Z^+$.it follows that either $(q_1, q_2) \in rad_{Q_1}(E_1) \oplus rad_{Q_2}(Q_2) + soc(Q_1 \oplus Q_2)$ or $r^n(Q_1 \oplus Q_2) \subseteq E_1 \oplus Q_2 + soc(Q_1 \oplus Q_2)$, that is either $(q_1, q_2) \in rad_{Q_1}(E_1) \oplus rad_{Q_2}(Q_2) + soc(Q_1) \oplus soc(Q_2)$ or $r^n(Q_1 \oplus Q_2) \subseteq E_1 \oplus Q_2 + soc(Q_1) \oplus soc(Q_2)$. Since $soc(Q_2) = Q_2$ implies that either $(q_1, q_2) \in rad_{Q_1}(E_1) + soc(Q_1) \oplus rad_{Q_2}(Q_2) + Q_2$ or $r^n(Q_1 \oplus Q_2) \subseteq E_1 + soc(Q_1) \oplus Q_2$, that is either $q_1 \in rad_{Q_1}(E_1) + soc(Q_1)$ or $r^n Q_1 \subseteq E_1 + soc(Q_1)$ for some $n \in Z^+$. Hence E_1 is an App-qp submodule of Q_1.

2) Its follows as in part (1).

3. Conclusion

In this paper, we introduce a new generalization of prime and primary submodules called an approximaitly quasi-primary submodule. Many characterizations of this generalization are introduced. Relationships of this generalization with other classes of modules are given.

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