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# Weak Pseudo – 2 – Absorbing Submodules And Related Concepts

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## Abstract

Let R be a commutative ring with identity and E be a unitary left R – module .We introduce and study the concept Weak Pseudo – 2 – Absorbing submodules as generalization of weakle – 2 – Absorbing submodules , where a proper submodule A of an R – module E is called Weak Pseudo – 2 – Absorbing if $0 \neq rsx \in A$ for $r, s \in R$ , $x \in E$ , implies that $rx \in A + soc(E)$ or $sx \in A + soc(E)$ or $rs \in [A + soc(E) :_R E]$. Many basic properties, characterizations and examples of Weak Pseudo – 2 – Absorbing submodule in some types of modules are introduced .

Key word : weakly – 2 – Absorbing submodules , essential submodule , socal of modules , multiplication modules , Z – regular modules , WP – 2 – Absorbing submodules .

## 1. Introduction

The concept of weakly – 2 – Absorbing submodule was first introduced by Darani and Soheilnia as generalization of weakly prime submodule , where a proper submodule A of an R – module E is called a weakly prime submodule of E if $0 \neq te \in A$ , for $t \in R$ , $e \in E$ implies that either $e \in A$ or $t \in [A :_R E]$ , where $[A :_R E] = \{ s \in R : sE \subseteq A \}$ [1] , and a proper submodule A of an R- module E is called a weakly – 2 – Absorbing submodule of E , if $0 \neq ste \in A$ , for $s, t \in R$ , $e \in E$ implies that either $se \in A$ or $te \in A$ or $st \in [A :_R E]$ [2] .

Recently , several generalizations of weakly – 2 – Absorbing submodules have been introduced [ 3,4,5 ] . In our paper , we introduce a new generalization of weakly – 2 – Absorbing submodule which we callWeak Pseudo – 2 – Absorbing submodule , where a proper submodule A of an R – module E is said to be Weak Pseudo – 2 – Absorbing submodule if $0 \neq ste \in A$ , for s , t $\in R$ , $e \in E$ , implies that either $se \in A + soc(E)$ or $te \in A + soc(E)$ or $st \in [A + soc(E) :_R E]$ . Soc ( E ) is the intersection of all essential submodules of E [6] . A nonzero submodule N of an R – module E is called an essential if $N \cap K \neq (0)$ for all nonzero submodules K of E [6]. Every weakly prime submodule of an R – module is weakly – 2 – Absorbing [7] . Recall that an R – module E is cyclic if $E = < x >$ for $x \in E$ [8]. Recall that an R – module E is a semi simple if soc ( E ) = (0) [6]. It is well known that an R – module E is a semi simple if and only if $soc\left(\frac{E}{N}\right) = \left(\frac{soc(E)+N}{N}\right)$ for each submodule N of E [ 6 , Ex.12(c) ] . The set $[N :_E I] = [x \in E : xI \subseteq N\}$ , where N is a submodule of E , and I is an ideal of R $[N :_E I]$ and is a submodule of E containing N. $[N :_E R] = N$ and $[I :_E R] = I$ [9]. Recall that an R – module E is a multiplication if every submodule A of E is of the form A = IE for some ideal I of R , Equivalently $A = [A :_R E]E$ [10]. Recall that an R – module E is a faithful if $Ann(E) = \{r \in R : rE = (0)\}$ [8].

**2.** Basic properties of WP – 2 – Absorbing submodules . In this part of the paper , we introduce the definition of WP – 2 – Absorbing submodules and, thus truth some of it's basic properties , examples and characterizations.

**Definition .1.**

A proper submodule A of an R – module E is said to be Weak Pseudo – 2 – Absorbing (for shorten WP – 2 – Absorbing ) submodule of E , if $0 \neq ste \in A$ , for $s, t \in R$ , $e \in E$ , implies that $se \in A + soc(E)$ or $te \in A + soc(E)$ or $st \in [A + soc(E) :_R E]$ . And an ideal J of a ring R is said to be WP – 2 – Absorbing ideal of R, if J is a WP – 2 – Absorbing R – submodule of an R – module R.

**Example and Remarks .2.**

**1.** In the Z – module $Z_{36}$ , the only essential submodules are $<\bar{2}>$, $<\bar{3}>$, $<\bar{6}>$ and $Z_{36}$ itself thus $Soc(Z_{36}) = <\bar{6}> = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}, \overline{24}, \overline{30}\}$

**2 .** It is clear that the submodules of the Z – module $Z_{36}$ are $<\bar{4}>$ ,$<\bar{6}>$, $<\bar{9}>$ , $<\overline{12}>$ and $<\overline{18}>$ are WP – 2 – Absorbing submodules .

**3 .** The submodules $<\overline{12}>$ and $<\overline{18}>$ of the Z – module $Z_{36}$ are not weakly – 2 – Absorbing submodules , since $0 \neq 2 . 3 . \bar{2} \in <\overline{12}>$ for $2, 3 \in Z$ , $\bar{2} \in Z_{36}$ but $2 . \bar{2} = \bar{4} \notin <\overline{12}>$ and $3 . \bar{2} = \bar{6} \notin <\overline{12}>$ and $2. 3 = 6 \notin [<\overline{12}> :_Z Z_{36}] = 12Z$ .

**4 .** The submodule $<\bar{2}>$, $<\bar{3}>$ of the Z- module $Z_{36}$ are weakly – 2 – Absorbing submodules of $Z_{36}$ because they are weakly prime submodules of Z - $Z_{36}$ .

**5 .** It is clear that the submodules $<\bar{4}>$ , $<\bar{6}>$ and $<\bar{9}>$ of the Z – module $Z_{36}$ are weakly – 2 – Absorbing submodules .

**6 .** It is clear that every weakly – 2 – Absorbing submodule of an R – module E is a WP – 2 – Absorbing , but not conversely , the following example shows that : -- In the Z – module $Z_{36}$ , the submodule $<\overline{18}>$ is a WP – 2- Absorbing by (2) , but $<\overline{18}>$ is not weakly – 2 – Absorbing submodule by (3) .

**7 .** It is clear that every weakly prime submodule of an R – module E is a WP – 2 – Absorbing but not conversely. The following example explains that in the Z- module $Z_{36}$ , the submodule $<\overline{4}>$ is a WP – 2 – Absorbing by (2) . But $<\overline{4}>$ is not weakly prime submodule, since $0 \neq 2 . \overline{2} \in <\overline{4}>$ for $2 \in Z$ , $\overline{2} \in Z_{36}$, but $\overline{2} \notin <\overline{4}>$ and $2 \notin [<\overline{4}> :_Z Z_{36}] = 4Z$.

**8 .** In general ,the submodule nZ of the Z – module Z is weakly – 2 – Absorbing if n = 0 , P, $P^2$ and pq by [7, Rems. And Exs. ( 1.2.2 ) (3) ] . Hence the submodules nZ of the Z – module Z is a WP – 2 – Absorbing if n = 0 , P , $P^2$ and pq by ( 6 ) .

**9 .** The submodules 12Z and 18Z of the Z – module Z are not WP – 2 – Absorbing because soc (Z) = ( 0 ) [8]. That is , $0 \neq 2 . 3 . 2 \in 12Z$ for 2 , 3 , 2 $\in$ Z , but $2 . 2 \notin 12Z + soc ( Z )$ and $3 .2 \notin 12Z + soc ( Z )$ and $2 . 3 \notin [ 12Z + soc ( Z ) :_Z Z ] = 12Z$ .

Also , $0 \neq 2 . 3 . 3 \in 18Z$ for 2 , 3 $\in$ Z , but $2.3 \notin 18Z + soc ( Z )$ and $3.3 \notin 18Z + soc ( Z )$ and $2. 3 \notin [ 18Z + soc ( Z ) :_Z Z ] = 18Z$ .

**10 .** If A is a WP – 2 – Absorbing submodule of an R – module E , then $[ A :_R E ]$ need not to be WP – 2 – Absorbing ideal of R . For example the submodule $<\overline{18}>$ of the Z – module $Z_{36}$ is a WP – 2 – Absorbing submodule by ( 2 ) , but $[<\overline{18}> :_Z Z_{36}] = 18Z$ is not WP – 2 – Absorbing ideal of Z by ( 9 ) .

**11 .** The intersection of two WP – 2 – Absorbing submodules of an R- module E need not to be WP – 2 – Absorbing submodule .For example the submodules 3Z , 4Z are WP – 2 – Absorbing submodule of the Z – module Z by (8) , but $3Z \cap 4Z = 12Z$ is not WP – 2 – Absorbing submodul by (9) . The following results are characterizations of WP – 2 – Absorbing submodules .

**Proposition 3.**

A proper submodule A of an R- module E is a WP – 2 – Absorbing submodule of E if and only if for any t , s $\in$ R with $ts \notin [ A + soc ( E ) :_R E ]$ we have $[ A :_E ts ] \subseteq [ 0 :_E ts ] \cup [ A + soc ( E ) :_E t ] \cup [ A + soc ( E ) :_E s ]$

**Proof : ( $\Rightarrow$)**

Let $e \in [ A :_E ts ]$ with $ts \notin [ A + soc ( E ) :_R E ]$ ,then $tse \in A$ . If $0 \neq tse$ , it follows that either $te \in A + soc ( E)$ or $se \in A + soc ( E )$ , that is either $e \in [ A + soc ( E ) :_E t ]$ or $e \in [ A + soc ( E ) :_E s ]$ . If tse = 0 then $e \in [ 0 :_E ts ]$ . Hence $e \in [0 :_E ts ] \cup [ A + soc ( E ) :_E t ] \cup [ A + soc ( E ) :_E s ]$ . Therefore $[ A :_E st ] \subseteq [ 0 :_E ts ] \cup [ A + soc ( E ) :_E t ] \cup [ A + soc ( E ) :_E s ]$ .

**($\Leftarrow$)** Let $0 \neq tse \in A$ for t, s $\in$ R , e $\in$ E with $ts \notin [ A + soc ( E ) :_R E ]$ . It follows by hypothesis $e \in [ A :_E ts ]$ and $e \notin [ 0 :_E ts ]$ , implies that $e \in [A + soc ( E ) :_E t ] \cup [ A + soc ( E ) :_E s ]$. Hence either $te \in A + soc ( E )$ or $se \in A + soc ( E )$ . Therefore A is aWP – 2 – Absorbing submodule of E .

**Proposition 4.**

A proper submodule A of an R – module E is a WP – 2 – Absorbing if $0 \neq tsK \subseteq A$ for t , s $\in$ R and K is a submodule of E , implies that either $tK \subseteq A + soc(E)$ or $sK \subseteq A + soc(E)$ or $ts \in [A + soc(E) :_R E]$.

**Proof : ( $\Rightarrow$)**

Let $0 \neq tsK \subseteq A$ , for t , s $\in$ R , K is a submodule of E . Suppose that $ts \notin [A + soc(E) :_R E]$ , $tK \not\subseteq A + soc(E)$ and $sK \not\subseteq A + soc(E)$.Then,there exists $e_1 , e_2 \in K$ such that $te_1 \notin A + soc(E)$ and $se_2 \notin A + soc(E)$. Now $0 \neq tse_1 \in A$ and $ts \notin [A + soc(E) :_R E]$ , then by proposition (3) we have $e_1 \in [A :_E st] \subseteq [0 :_E ts] \cup [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$ . But $e_1 \notin [0 :_E ts]$ and $e_1 \notin [A + soc(E) :_E t]$ .It follows that $e_1 \in [A + soc(E) :_E s]$ , that is, $se_1 \in A + soc(E)$. Again $0 \neq tse_2$ and $ts \notin [A + soc(E) :_R E]$ and $e_2 \notin [A + soc(E) :_E s]$, it follow tha $te_2 \in A + soc(E)$. Now, $0 \neq ts(e_1 + e_2) \in A$ and $ts \notin [A + soc(E) :_R E]$ , then $(e_1 + e_2) \in [A :_E ts]$ and $(e_1 + e_2) \notin [0 :_E ts]$, it follows by proposition ( 3 ) either $(e_1 + e_2) \in [A + soc(E) :_E t]$ or $(e_1 + e_2) \in [A + soc(E) :_E s]$. That is either $t(e_1 + e_2) \in A + soc(E)$ or $s(e_1 + e_2) \in A + soc(E)$ . If $t(e_1 + e_2) = te_1 + te_2 \in A + soc(E)$ , and $te_2 \in A + soc(E)$ , then $te_1 \in A + soc(E)$ which is a contradiction .

If $s(e_1 + e_2) = se_1 + se_2 \in A + soc(E)$ and $se_1 \in A + soc(E)$, then $se_2 \in A + soc(E)$ which is a contradiction.

Hence either $tK \subseteq A + soc(E)$ or $sK \subseteq A + soc(E)$ or $ts \in [A + soc(E) :_R E]$ .

**( $\Leftarrow$ )** Trivial , so we omitted it.

**Proposition 5.**

A proper submodule A of a cyclic R – module E is a WP – 2 – Absorbing if and only if for each t , s $\in$ R with $ts \notin [A + soc(E) :_R E]$ we have $[A :_R tse] \subseteq [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$.

**Proof : ( $\Rightarrow$)**

Let t , s $\in$ R ,with $ts \notin [A + soc(E) :_R E]$ and let $r \in [A :_R tse]$, it follows that $ts(re) \in A$. If $0 \neq ts(re) \in A$ and A is a WP – 2 – Absorbing and $ts \notin [A + soc(E) :_R E]$, then either $tre \in A + soc(E)$ or $sre \in A + soc(E)$ , that is either $r \in [A + soc(E) :_R te]$ or $r \in [A + soc(E) :_R se]$. If $tsre = 0$ , implies that $r \in [0 : tse]$. Hence $r \in [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$ . Therefore $[A :_R tse] \subseteq [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$.

**( $\Leftarrow$ )** Since E is cyclic , then $E = \langle e_1 \rangle$ for some $e_1 \in E$ . Let $0 \neq tse \in A$ for t , s $\in$ R , e $\in$ E with $ts \notin [A + soc(E) :_R E]$.Since e $\in$ E then $e = re_1$ for some r $\in$ R , that is $0 \neq ts(re_1) \in A$ ,it follows that $r \in [A :_R tse_1] \subseteq [0 :_R tse_1] \cup [A + soc(E) :_R te_1] \cup [A + soc(E) :_R se_1]$. But $r \notin [0 :_R tse_1]$ ( since $0 \neq tsre_1$ ) , therefore , $r \in [A + soc(E) :_R te_1]$ or $r \in [A +$

soc ( E ) $:_R$ $se_1$ ], it follows that $tre_1 \in A + soc(E)$ or $sre_1 \in A + soc(E)$. That is $te \in A + soc(E)$ or $se \in A + soc(E)$. Therefore A is a WP – 2 – Absorbing submodule of E .

**Proposition 6.**

A proper submodule N of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq IJL \subseteq N$ for some ideals I , J of R and some submodule L of E implies that either $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$ or $IJ \subseteq [N + soc(E) :_R E]$ .

**Proof : ( ⇒)**

Let $(0) \neq IJL \subseteq N$ for some ideals I,J of R and some submodule L of E with $IJ \not\subseteq [N + soc(E) :_R E]$. To prove that $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$ .Suppose that $IL \not\subseteq N + soc(E)$ and $JL \not\subseteq N + soc(E)$ , that is there exist $a_1 \in I$ and $a_2 \in J$ such that $a_1L \not\subseteq N + soc(E)$ and $a_2L \not\subseteq N + soc(E)$ .Now , $(0) \neq a_1a_2L \subseteq N$ , and N is aWP – 2 – Absorbing submodule of E , then by proposition ( 4 ) either $a_1 L \subseteq N + soc(E)$ or $a_2L \subseteq N + soc(E)$ or $a_1a_2 \in [N + soc(E) :_R E]$. Since $IJ \not\subseteq [N + soc(E) :_R E]$ , there exists $b_1 \in I$ and $b_2 \in J$ such that $b_1b_2 \notin [N + soc(E) :_R E]$.But $(0) \neq b_1b_2L \subseteq N$ and N is a WP – 2 – Absorbing submodule of E , and $b_1b_2 \notin [N + soc(E) :_R E]$ , then by proposition ( 4 ) either $b_1L \subseteq N + soc(E)$ or $b_2L \subseteq N + soc(E)$.

Now : -- **( 1 )** If $b_1L \subseteq N + soc(E)$ and $b_2L \not\subseteq N + soc(E)$ . Since $(0) \neq a_1b_2L \subseteq N$ and $b_2L \not\subseteq N + soc(E)$ and $a_1L \not\subseteq N + soc(E)$ , then by proposition ( 4 ) $a_1b_2 \in [N + soc(E) :_R E]$. Since $b_1L \subseteq N + soc(E)$ and $a_1L \not\subseteq N + soc(E)$ , we get $(a_1 + b_1) L \not\subseteq N + soc(E)$ .For there more $(0) \neq (a_1 + b_1)b_2 L \subseteq N$ and N is a WP – 2 – Absorbing with $(a_1 + b_1) L \not\subseteq N + soc(E)$ , $b_2L \not\subseteq N + soc(E)$ , it follows that by proposition ( 4 ) $(a_1 + b_1) b_2 = a_1b_2 + b_1b_2 \in [N + soc(E) :_R E]$ , but $a_1b_2 \in [N + soc(E) :_R E]$ , then $b_1b_2 \in [N + soc(E) :_R E]$ , this is a contradiction .

**( 2 )** If $b_2L \subseteq N + soc(E)$ and $b_1L \not\subseteq N + soc(E)$ , so by similar steps of ( 1 ) we get a contradiction .

**( 3 )** If $b_1L \subseteq N + soc(E)$ and $b_2L \subseteq N + soc(E)$ , since $b_2L \subseteq N + soc(E)$ and $a_2L \not\subseteq N + soc(E)$ , we get $(a_2 + b_2)L \not\subseteq N + soc(E)$ . But $(0) \neq a_1 (a_2 + b_2) L \subseteq N$ and N is a WP – 2 – Absorbing with $a_1L \not\subseteq N + soc(E)$ and $(a_2 + b_2) L \not\subseteq N + soc(E)$ then , we get $a_1 (a_2 + b_2) \in [N + soc(E) :_R E]$ . Since $a_1a_2 \in [N + soc(E) :_R E]$ and $a_1a_2 + a_1b_2 \in [N + soc(E) :_R E]$ , it follows that $a_1b_2 \in [N + soc(E) :_R E]$.

Now , $(0) \neq (a_1 + b_1) a_2 \in N$ and $a_2L \not\subseteq N + soc(E)$ and $(a_1 + b_1) L \not\subseteq N + soc(E)$ , it follows by proposition ( 4 ) $(a_1 + b_1)a_2 = a_1a_2 + b_1a_2 \in [N + soc(E) :_R E]$ and since $a_1a_2 \in [N + soc(E) :_R E]$ , we get $b_1a_2 \in [N + soc(E) :_R E]$. Since $(0) \neq (a_1 + b_1) (a_2 + b_2) L \subseteq N$ and $(a_1 + b_1) L \not\subseteq N + soc(E)$ and $(a_2 + b_2) L \not\subseteq N + soc(E)$ then by proposition ( 4 ) we have $(a_1 + b_1) (a_2 + b_2) = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 \in [N + soc(E) :_R E]$. But $a_1a_2 , b_1a_2 , a_1b_2 \in [N + soc(E) :_R E]$ , we get $b_1b_2 \in [N + soc(E) :_R E]$ which is a contradiction . Thus $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$ .

**( $\Leftarrow$ )** Trivial, so we omittedit

The following corollaries are adirect consequence of proposition ( 6 ) .

**Corollary 7.**

A proper submodule A of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $( 0 ) \neq IJx \subseteq A$ for some ideals I , J of R and $x \in E$ , implies that either $Ix \subseteq A + soc ( E )$ or $Jx \subseteq A + soc ( E )$ or $IJ \subseteq [A + soc ( E ) :_R E ]$ .

**Corollary 8.**

A proper submodule A of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $( 0 ) \neq sIL \subseteq A$ for some $s \in R$ and ideal I of R and some submodule L of E , implies that either $sL \subseteq A + soc ( E )$ or $IL \subseteq A + soc ( E )$ or $sI \subseteq [ A + soc ( E ) :_R E ]$ .

**Corollary 9.**

A proper submodule A of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $( 0 ) \neq sIx \subseteq A$ for some $s \in R$ , ideal I of R and some $x \in E$ , implies that either $sx \in A + soc ( E )$ or $Ix \subseteq A + soc ( E )$ or $sI \subseteq [A + soc ( E ) :_R E ]$ .

**Proposition 10.**

Let A be a WP – 2 – Absorbing submodule of an R – module E and B is a submodule of E with $B \subseteq A$ then $\frac{A}{B}$ is a WP – 2 – Absorbing submodule of an R – module $\frac{E}{B}$ .

**Proof :** Let $0 \neq ts ( x + B ) = stx + B \in \frac{A}{B}$ for $s , t \in R$ , $x + B \in \frac{E}{B}$ , $x \in E$ .It follows that $tsx \in A$. If $tsx = 0$ then $ts( x + B ) = 0$ which is a contradiction . thus $0 \neq tsx \in A$ implies that either $tx \in A + soc ( E )$ or $sx \in A + soc ( E )$ or $tsE \subseteq A + soc ( E )$ .It follow that either $t( x + B ) \in \frac{A + soc(E)}{B}$ or $s(x + B ) \in \frac{A + soc(E)}{B}$ or $\frac{tsE}{B} \subseteq \frac{A + soc(E)}{B}$ . That is either $t( x + B ) \in \frac{A}{B} + \frac{A + soc\ (E)}{B} \subseteq \frac{A}{B} + soc\ ( \frac{E}{B} )$ or $s( x + B ) \in \frac{A}{B} + \frac{A + soc(E)}{B} \subseteq \frac{A}{B} + soc\ ( \frac{E}{B} )$ or $ts \frac{E}{B} \subseteq \frac{A}{B} + \frac{A + soc(E)}{B} \subseteq \frac{A}{B} + soc\ ( \frac{E}{B} )$. Hence , $\frac{A}{B}$ is a WP – 2 – Absorbing submodule of an R – module $\frac{E}{B}$ .

**Proposition .11.**

Let A , B be submodules of semi simple R – module E with $B \subseteq A$. If B and $\frac{A}{B}$ are WP – 2 – Absorbing submodules of E , $\frac{E}{B}$ respectively , then A is a WP – 2 – Absorbing submodule of E .

**Proof :**

Let $0 \neq tsx \notin A$ for $t , s \in R$ , $x \in E$ , then $0 \neq ts ( x + B ) = tsx + B \in \frac{A}{B}$ . If $0 \neq tsx \in B$ and B is a WP – 2 – Absorbing , implies that either $tx \in B + soc ( E ) \subseteq A + soc ( E )$ or $sx \in B + soc ( E ) \subseteq A + soc ( E )$ or $tsE \subseteq B + soc ( E ) \subseteq A + soc ( E )$ .Thus A is a WP – 2 –

Absorbing submodule of E . Assume that tsx $\notin$ B , it follows that $0 \neq ts(x + B) \in \frac{A}{B}$ . But $\frac{A}{B}$ is a WP – 2 – Absorbing submodule of $\frac{E}{B}$ implies that either $t(x + B) \in \frac{A}{B} + soc(\frac{E}{B})$ or $s(x + B)\ \frac{A}{B} + soc(\frac{E}{B})$ or $\frac{tsE}{B} \subseteq \frac{A}{B} + soc(\frac{E}{B})$ . Since E is a semi simple then $soc(\frac{E}{B}) = \frac{soc(E)+B}{B}$ . It follows that either $t(x + B) \in \frac{A}{B} + \frac{B+soc(E)}{B}$ or $s(t + B) \subseteq \frac{A}{B} + \frac{B+soc(E)}{B}$ or

$\frac{tsE}{B} \subseteq \frac{A}{B} + \frac{B+soc(E)}{B}$ . But B $\subseteq$ A , implies that B + soc ( E ) $\subseteq$ A + soc ( E ) , hence $\frac{A}{B} + \frac{B+soc(E)}{B} \subseteq \frac{A}{B} + \frac{A+soc(E)}{B}$. Since $\frac{A}{B} \subseteq \frac{A+soc(E)}{B}$ implies that $\frac{A}{B} + \frac{A+soc(E)}{B} = \frac{A+soc(E)}{B}$ . that is either $t(x + B) \in \frac{A+soc(E)}{B}$ or $s(x + B) \in \frac{A+soc(E)}{B}$ , $\frac{tsE}{B} \subseteq \frac{A+soc(E)}{B}$ , it follows that either tx $\in$ A + soc ( E ) or sx $\in$ A + soc ( E ) or tsE $\subseteq$ A + soc ( E ) . Thus A is WP – 2 – Absorbing submodule of E .

**Proposition 12.**

Let A be a proper submodule of an R – module E with soc ( E ) $\subseteq$ A. Then A is a WP – 2 – Absorbing submodule of E so if and only if $[A :_E I]$ is a WP – 2 – Absorbing submodule of E for each ideal I of R .

**Proof : ( $\Rightarrow$ )**

Let $(0) \neq tsB \subseteq [A :_E I]$ for t , s $\in$ R , B is a submodule of E , then $(0) \neq tsIB \subseteq A$ , implies that either tIB $\subseteq$ A + soc ( E ) or sIB $\subseteq$ A + soc ( E ) or tsE $\subseteq$ A + soc ( E ) . But soc ( E ) $\subseteq$ A , then A + soc ( E ) = A . that is either tIB $\subseteq$ A or sIB $\subseteq$ A or tsE $\subseteq$ A. Thus , either $tB \subseteq [A :_E I]$ or $sB \subseteq [A :_E I]$ or $tsE \subseteq A \subseteq [A :_E I]$ . It follows that either $tB \subseteq [A :_E I] \subseteq [A :_E I] + soc(E)$ or $sB \subseteq [A :_E I] \subseteq [A :_E I] + soc(E)$ or $tsE \subseteq [A :_E I] \subseteq [A :_E I] + soc(E)$. Hence , $[A :_E I]$ is a WP – 2 – Absorbing submodule of E .

**( $\Leftarrow$ )**Since $[A :_E I]$ is a WP – 2 – Absorbing subodule for every non zero ideal I of R . Put I = R , we get $[A :_E R] = A$ is a WP – 2 – Absorbing submodule of E .

We need to introduce the following definition.

**Definition 13.** Let A be a WP – 2 – Absorbing submodule of an R – module E and r, s $\in$ R , e $\in$ E , we say that ( r , s , e ) is WP – triple zero of A if rse = 0 , re $\notin$ A + soc ( E ) , se $\notin$ A + soc ( E ) and rs $\notin$ $[A + soc(E) :_R E]$ .

**Proposition 14.** If A is a WP – 2 – Absorbing submodule of E with ( r , s , e ) is a WP – trible zero of A for some r, s $\in$ R , e $\in$ E . Then rsA = ( 0 ) .

**Proof :** Suppose rsA $\neq$ ( 0 ) , then rsa $\neq$ 0 for some a $\in$ A. Since ( r , s , e ) is a WP – triple zero of A then rse = 0 , re $\notin$ A + soc ( E ) , se $\notin$ A + soc ( E ) and rs $\notin$ $[A + soc(E) :_R E]$. Since 0 $\neq$ rsa $\in$ A and A is a WP – 2 – Absorbing submodule of E and rs $\notin$ $[A + soc(E) :_R E]$, then either ra$\in$ A + soc ( E ) or sa $\in$ A + soc ( E).

Now, $0 \neq rs(e + a) = rse + rsa = rsa \in A$, and $rs \notin [A + soc(E) :_R E]$, then either $r(e + a) = re + ra \in A + soc(E)$ or $s(e + a) = se + sa \in A + soc(E)$. If $re + sa \in A + soc(E)$ and $ra \in A + soc(E)$ implies that $re \in A + soc(E)$ contradiction. If $se + sa \in A + soc(E)$ and $sa \in A + soc(E)$, implies that $se \in A + soc(E)$ contradiction. Hence, $rsA = (0)$.

**Proposition 15.** If A is a WP – 2 – Absorbing submodule of E with $(r, s, e)$ is a WP – triple zero of A for some $r, s \in R$, $e \in E$, then $[A :_R E]re = [A :_R E]se = (0)$.

**Proof :** Suppose that $[A :_R E]se \neq (0)$ then $yse \neq o$ for some $y \in [A :_R E]$. Since $(r, s, e)$ is a WP – triple zero of A, $rse = 0$ and $re \notin A + soc(E)$, $se \notin A + soc(E)$ and $rs \notin [A + soc(E) :_R E]$. We have $0 \neq yse \in A$ and A is a WP – 2 – Absorbing submodule of E, then either $ye \in A + soc(E)$ or $se \in A + soc(E)$ or $ys \in [A + soc(E) :_R E]$. Now, $0 \neq (r + y)se = rse + yse = yse \in A$ and A is a WP – 2 – Absorbing submodule, then either $(r + y)e = re + ye \in A + soc(E)$ or $se \in A + soc(E)$ or $(r + y)s \in [A + soc(E) :_R E]$.Since $ye \in A + soc(E)$ and if $re + ye \in A + soc(E)$, it follows that $re \in A + soc(E)$ a contradiction. If $(r + y)s = rs + ys \in [A + soc(E) :_R E]$ and $ys \in [A + soc(E) :_R E]$, then $rs \in [A + soc(E) :_R E]$ a contradiction. Thus $[A :_R E]se = (0)$. Similarly we can prove $[A :_R E]re = (0)$.

**Proposition 16.** If A is a WP – 2 – Absorbing submodule of E with $(r, s, e)$ is a WP – triple zero of A for some $r, s \in R$, $e \in E$. Then $r[A :_R E]e = s[A :_R E]e = (0)$.

**Proof :** Suppose that $r[A :_R E]e \neq (0)$, then there exists $x \in [A :_R E]$ such that $rxe \neq 0$. But $(r, s, e)$ is a WP – triple zero of A, $rse = 0$, $re \notin A + soc(E)$ or $se \notin A + soc(E)$ and $rs \notin [A + soc(E) :_R E]$.

For $0 \neq rxe \in A$ and A is a WP – 2 – Absorbing submodule of E, then either $re \in A + soc(E)$ or $xe \in A + soc(E)$ or $rx \in [A + soc(E) :_R E]$. Now, $0 \neq r(s + x)e = rse + rxe = rxe \in A$, and A is a WP – 2 – Absorbing submodule, then either $re \in A + soc(E)$ or $(s + x)e = se + xe \in A + soc(E)$ or $r(s + x) = rs + rx \in [A + soc(E) :_R E]$. That is $re \in A + soc(E)$ a contradiction. If $(s + x)e = se + xe \in A + soc(E)$, implies that $se \in A + soc(E)$ a contradiction. If $rs + rx \in [A + soc(E) :_R E]$, implies that $rs \in [A + soc(E) :_R E]$ a contradiction. Thus $r[A :_R E]e = (0)$. In similary way $s[A :_R E]e = (0)$.

As direct consequence of proposition ( 16 ), we get the following corollary :

**Corollary 17.** If A is a WP – 2 – Absorbing submodule of an R – module E with $(r, s, e)$ is a WP – triple zero of A for some $r, s \in R$, $e \in E$, then $r[A :_R E]A = s[A :_R E]A = (0)$.

**Proposition 18.** If A is a WP – 2 – Absorbing submodule of E with $(r, s, e)$ is a WP – triple zero of A for some $r, s \in R$, $e \in E$, then $[A :_R E]sA = [A :_R E]rA = (0)$.

**Proof :** Suppose that $[A :_R E]sA \neq (0)$, then $xsa \neq (0)$ for some $x \in [A :_R E]$, $a \in A$. Since $(r, s, e)$ is a WP – triple zero of A, then $rse = 0$, $re \notin A + soc(E)$, $se \notin A + soc(E)$ and $rs \notin [\notin A + soc(E) :_R E]$. For $0 \neq xsa \in A$, it follows that either $xa \in A + soc(E)$ or $sa \in A + soc(E)$ or $xs \in [A + soc(E) :_R E]$. We have $(r + x)s(a + e) = rsa + rse +$

$xsa + xse = xsa \in A$ ( since $rse = 0$ , and $rsa = 0$ for proposition ( 14 ) and $xsa = 0$ from proposition ( 16 )). That is $0 \neq (r + x)(a + e) = ra + re + xa + xe \in A$ , implies that $re \in A + soc(E)$ a contradiction or $s(a + e) = sa + se \in A + soc(E)$ implies that $se \in A + soc(E)$ a contradiction or $(r + x)s = rs + xs \in [A + soc(E) :_R E]$, implies that $rs \in [A + soc(E) :_R E]$ a contradiction .Thus $[A :_R E]\, sA = (0)$ .

In similar steps , we can show that $[A :_R E]rA = (0)$ .

**Proposition 19.** If A is a WP – 2 – Absorbing submodule of E with ( r , s , e ) is a WP – triple zero of A for some $r, s \in R$ , $e \in E$ , then $[A :_R E][A :_R E]\, e = (0)$ .

**Proof :** Suppose that $[A :_R E][A :_R E]\, e \neq (0)$, then $0 \neq xye \in A$ for some $x , y \in [A :_R E]$. For ( r , s , e ) is a WP – triple zero of A, then $rse = 0$, $re \notin A + soc(E)$ , $se \notin A + soc(E)$ and $rs \notin [A + soc(E) :_R E]$.

Now, $0 \neq xye \in A$ , implies that either $xe \in A + soc(E)$ or $ye \in A + soc(E)$ or $xy \in [A + soc(E) :_R E]$.

Now , $0 \neq (r + x)(s + y)e = rse + rye + xse + xye = xye \in A$ (since $rse = 0$ , $rye = 0$ , $xse = 0$ by proposition (16)).It follows that either $(r + x)e = re + xe \in A + soc(E)$, implies that $re \in A + soc(E)$ a contradiction . or $(s + y)e = se + ye \in A + soc(E)$, implies that $se \in A + soc(E)$ a contradiction, or $(r + x)(s + y) = rs + ry + xs + xy \in [A + soc(E) :_R E]$, implies that $rs \in [A + soc(E) :_R E]$ a contradiction.

Hence $[A :_R E][A :_R E]\, e = (0)$ .

**Proposition 20.** If A is a WP – 2 – Absorbing submodule of E with ( r , s , e ) is a WP – triple zero of A for some $r, s \in R$ , $e \in E$ , then $[A :_R E][A :_R E]A = (0)$ .

**Proof :** By proposition ( 14 ) and proposition ( 19 ) .

**Proposition 21 .** Let A be a WP – 2 – Absorbing submodule of E and $rsB \subseteq A$ for some $r, s \in R$, and some submodule B of E with ( r , s , x ) is not WP – triple zero of A for every $x \in B$ . If $rs \notin [A + soc(E) :_R E]$, then $rx \in A + soc(E)$ or $sx \in A + soc(E)$ .

**Proof :** Suppose that ( r , s , x ) is not WP – triple zero of A for every $x \in B$ and suppose that $rB \not\subseteq A + soc(E)$ and $sB \not\subseteq A + soc(E)$, then $ry_1 \notin A + soc(E)$ or $sy_2 \notin A + soc(E)$ for some $y_1, y_2 \in B$. If $0 \neq rsy_1 \in A$ with $rs \notin [A + soc(E) :_R E]$ and since $ry_1 \notin A + soc(E)$ then $sy_1 \in A + soc(E)$ ( for A is a WP – 2 – Absorbing submodule ) . If $rsy_1 = 0$ and $ry_1 \notin A + soc(E)$ , $rs \notin [A + soc(E) :_R E]$ and ( r , s , $y_1$ ) is not WP – triple zero of A , we get $sy_1 \in A + soc(E)$. By similar arguments since ( r , s , $y_2$ ) is not WP – triple zero of A , we get $ry_2 \in A + soc(E)$. Now , $rs(y_1 + y_2) \in A$ and ( r , s , $y_1 + y_2$ ) is not WP – triple zero of A and $rs \notin [A + soc(E) :_R E]$, we get $r(y_1 + y_2) \in A + soc(E)$ or $s(y_1 + y_2) \in A + soc(E)$ .

If $r(y_1 + y_2) = ry_1 + ry_2 \in A + soc(E)$ and $ry_2 \in A + soc(E)$, we get $ry_1 \in A + soc(E)$ is a contradiction .

If $s(y_1 + y_2) = sy_1 + sy_2 \in A + soc(E)$ and $sy_1 \in A + soc(E)$ then $sy_2 \in A + soc(E)$ is a contradiction .

Hence $rB \subseteq A + soc(E)$ or $sB \subseteq A + soc(E)$ .

**Proposition 22.** Let A , B be WP – 2 – Absorbing submodule of E with B is not contained in A and either $soc(E) \subseteq A$ or $soc(E) \subseteq B$ . Then $A \cap B$ is a WP – 2 – Absorbing submodule of E .

**Proof :** It is clear that $A \cap B$ is a proper submodule of B and B is a proper submodule of E , implies that $A \cap B$ is a proper submodule of E. Let $(0) \neq rsL \subseteq A \cap B$ for $r, s \in R$ , L is a submodule of E, it follows that $(0) \neq rsL \subseteq A$ and $(0) \neq rsL \subseteq B$ . But A, B are WP – 2 – Absorbing submodule of E, then either $rL \subseteq A + soc(E)$ or $sL \subseteq A + soc(E)$ or $rsE \subseteq A + soc(E)$ and $rL \subseteq B + soc(E)$ or $sL \subseteq B + soc(E)$ or $rsE \subseteq B + soc(E)$. Thus , either $rL \subseteq (A + soc(E)) \cap (B + soc(E))$ or $sL \subseteq (A + soc(E)) \cap (B + soc(E))$ or $rsE \subseteq (A + soc(E)) \cap (B + soc(E))$. If $soc(E) \subseteq B$ then $B + soc(E) = B$, it follows that either $rL \subseteq (A + soc(E)) \cap B$ or $sL \subseteq (A + soc(E)) \cap B$ or $rsE \subseteq (A + soc(E)) \cap B$. Again Since $soc(E) \subseteq B$, then by Modular Law $(A + soc(E)) \cap B = (A \cap B) + soc(E)$. Thus either $rL \subseteq (A \cap B) + soc(E)$ or $sL \subseteq (A \cap B) + soc(E)$ or $rsE \subseteq (A \cap B) + soc(E)$. Thus $A \cap B$ is a WP – 2 – Absorbing submodule of E .

Recall that for any submodules A , K a multiplication R – module E with A = IE , B = JE , for some ideals I, J of R, the product AB = IJE = IB. In particular AE = IEE = IE = A, and for any $x \in E$ , A = Ix [2] .

The following propositions are characterizations of WP – 2 – Absorbing submodules is class of multiplication modules.

**Proposition 23.** Let E be a multiplication R – module, and A be a proper submodule of E. Then A is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq L_1L_2L_3 \subseteq A$ for some submodules $L_1, L_2, L_3$ of E implies that either $L_1L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$ .

**Proof : ( $\Rightarrow$ )** Let $(0) \neq L_1L_2L_3 \subseteq A$ for some submodules $L_1, L_2, L_3$ of E . But E is a multiplication , then $L_1 = I_1E$ , $L_2 = I_2E$ , $L_3 = I_3E$ for some ideals $I_1, I_2, I_3$ of R . That is $(0) \neq L_1L_2L_3 = I_1 I_2 I_3 E \subseteq A$. But A is a WP – 2 – Absorbing submodule of E, then by proposition ( 6 ) either $I_1 I_3E \subseteq A + soc(E)$ or $I_2 I_3E \subseteq A + soc(E)$ or $I_1I_2 \subseteq [A + soc(E) :_R E]$ ( ie $I_1 I_2E \subseteq A + soc(E)$ ). It follows that either $L_1L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1 L_2 \subseteq A + soc(E)$ .

**( $\Leftarrow$ )** Let $(0) \neq I_1 I_2L \subseteq A$ for $I_1, I_2$ are ideals of R , L is submodule of E. Since E is a multiplication, then $L = I_3E$ for some ideal $I_3$ of R. That is $(0) \neq I_1 I_2 I_3 E \subseteq A$. Put $L_1 = I_1E$ and $L_2 = I_2E$, then $(0) \neq L_1L_2L \subseteq A$, it follows by hypothesis that either $L_1L \subseteq A + soc(E)$ or $L_2L \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$. That is either $I_1L \subseteq A + soc(E)$ or $I_2L \subseteq A + soc(E)$ or $I_1I_2E \subseteq A + soc(E)$, ( ie $I_1I_2 \subseteq [A + soc(E) :_R E]$. Thus , by proposition ( 6 ) A is a WP – 2 – Absorbing submodule of E .

The following corollary is a direct consequence of proposition ( 23 ) .

**Corollary 24.** Let E be a multiplication R – module and A be a proper submodule of E. Then A is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq L_1L_2 e \subseteq A$ for some submodules $L_1, L_2$ of E and $e \in E$, implies that either $L_1e \subseteq A + soc(E)$ or $L_2e \subseteq A + soc(E)$ or $L_1 L_2 \subseteq A + soc(E)$ .

It is well known that if E is a faithful multiplication R – module then soc (E) = soc (R)E [11,coro. (2.14) (i)].

**Proposition 25.** Let E be a faithful multiplication R – module and A be a proper submodule of E. Then A is a WP – 2 – Absorbing submodule of E if and only if $[A :_R E]$ is a WP – 2 – Absorbing ideal of R .

**Proof : ( $\Rightarrow$ )** Let $(0) \neq I_1I_2 I_3 \subseteq [A :_R E]$ for $I_1, I_2, I_3$ are ideals of R , it follows that $(0) \neq I_1I_2 I_3 E \subseteq A$. But E is a multiplication then $(0) \neq I_1I_2 I_3 E = L_1L_2L_3 \subseteq A$ by taking $L_1 = I_1E$ , $L_2 = I_2E$ and $L_3 = I_3E$ . Now since A is a WP – 2 – Absorbing , then by proposition (23) either $L_1 L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$ . But E is a faithful multiplication then soc (E) = soc (R)E . Thus either $I_1 I_3E \subseteq [A :_R E]E + soc(R)E$ or $I_2 I_3 E \subseteq [A :_R E]E + soc(R)E$ or $I_1I_2E \subseteq [A :_R E]E + soc(R)E$ . That is either $I_1 I_3 \subseteq [A :_R E] + soc(R)$ or $I_2 I_3 \subseteq [A :_R E] + soc(R)$ or $I_1I_2 \subseteq [A :_R E] + soc(R) =[[A :_R E] + soc(R) :_E R]$. Therefore by proposition ( 6 ) $[A :_R E]$ is a WP – 2 – Absorbing ideal of R .

**( $\Leftarrow$ )** Let $(0) \neq I_1I_2L \subseteq A$ for $I_1, I_2$ are ideals of R and L is submodule of E. Since E is a multiplication , then $L = I_3E$ for some ideal $I_3$ of R . That is $(0) \neq I_1 I_2 I_3 E \subseteq A$ , it follows that $(0) \neq I_1 I_2 I_3 \subseteq [A:_R E]$ . But $[A:_R E]$ is a WP – 2 – Absorbing ideal of R , then by proposition (6) either $I_1 I_3 \subseteq [A :_R E] + soc(R)$ or $I_2 I_3 \subseteq [A :_R E] + soc(R)$ or $I_1I_2 \subseteq [A :_R E] + soc(R)$ .Thus either $I_1 I_3E \subseteq [A :_R E]E + soc(R)E$ or $I_2 I_3 E \subseteq [A :_R E]E + soc(R)E$ or $I_1I_2E \subseteq [A :_R E]E + soc(R)E$ . That is either $I_1L \subseteq A + soc(E)$ or $I_2L \subseteq A + soc(E)$ or $I_1I_2E \subseteq A + soc(E)$, ( ie $I_1I_2 \subseteq [A + soc(E) :_R E]$. Hence by proposition (6) A is a WP – 2 – Absorbing submodule of E .

It is well known that cyclic R- module is multiplication [10]. We get the following corollary:

**Corollary 26.** Let E be faithful cyclic R – module and A be a proper submodule of E. Then A is a WP – 2 – Absorbing if and only if $[A :_R E]$ is a WP – 2 – Absorbing ideal of R .

**Proposition .27.** Let E be a faithful finitely generated multiplication R – module and I be a WP – 2 – Absorbing ideal of R . Then , IE is a WP – 2 – Absorbing submodule of E .

**Proof :** Let $(0) \neq rI_1K \subseteq IE$ for $r \in R$ , $I_1$ be an ideal of R , K is a submodule of E. It follows that $0 \neq rI_1I_2E \subseteq IE$ f or some ideal $I_2$ of R. Since E is a finitely generated multiplication , then by[2 coro. of Theo. ( 9 )] we have $0 \neq r I_1I_2 \subseteq I + ann(E) = I$ . But I is a WP – 2 – Absorbing , then,by corollary (8) either $rI_2 \subseteq I + soc(R)$ or $I_1I_2 \subseteq I + soc(R)$ or $rI_1 \subseteq [I + soc(R) :_R R] = I + soc(R)$. That is either $rI_2E \subseteq IE + soc(R)E$ or $I_1I_2E \subseteq IE + soc(R)E$ or $rI_1E \subseteq IE + soc(R)E$ . Thus either $rK \subseteq IE + soc(E)$

or $I_1K \subseteq IE + soc(E)$ or $r I_1 \subseteq [IE + soc(E) :_R E]$. Therefore ,by corollary (8) IE is a WP – 2 – Absorbing submodule of E .

It is well known that cyclic R – modules are finitely generated [8], we get the following corollary which is a direct consequence of proposition (27)

**Corollary 28.** Let E be a faithful cyclic R – module , and I be a WP – 2 – Absorbing ideal of R. Then , IE is a WP – 2 – Absorbing submodule of E .

**3 . Conclusion . …** A new generalization of weakly – 2 – Absorbing submodule was introduced , and many characterizations were given. The definition of WP – triple zero of WP – 2 – Absorbing submodules were introduced. A lot of basic properties of these concepts were established. Among the main new characterizations of WP – 2 – Absorbing submodules are the following :

• A proper submodule A of E is a WP- 2 – Absorbing if and only if for any $t , s \in R$ with $ts \notin [A + soc(E) :_R E]$ ; we have $[A :_E ts] \subseteq [0 :_E ts] \cup [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$.

• A proper submodule A of E is a WP- 2 – Absorbing if and only if $0 \neq tsK \subseteq A$ for $t , s \in R$ and K is a submodule of E , implies that either $tK \subseteq A + soc(E)$ or $sK \subseteq A + soc(E)$ or $ts \in [A + soc(E) :_R E]$.

• A proper submodule A of a cyclic R – module E is a WP- 2 – Absorbing if and only if for each $t , s \in R$ with $ts \notin [A + soc(E) :_R E]$ , we have $[A :_R tse] \subseteq [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$.

• A proper submodule N of E is a WP- 2 – Absorbing if and only if $(0) \neq IJL \subseteq N$ for some ideal I , J of R and submodule L of E implies that either $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$ or $IJ \subseteq [N + soc(E) :_R E]$ .

• If A is a WP- 2 – Absorbing submodule of E with ( r , s , e ) is a WP – triple zero of A for some $t , s \in R$, $e \in E$. Then , $rsA = (0)$ , $[A :_R E] re = (0)$, $r [A :_R E]e = (0)$, $r [A :_R E] A = (0)$, $[A :_R E]sA = (0)$ and $[A :_R E] [A :_R E] A = (0)$ .

• A proper submodule A of multiplication module E is a WP- 2 – Absorbing if and only if $(0) \neq L_1L_2L_3 \subseteq A$ for some submodules $L_1 , L_2 , L_3$ of E implies that either $L_1L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$ .

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