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Constraints Optimal Classical Continuous Control Vector Problem for Quaternary Nonlinear Hyperbolic System

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Abstract

This paper is concerned with the quaternary nonlinear hyperbolic boundary value problem (QNLHBVP) studding constraints quaternary optimal classical continuous control vector (CQOCCCV), the cost function (CF), and the equality and inequality quaternary state and control constraints vector (EIQSCCV). The existence of a CQOCCCV dominating by the QNLHBVP is stated and demonstrated using the Aubin compactness theorem (ACTH) under appropriate hypotheses (HYPs). Furthermore, mathematical formulation of the quaternary adjoint equations (QAEs) related to the quaternary state equations (QSE) are discovere so as its weak form (WF) . The directional derivative (DD) of the Hamiltonian (Ham) is calculated. The necessary and sufficient conditions for optimality (NCSO) theorems for the proposed problem are stated and proved.

Keywords: Necessary and Sufficient Conditions fro optimality, Nonlinear Hyperbolic System, Quaternary Optimal Classical Continuous Control vector.

1. Introduction

Optimal control problems (OCPs) are important in a wide range of practical applications, including robotics robotics[1], economics[2], weather conditions[3], community health[4], and a variety of other scientific fields. Nonlinear ODEs [5]or nonlinear PDEs (NLPDEs) [6] usually dominate OCPs. This significance pushed many researchers to be concerned about OCPs in general and optimal classical continuous control problems (OCCCPs) in particular. During the last decade much emphasis has been place on studying the OCPs for system dominating by nonlinear PDEs (NLPDEs) of the three types in general; hyperbolic, elliptic and parabolic [7-9]. Later the study of this subject, in particular for hyperbolic type of PDEs was generalized to deal with CCOCPs dominated by coupled NLPDEs of it [10], and then to CCOCPs dominating by triple NLPDEs of it [11]. the problem in each type of these OCCCPs was typically comprised of an initial and boundary value problem, the CF and the constraints on the state and the control vectors (CSCV). The study of each one of these problems had been included of; the existence theorem of constraints OCCC vector satisfying the SCCV had been stated and demonstrated under appropriate HYPs, the mathematical formulation for the QAEs related to the given QSEs had been obtained, and the DD for the Ham had been derived. The theorems of necessity and sufficient conditions for optimality had been stated and demonstrated.

All of these concerns motivated us to consider extending the study of the CCOCP dominating by triple NLPDEs of hyperbolic type to a CCOCP dominating by QNLHBVP. As a result of this expansion, there was a need to generalize the mathematical model and then to generalize all the proofs related to this generalization, and accordingly. The authors created new Theorems, Lemma and then proved them in this paper. The existence theorem (ETH) of a CQOCCCV dominating by the QNLHBVPs with EIQSCCV was stated and demonstrated in this work using the ACTH under appropriate HYPs. Moreover mathematical formulation of the QAEs related to QSEs was discovered as was the WF of the QAEs. The derivative of DD was obtained. Lastly, both the theorems for the NCSO of the proposed problems were stated and demonstrated.

2. Problem Description:

Let $I=[0,T]$, $T<\infty, \Omega\subset\mathbb{R}^2$, be an open bounded regular region with boundary $\Gamma=\partial\Omega$, $Q=\Omega\times I$, $\Sigma=\Gamma\times I$. The CQOCCCV including of the QSEs are given by the following QNLHBVP:

$y_{1tt}-\Delta y_1+y_1-y_2+y_3+y_4=f_1(x,t,y_1,u_1)$, in Q, (1)

$y_{2tt}-\Delta y_2+y_1+y_2-y_3-y_4=f_2(x,t,y_2,u_2)$, in Q, (2)

$y_{3tt}-\Delta y_3-y_1+y_2+y_3+y_4=f_3(x,t,y_3,u_3)$, in Q, (3)

$y_{4tt}-\Delta y_4-y_1+y_2-y_3+y_4=f_4(x,t,y_4,u_4)$, in Q, (4)

with the following boundary conditions (BCs) and the initial conditions (ICs)

$y_i(x,t)=0$, on Σ, for $i=1,2,3,4$. (5)

$y_1(x,0)=y_i^0(x)$,and $y_{it}(x,0)=y_i^1(x)$, in Ω for $i=1,2,3,4$. (6)

where $\vec{y}=(y_1,y_2,y_3,y_4)\in \boldsymbol{H^1(\Omega)}=(H^1(\Omega))^4$is the quaternary solution vectors (QSVs), corresponding to the quaternary classical continuous control vector (QCCCV) $\vec{u}=(u_1,u_2,u_3,u_4)\in \boldsymbol{L^2(Q)}=(L^2(Q))^4$ and $(f_1,f_2,f_3,f_4)\in \boldsymbol{L^2(Q)}$ is a vector of a given function on $(Q\times\mathbb{R}\times U_1)\times(Q\times\mathbb{R}\times U_2)\times(Q\times\mathbb{R}\times U_3)\times(Q\times\mathbb{R}\times U_4)$, with $U_i\subset\mathbb{R}$, $\forall i=1,2,3,4$.

The QSCCs are $\vec{u}\in\vec{W}$, $\vec{W}\subset \boldsymbol{L^2(Q)}$ where $\vec{W}=\{\vec{w}\in\vec{U}\subset\mathbb{R}^4, a.e\ in\ Q\}$, with is a convex (CO).

The CF is given and The EINEQSCC on the QSCCs are resp.

$$G_0(\vec{u})=\sum_{i=1}^{4}\int_Q g_{0i}\,(x,t,y_i,u_i)dxdt\,, \quad (7)$$

$$G_1(\vec{u})=\sum_{i=1}^{4}\int_Q g_{1i}\,(x,t,y_i,u_i)dxdt=0, \quad (8)$$

$$G_2(\vec{u})=\sum_{i=1}^{4}\int_Q g_{2i}\,(x,t,y_i,u_i)dxdt\le 0, \quad (9)$$

The set of admissible quaternary control (AQC) is:

$\vec{W}_A=\{\vec{u}\in\vec{W}\mid G_1(\vec{u})=0, G_2(\vec{u})\le 0\}$.

The CQOCCCV is to find $\vec{u}\in\vec{W}_A$, s.t. $G_0(\vec{u})=\min_{\vec{w}\in\vec{W}_A} G_0(\vec{w})$.

Let$\vec{V}=\{\vec{v}=(v_1,v_2,v_3,v_4)\in \boldsymbol{H^1(\Omega)}, v_1=v_2=v_3=v_4=0\ on\ \partial\Omega\}$,$\vec{V}=\boldsymbol{H_0^1(\Omega)}=(H_0^1(\Omega))^4$, $\boldsymbol{L^2(I,V)}=(L^2(I,V))^4$ and $V=H_0^1(\Omega)$, the inner product (IP) and the norm(Nr) in $\boldsymbol{L^2(Q)}$ are denoted by $(\vec{v},\vec{v})$ and $\|\vec{v}\|_{\boldsymbol{L^2(Q)}}=\sum_{i=1}^{4}\|v_1\|^2_{L^2(Q)}$ resp., the Nr in $\boldsymbol{L^2(I,V)}$ by $\|\vec{v}\|_{\boldsymbol{L^2(I,V)}}=\sum_{i=1}^{4}\|v_1\|^2_{L^2(I,V)}$, and $\boldsymbol{L^2(I,V^*)}$ is the dual of $\boldsymbol{L^2(I,V)}$.

The WF of ((1)-(6)) with $\vec{y}\in \boldsymbol{H_0^1(\Omega)}$ is given (a.e. on I and $\forall v_i, y_i(0,t)\in V$,$\forall i=1,2,3,4$) by :

$(y_{1tt},v_1)+(\nabla y_1,\nabla v_1)+(y_1,v_1)-(y_2,v_1)+(y_3,v_1)+(y_4,v_1)=(f_1,v_1)$, (10)

$(y_1^0, v_1) = (y_1(0), v_1)$, and $(y_{1t}^1, v_1) = (y_{1t}(0), v_1)$, (11)

$(y_{2tt}, v_2) + (\Delta y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2, v_2)$, (12)

$(y_2^0, v_2) = (y_2(0), v_2)$, and $(y_{2t}^1, v_2) = (y_{2t}(0), v_2)$, (13)

$(y_{3tt}, v_3) + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3, v_3)$, (14)

$(y_3^0, v_3) = (y_3(0), v_3)$, and $(y_{3t}^1, v_3) = (y_{3t}(0), v_3)$, (15)

$(y_{4tt}, v_4) + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4, v_4)$, (16)

$(y_4^0, v_4) = (y_4(0), v_4)$, and $(y_{4t}^1, v_4) = (y_{4t}(0), v_4)$, (17)

Assums (A): Suppose that f_i is of Carathéodory type (CaraT) on $Q \times (\mathbb{R} \times U_i)$ satisfies (w.r.t. $y_i \& u_i$) the following

(i) $|f_i(x,t,y_i,u_i)| \leq F_i(x,t) + |u_i| + \beta_i |y_i|$, where $y_i, u_i \in \mathbb{R}$, $\beta_i > 0$ and $F_i \in L^2(\mathrm{Q})$.

(ii) f_i is satisfied Lipschitz condition (LPC) w.r.t. y_i, i.e.

$$|f_i(x,t,y_i,u_i) - f_i(x,t,\bar{y}_i,u_i)| \leq L_i |y_i - \bar{y}_i| \,,\ y_i, \bar{y}_i, u_i \in \mathbb{R},\ L_i > 0, \text{ for } (x,t) \in Q.$$

Proposition 2.1[12]: Let $D \subset \mathbb{R}^2$ be measurable, $f: D \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of CaraT satisfies:

$\|f(v,x)\| \leq \zeta(v) + \eta(v)\|x\|^{\alpha}$,

where $x \in L^p(D \times \mathbb{R}^n)$, $\zeta \in L^1(D \times \mathbb{R})$, $\eta \in L^{\frac{p}{p-\alpha}}(D \times \mathbb{R})$, $\alpha \in [0, \infty)$.

Then the functional (funl) $F(x) = \int_D f(v, x(v))dv$ is continuous (cont.).

Theorem2.1 (ETH of a Unique QSVs)[13]: If Assums (A) hold, then for each given $\vec{u} \in \boldsymbol{L^2}(\mathbf{Q})$, the WF (10- 17) has a unique QSVs $\vec{y} = (y_1, y_2, y_3, y_4) \in \boldsymbol{L^2}(\boldsymbol{I}, \boldsymbol{V})$ with $\vec{y}_t = (y_{1t}, y_{2t}, y_{3t}, y_{4t}) \in \boldsymbol{L^2}(\mathbf{Q})$, $\vec{y}_{tt} = (y_{1tt}, y_{2tt}, y_{3tt}, y_{4tt}) \in \boldsymbol{L^2}(\boldsymbol{I}, \boldsymbol{V}^*)$.

Assums (B): Consider g_{li} (for $i = 1,2,3,4$ & $l = 0,1,2$) is of the CaraT on $Q \times (\mathbb{R} \times U_i)$ and satisfies: $|g_{li}(x,t,y_i,u_i)| \leq G_{li}(x,t) + C_{li}(y_i)^2 + C_{li}(u_i)^2$, where $G_{li} \in L^1(Q)$, $y_i \in \mathbb{R}$ & $u_i \in U_i$.

Lemma 2.1: With Assums (B), the funl $\vec{u} \to G_l(\vec{u}), \forall l = 0,1,2$ is cont. on $\boldsymbol{L^2}(\mathbf{Q})$.

Proof: The proof is obtained from the Assums (B) and Proposition 1.

Lemma 2.2[12]: Let $g: Q \times \mathbb{R} \to \mathbb{R}$ is of CaraT on $Q \times (\mathbb{R} \times \mathbb{R})$ and satisfies $|g\,(x,t,y,u)| \leq G\,(x,t) + c\,y^2 + \acute{c}u^2$, where $G(x,t) \in L^1(\mathrm{Q})$, $u \in U, \acute{c}, c \geq 0, U \subset \mathbb{R}$, is compact(COM). Then $\int_Q g\,(x,y,u\,)dx$ is cont. on $L^2(Q)$ w.r.t. y.

Theorem 2.2 (LP Cont. Theorem)[13]: In addition to Assums (A), if $\vec{y}$ and $\vec{y} + \delta\vec{y}$ are the QSVs corresponding to the bounded QCCCVs $\vec{u}$ and $\vec{u} + \delta\vec{u}$ resp. in $L^2(Q)$, then for $\delta \in \mathbb{R}^+$

$\| \delta\vec{y}_{\varepsilon} \|_{L^\infty(I,\boldsymbol{L^2}(\boldsymbol{\Omega}))} \leq \delta \| \delta\vec{u} \|_{\boldsymbol{L^2}(\boldsymbol{Q})}$, $\| \delta\vec{y}_{\in} \|_{\boldsymbol{L^2}(\boldsymbol{I},\boldsymbol{V})} \leq \delta \| \delta\vec{u} \|_{\boldsymbol{L^2}(\boldsymbol{Q})}$ and $\| \delta\vec{y}_{\varepsilon} \|_{L^2 Q)} \leq \delta \| \delta\vec{u} \|_{\boldsymbol{L^2}(\boldsymbol{Q})}$.

Assums (C): Assume that for each ($l = 0,1,2$ & $i = 1,2,3,4$), the functions $f_i, f_{iy_i}, f_{iu_i}, g_{liy_i}, g_{liu_i}$ are of CaraT on $Q \times (\mathbb{R} \times U')$, where ($U'$ is an open set containing U), s.t.(for $(x,t) \in Q$) :

$|f_{iy_i}(x,t,y_i,u_i)| \leq L_i$, $|f_{iy_i}(x,t,y_i,u_i)| \leq L'_i$, $|g_{iy_i}(x,t,y_i,u_i)| \leq G_{li5}(x,t) + G_{li5}(x,t)\ |\ y_i|$,

$|g_{iu_i}(x,t,y_i,u_i)| \leq G_{li6}(x,t) + G_{li6}(x,t)\ |\ y_i|$, where $y_i, u_i \in \mathbb{R}$, $G_{li5}, G_{li6} \in L^2(Q)$, $G_{li5}, G_{li6} \geq 0$.

Main Results

3.Existence of the CQOCCCV

Theorem 3.1: In addition to Assums ((A) & (B)), if the set $\vec{U}$ is CO and com., $\overrightarrow{W_A} \neq \phi$, the function f_i ($\forall i = 1,2,3,4$) has the form: $f_i(x,t,y_i,u_i) = f_{i1}(x,t,y_i) + f_{i2}(x,t)u_i$,

with $|f_{i1}(x,t,y_i)| \leq \eta_i(x,t) + c_i\ |\ y_i\ |$, $|f_{i2}(x,t)| \leq K_i$, $\eta_i \in L^2(\mathrm{Q})$, $c_i \geq 0$.

g_{1i} is independent of u_i, g_{0i} and $g_{2i}$are CO w.r.t. u_i for fixed (x,t,y_i), $\forall\ i=1,2,3,4$. Then there is a CQOCCCV.

Proof: From the Assum on $\vec{U}\subset\mathbb{R}$, $\vec{W}$ is weakly compact (WCOM), since $\vec{W}_A\neq\phi$, then there is a minimum sequence(Seq.) $\{\vec{u}_k\}=\{(u_{1k},u_{2k},u_{3k},u_{4k})\}\in\vec{W}_A$, $\forall k$ s.t.

$$\lim_{k\to\infty} G_0(\vec{u}_k)=\inf_{\vec{u}_k\in\vec{W}_A} G_0(\vec{\vec{u}}).$$

Since $\vec{u}_k\in\vec{W}_A$, $\forall k$ and $\vec{W}$ is WCOM, there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ s.t. . $\vec{u}_k\to\vec{u}$ weakly (WK) in $\boldsymbol{L^2(Q)}$ and $\|\vec{u}_k\|_{L^2(\mathbf{Q})}\leq d$,$\forall k$. From Theorem 1, corresponding to the Seq. QCV $\{\vec{u}_k\}$ the WF of the QSEs has “a unique” solution $\{\vec{y}_k=\vec{y}_{u_k}\}$ and $\|\vec{y}_k\|_{L^2(I,V)}$, $\|\vec{y}_{kt}\|_{L^2(Q)}$ are bounded, then by Alaoglu’s theorem (ATH), there exists a Subsequence of $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$, say again $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$, s.t. $\vec{y}_k\to\vec{y}$ WK in $\boldsymbol{L^2(I,V)}$, $\vec{y}_{kt}\to\vec{y}_t$ WK in $(L^2(Q))^4$. Now for each k. and by applying the ACTH[14] , there is a Subsequence of$\{\vec{y}_k\}$ say a gain $\{\vec{y}_k\}$ s.t. $\vec{y}_k\to\vec{y}$ strongly (ST) in $\boldsymbol{L^2(Q)}$.

Now, for each k, substituting the QSVs $\vec{y}_k$ in the WF ((10), (12), (14), (16)), multiplying both sides (MBSs) of each one by $\phi_i(t)$, $\forall i=1,2,3,4$ (with $\phi_i\in C^2[0,T]$, s.t. $\phi_i(T)=\phi_i'(T)=0,\phi_i(0)\neq 0,\ \phi_i'(0)\neq 0$), rewriting the 1st terms in the LHS of each one, then integrating both sides (IBS) on $[0,T]$, and then integrating by parts (IBPs) for the 1st terms, yield to

$$\int_0^T \frac{d}{dt}(y_{1kt},v_1)\phi_1 dt+\int_0^T [(\nabla y_{1k},\nabla v_1)+(y_{1k},v_1)-(y_{2k},v_1)+(y_{3k},v_1)+(y_{4k},v_1)]\phi_1 dt$$
$$=\int_0^T (f_{11}(x,t,y_{1k}),v_1)\phi_1(t)dt+\int_0^T (f_{12}(x,t)u_{1k},v_1)\phi_1(t)dt\ , \tag{18}$$

$$\int_0^T \frac{d}{dt}(y_{2kt},v_2)\phi_2 dt+\int_0^T [(\nabla y_{2k},\nabla v_2)+(y_{1k},v_2)+(y_{2k},v_2)-(y_{3k},v_2)-(y_{4k},v_2)]\phi_2 dt$$
$$=\int_0^T (f_{21}(x,t,y_{2k}),v_2)\phi_2(t)dt+\int_0^T (f_{22}(x,t)u_{2k},v_2)\phi_2(t)dt, \tag{19}$$

$$\int_0^T \frac{d}{dt}(y_{3k},v_3)\phi_3 dt+\int_0^T [(\nabla y_{3k},\nabla v_3)-(y_{1k},v_3)+(y_{2k},v_3)+(y_{3k},v_3)+(y_{4k},v_3)]\phi_3 dt$$
$$=\int_0^T (f_{31}(x,t,y_{3k}),v_3)\phi_3(t)dt+\int_0^T (f_{32}(x,t)u_{3k},v_3)\phi_3(t)dt, \tag{20}$$

$$\int_0^T \frac{d}{dt}(y_{4k},v_4)\phi_4 dt+\int_0^T [(\nabla y_{4k},\nabla v_4)-(y_{1k},v_4)+(y_{2k},v_4)-(y_{3k},v_4)+(y_{4k},v_4)]\phi_4 dt$$
$$=\int_0^T (f_{41}(x,t,y_{4k}),v_4)\phi_4(t)dt+\int_0^T (f_{42}(x,t)u_{4k},v_4)\phi_4(t)dt, \tag{21}$$

At this point, the same steps which were utilized in the proof of Theorem 2.1, can be utilized here to passage the limit in the WF of ((18) – (21)), to acquire

$$(y_{1t},v_1)+(\nabla y_1,\nabla v_1)+(y_1,v_1)-(y_2,v_1)+(y_3,v_1)+(y_4,v_1)$$
$$=(f_{11}(x,t,y_1)+f_{12}(x,t)u_1,v_1),\forall v_1\in V\ \text{ a.e. on I}, \tag{22}$$

$$(y_{2t},v_2)+(\Delta y_2,\nabla v_2)+(y_1,v_2)+(y_2,v_2)-(y_3,v_2)-(y_4,v_2)$$
$$=(f_{21}(x,t,y_2)+f_{22}(x,t)u_2,v_2),\forall v_2\in V\ \text{a.e. on I}, \tag{23}$$

$$(y_{3t},v_3)+(\nabla y_3,\nabla v_3)-(y_1,v_3)+(y_2,v_3)+(y_3,v_3)+(y_4,v_3)$$
$$=(f_{31}(x,t,y_3)+f_{32}(x,t)u_3,v_3),\forall v_3\in V\quad \text{a.e. on I}, \tag{24}$$

$$(y_{4t},v_4)+(\nabla y_4,\nabla v_4)-(y_1,v_4)+(y_2,v_4)-(y_3,v_4)+(y_4,v_4)$$
$$=(f_{41}(x,t,y_4)+f_{42}(x,t)u_4,v_4),\ \forall v_4\in V\ \ \text{a.e. on I}, \tag{25}$$

Same manner also can be utilized to that the ICs are held. Thus $\vec{y}$ is QSVs

From the other side, since

$G_1(\vec{u}) = \sum_{i=1}^{4}\int_Q g_{1i}\,(x,t,y_{ik})dxdt$, with g_{1i} ($\forall i = 1,2,3,4$) is cont. w.r.t. y_i, then by Lemma 2.1, $\int_Q g_{1i}\,(x,t,y_{ik})dxdt$ is cont. w.r.t. y_i but $\vec{y}_k \to \vec{y}$ ST in $\boldsymbol{L^2(Q)}$, therefore $\int_Q g_{1i}(x,t,y_{ik})\,dxdt \to \int_Q g_{1i}\,(x,t,y_i)dxdt$. Thus $G_1(\vec{u}) = \lim_{k\to\infty} G_1(\vec{u}_k) = 0$.

As well, since for $l = 0,2$ &$i = 1,2,3,4$, $g_{li}(x,t,y_i,u_i)$ is cont. w.r.t. (y_i,u_i) and U_i is COM with $u_i \in U_i$ a.e. in Q, then using Lemma 2.2 to get

$$\int_Q g_{li}(x,t,y_{ik},u_{ik})\,dxdt \to \int_Q g_{li}\,(x,t,y_i,u_{ik})dxdt\,, \tag{26}$$

But $g_{li}(x,t,y_i,u_i)$ is CO and cont. w.r.t. u_i , then

$\int_Q g_{li}\,(x,t,y_i,u_i)dxdt$ is weakly lowe semi cont. (WLSC) w.r.t. u_i, $\forall\ l = 0,2$ &$i = 1,2,3,4$, i.e.

$$\int_Q g_{li}\,(x,t,y_i,u_i)dxdt \le \liminf_{k\to\infty}\int_Q [g_{li}\,(x,t,y_i,u_{ik}) - g_{li}\,(x,t,y_{ik},u_{ik})]dxdt$$

$$+\liminf_{k\to\infty}\int_Q g_{li}\,(x,t,y_{ik},u_{ik})dxdt \le \liminf_{k\to\infty}\int_Q g_{li}\,(x,t,y_{ik},u_{ik})dxd$$

$$\Longrightarrow \sum_{i=1}^{4}\int_Q g_{li}\,(x,t,y_i,u_i)dxdt \le \sum_{i=1}^{4}\int_Q g_{li}\,(x,t,y_{ik},u_{ik})dxdt.$$

Thus $G_l(\vec{u}) \le \lim_{k\to\infty} \inf_{\vec{u}_k\in\vec{W}_A} G_l(\vec{u}_k)$, then $G_2(\vec{u}) \le 0$, since $\vec{u}_k \in \vec{W}_A, \forall k$, and

$G_0(\vec{u}) \le \lim_{k\to\infty} \inf_{\vec{u}_k\in\vec{W}_A} G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}}_k\in\vec{W}_A} G_0(\vec{\vec{u}}) \Longrightarrow G_0(\vec{u}) = \min_{\vec{\vec{u}}_k\in\vec{W}_A} G_0(\vec{\vec{u}})$, then $\vec{u}$ is a QOCCCV.

Theorem 3.2: Neglecting the index l from G_l and g_{li}. The QAEs $\vec{Z} = (Z_1, Z_2, Z_3, Z_4)$ of the QSEs in ((1)-(6)) can be formulated as

$$Z_{1tt} - \Delta Z_1 + Z_1 + Z_2 - Z_3 - Z_4 = Z_1 f_{1y_1}(x,t,y_1,u_1) + g_{1y_1}(x,t,y_1,u_1),\ \text{in } Q, \tag{27}$$
$$Z_1 = 0\ \text{ on } \Sigma,\ \ Z_1(x,T) = Z_{1t}(x,T) = 0 \text{ on } \Omega\,, \tag{28}$$
$$Z_{2tt} - \Delta Z_2 - Z_1 + Z_2 + Z_3 + Z_4 = Z_2 f_{2y_2}(x,t,y_2,u_2) + g_{2y_2}(x,t,y_2,u_2),\ \text{in } Q\,, \tag{29}$$
$$Z_2 = 0 \text{ on } \Sigma\,,\ Z_2(x,T) = Z_{2t}(x,T) = 0 \text{ on } \Omega, \tag{30}$$
$$Z_{3tt} - \Delta Z_3 + Z_1 - Z_2 + Z_3 - Z_4 = Z_3 f_{3y_3}(x,t,y_3,u_3) + g_{3y_3}(x,t,y_3,u_3)\,,\ \text{in } Q, \tag{31}$$
$$Z_3 = 0 \text{ on } \Sigma,\ Z_3(x,T) = Z_{3t}(x,T) = 0 \text{ on } \Omega\,, \tag{32}$$
$$Z_{4tt} - \Delta Z_4 + Z_1 - Z_2 + Z_3 + Z_4 = Z_4 f_{4y_4}(x,t,y_4,u_4) + g_{4y_4}(x,t,y_4,u_4)\,,\ \text{in } Q, \tag{33}$$
$$Z_4 = 0 \text{ on } \Sigma,\ Z_4(x,T) = Z_{4t}(x,T) = 0 \text{ on } \Omega, \tag{34}$$

And the Ham is defined as: $H(x,t,\vec{y},\vec{u},\vec{Z}) = \sum_{i=1}^{4}(Z_i f_i(x,t,y_i,u_i) + g_i(x,t,y_i,u_i))$,

Where $G\,(\vec{u}) = \sum_{i=1}^{4}\int_Q g_{\,i}\,(x,t,y_i,u_i)dxdt.$

Then the DD of G is

$$DG(\vec{u},\vec{\vec{u}} - \vec{u}) = \lim_{\varepsilon\to 0}\frac{G(\vec{u}+\varepsilon\delta\vec{u}) - G(\vec{u})}{\varepsilon} = \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{u},\vec{Z})(\vec{\vec{u}} - \vec{u})dxdt\,.$$

Proof: The WF of the QAEs $\forall v_i \in V$ and $i = 1,2,3,4$ is

$$(Z_{1tt},v_1) + (\nabla Z_1,\nabla v_1) + (Z_1,v_1) + (Z_2,v_1) - (Z_3,v_1) - (Z_4,v_1)$$
$$= (Z_1 f_{1y_1},v_1) + (g_{1y_1},v_1)\,,\ \forall v_1 \in V\ \text{ a.e. on } I, \tag{35}$$
$$(Z_1(T),v_1) = (Z_{1t}(T),v_1) = 0, \tag{36}$$
$$(Z_{2tt},v_2) + (\nabla Z_2,\nabla v_2) - (Z_1,v_2) + (Z_2,v_2) + (Z_3,v_2) + (Z_4,v_2)$$
$$= (Z_2 f_{2y_2},v_2) + (g_{2y_2},v_2)\,,\ \forall v_2 \in V\ \text{ a.e. on }\ , \tag{37}$$

$(Z_2(T), v_2) = (Z_{2t}(T), v_2) = 0,$ (38)

$(Z_{3tt}, v_3) + (\nabla Z_3, \nabla v_3) + (Z_1, v_3) - (Z_2, v_3) + (Z_3, v_3) - (Z_4, v_3)$
$= (Z_3 f_{3y_3}, v_3) + (g_{3y_3}, v_3)$, $\forall v_3 \in V$ a.e. on I, (39)

$(Z_3(T), v_3) = (Z_{3t}(T), v_3) = 0$, (40)

$(Z_{4tt}, v_4) + (\nabla Z_4, \nabla v_4) + (Z_1, v_4) - (Z_2, v_4) + (Z_3, v_4) + (Z_4, v_4)$
$= (Z_4 f_{4y_4}, v_4) + (g_{4y_4}, v_4)$, $\forall v_4 \in V$ a.e. on, (41)

$(Z_4(x,T), v_4) = (Z_{4t}(T), v_4) = 0,$ (42)

The WF ((35-(42)) has a unique solution $\vec{Z} = (Z_1, Z_2, Z_3, Z_4) \in (L^2(Q))^4$ (this it can proved so as the proof of existence a unique QSVs for the WF ((11)-(15)).

Now, replacing $v_i = \delta y_{i\varepsilon}$ in (35), (37), (39) and (41), for $i = 1,2,3,4$ resp.

$$\int_0^T (\delta y_{1\varepsilon}, Z_{1tt})dt + \int_0^T [(\nabla Z_1, \nabla \delta y_{1\varepsilon}) + (Z_1, \delta y_{1\varepsilon}) + (Z_2, \delta y_{1\varepsilon}) - (Z_3, \delta y_{1\varepsilon}) - (Z_4, \delta y_{1\varepsilon})]dt$$
$$= \int_0^T (Z_1 f_{1y_1}, \delta y_{1\varepsilon}) + (g_{1y_1}, \delta y_{1\varepsilon})dt, \quad (43)$$

$$\int_0^T (\delta y_{2\varepsilon}, Z_{2tt})dt + \int_0^T [(\nabla Z_2, \nabla \delta y_{2\varepsilon}) - (Z_1, \delta y_{2\varepsilon}) + (Z_2, \delta y_{2\varepsilon}) + (Z_3, \delta y_{3\varepsilon}) + (Z_4, \delta y_{2\varepsilon})]dt$$
$$= \int_0^T (Z_2 f_{2y_2}, \delta y_{2\varepsilon}) + (g_{2y_2}, \delta y_{2\varepsilon})dt, \quad (44)$$

$$\int_0^T (\delta y_{3\varepsilon}, Z_{3tt})dt + \int_0^T [(\nabla Z_3, \nabla \delta y_{3\varepsilon}) + (Z_1, \delta y_{3\varepsilon}) - (Z_2, \delta y_{3\varepsilon}) + (Z_3, \delta y_{3\varepsilon}) - (Z_4, \delta y_{3\varepsilon})]dt$$
$$= \int_0^T (Z_3 f_{3y_3}, \delta y_{3\varepsilon}) + (g_{3y_3}, \delta y_{3\varepsilon})dt, \quad (45)$$

$$\int_0^T (\delta y_{4\varepsilon}, Z_{4tt})dt + \int_0^T [(\nabla Z_4, \nabla \delta y_{4\varepsilon}) + (Z_1, \delta y_{4\varepsilon}) - (Z_2, \delta y_{4\varepsilon}) + (Z_3, \delta y_{4\varepsilon}) + (Z_4, \delta y_{4\varepsilon})]dt$$
$$= \int_0^T (Z_4 f_{4y_4}, \delta y_{4\varepsilon}) + (g_{4y_4}, \delta y_{4\varepsilon})dt, \quad (46)$$

Now, take $\vec{\vec{u}}, \vec{u} \in \boldsymbol{L^2}(\mathbf{Q})$,set $\overrightarrow{\delta u} = \vec{\vec{u}} - \vec{u}$, $\vec{u}_\varepsilon = \vec{u} + \varepsilon\overrightarrow{\delta u} \in \boldsymbol{L^2}(\mathbf{Q})$ for $\varepsilon > 0$, then by Theorem 1, $\vec{y} = \vec{y}_{\vec{u}}$ & $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon}$ are their corresponding QSVs. Setting $\overrightarrow{\delta y_\varepsilon} = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}, \delta y_{3\varepsilon}, \delta y_{4\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, substituting $v_i = Z_i$ for $i = 1,2,3,4$ in ((10)- (17)), IBSs on $[0, T]$, then integrating by parts twice (IBPs2) the 1st in the LHS of each obtained equation, finding the FrD of f_i ($\forall i = 1,2,3,4$) in the RHS of each one equation (which is exists from the Assums C), then from the result of Theorem 2.2 and the Minkowiski inequality (MIN), once get

$$\int_0^T (\delta y_{1\varepsilon}, Z_{1tt})dt + \int_0^T [(\nabla \delta y_{1\varepsilon}, \nabla Z_1) + (\delta y_{1\varepsilon}, Z_1) + (\delta y_{2\varepsilon}, Z_1) + (\delta y_{3\varepsilon}, Z_1) + (\delta y_{4\varepsilon}, Z_1)]dt$$
$$= \int_0^T \left(f_{1y_1}\delta y_{1\varepsilon} + f_{1u_1}\varepsilon\delta u_1, Z_1\right)dt + O_{11}(\varepsilon), \quad (47)$$

$$\int_0^T (\delta y_{2\varepsilon}, Z_{2tt})dt + \int_0^T [(\nabla \delta y_{2\varepsilon}, \nabla Z_2) + (\delta y_{1\varepsilon}, Z_2) + (\delta y_{2\varepsilon}, Z_2) + (\delta y_{3\varepsilon}, Z_2) + (\delta y_{4\varepsilon}, Z_2)]dt$$
$$= \int_0^T \left(f_{2y_2}\delta y_{2\varepsilon} + f_{2u_2}\varepsilon\delta u_2, Z_2\right)dt + O_{12}(\varepsilon), \quad (48)$$

$$\int_0^T (\delta y_{3\varepsilon}, Z_{3tt})dt + \int_0^T [(\nabla \delta y_{3\varepsilon}, \nabla Z_3) + (\delta y_{1\varepsilon}, Z_3) - (\delta y_{2\varepsilon}, Z_3) + (\delta y_{3\varepsilon}, Z_3) + (\delta y_{4\varepsilon}, Z_3)]dt$$
$$= \int_0^T \left(f_{3y_3}\delta y_{3\varepsilon} + f_{3u_3}\varepsilon\delta u_3, Z_3\right)dt + O_{13}(\varepsilon), \quad (49)$$

$$\int_0^T (\delta y_{4\varepsilon}, Z_{4tt})dt + \int_0^T [(\nabla \delta y_{4\varepsilon}, \nabla Z_4) - (\delta y_{1\varepsilon}, Z_4) - (\delta y_{2\varepsilon}, Z_4) + (\delta y_{3\varepsilon}, Z_4) + (\delta y_{4\varepsilon}, Z_4)]dt$$

$$= \int_0^T \left(f_{4y_4}\delta y_{4\varepsilon} + f_{4u_4}\varepsilon\delta u_4, Z_4\right)dt + O_{14}(\varepsilon), \quad (50)$$

where $O_{1i}(\varepsilon) = \| \delta y_{i\varepsilon} \|_Q^2 + \varepsilon \| \delta u_i \|_Q^2 \to 0$, as $\varepsilon \to 0$, $\forall i = 1,2,3,4$.

Subtracting ((47) – (50)) from ((43)- (46)) resp., collecting the obtain equations, to acquire

$$\varepsilon \int_0^T \sum_{i=1}^4 \left(f_{iu_i}\delta u_i, Z_i\right)dt + O_1(\varepsilon) = \varepsilon \int_0^T \sum_{i=1}^4 \left(g_{iy_i}, \delta y_i\right)dt \ , \forall i = 1,2,3,4 \ , \quad (51)$$

Where $O_1(\varepsilon) = \sum_{i=1}^4 O_{1i}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

From the other side, by employing the Assums (C), the definition of the FrD the result of Theorem 2.2, and using the MIN, one has

$$G(\vec{u}_\varepsilon) - G(\vec{u}) = \sum_{i=1}^4 \int_Q \left(g_{iy_i}\, \delta y_{i\varepsilon} + g_{iu_i}\, \varepsilon\delta u_i\right)dxdt + O_2(\varepsilon), \quad (52)$$

Where $O_2(\varepsilon) = \| \overrightarrow{\delta y_\varepsilon} \|_{L^2(\mathbf{Q})}^2 + \varepsilon \| \overrightarrow{\delta u} \|_{L^2(\mathbf{Q})}^2 \to 0$ as $\varepsilon \to 0$, $\forall i = 1,2,3,4$.

Now, by using (51) in (52), to obtain

$$G(\vec{u}_\varepsilon) - G(\vec{u}) = \varepsilon \int_0^T \sum_{i=1}^4 \left(Z_i f_{iu_i} + g_{iu_i}\right)\delta u_i dxdt + O_3(\varepsilon) \ ,$$

Where $O_3(\varepsilon) = O_1(\varepsilon) + O_2(\varepsilon)$.

Lastly, dividing both sides by ε, then taking the limit $\varepsilon \to 0$, yields to

$$DG\left(\vec{u}, \vec{\vec{u}} - \vec{u}\right) = \int_Q H_{\vec{u}}\left(x, t, \vec{y}, \vec{u}, \vec{Z}\right)\left(\vec{\vec{u}} - \vec{u}\right)dxdt \ .$$

4.The NCSO and SCSO

4.1Theorem:

(a) with Assums (A), (B) &(C), if $\vec{W}$ is CO., the $\vec{u} \in \overrightarrow{W_A}$ is CQOCCCV, then there exist $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^2 | \lambda_l | = 1$, s.t. the following Kuhn-Tucher Lagrange (KTL) conditions are held:

$$\sum_{l=0}^2 \lambda_l\, DG_l\left(\vec{u}, \vec{\vec{u}} - \vec{u}\right) \geq 0, \forall\, \vec{\vec{u}} \in \vec{W}, \quad (53)$$

$$\lambda_2 G_2(\vec{u}) = 0, \quad (54)$$

(b) Inequality (53) is equivalent to:

$$H_{\vec{u}}\left(x, t, \vec{y}, \vec{u}, \vec{Z}\right)\vec{u}(t) = \min_{\vec{\vec{u}} \in \vec{U}} H_{\vec{u}}\left(x, t, \vec{y}, \vec{u}, \vec{Z}\right)\vec{\vec{u}}(t) \, , a.e. on\ Q, \quad (55)$$

Where $H_{\vec{u}}\left(x, t, \vec{y}, \vec{u}, \vec{Z}\right) = \sum_{i=1}^4 \left(Z_i f_{iu_i}(x, t, y_i, u_i) + g_{iu_i}(x, t, y_i, u_i)\right)$.

Proof: From Lemma 2.1, the funl. $G_l(\vec{u})$ (for $l = 0,1,2$) is cont. w.r.t. $\vec{\vec{u}} - \vec{u}$ and linear in $\vec{\vec{u}} - \vec{u}$, the $DG_l(\vec{u})$ is M-differential for any M, then applying the KTL theorem[15], there exist $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0, \lambda_2 \geq 0$, $\sum_{l=0}^2 | \lambda_l | = 1$ s.t. ((53)-(54)) are satisfied, then by utilizing Theorem 3.2, (53) becomes

$$\int_Q \left(Z_1 f_{1u_1}, Z_2 f_{2u_2}, Z_3 f_{3u_3}, Z_4 f_{4u_4}\right).\left(\vec{\vec{u}} - \vec{u}\right)dxdt \geq 0, \forall\, \vec{\vec{u}} \in \vec{W}, \quad (56)$$

where $g_i = \sum_{l=0}^2 \lambda_l g_{li}$ and $Z_i = \sum_{l=0}^2 \lambda_l Z_{li}$, ($\forall i = 1,2,3,4$).

(b) Let $\{\overrightarrow{\vec{u}_k}\}$ be dense Seq (DSeq) in $\vec{W}$, μ is Lebesgue measure (LM) on Q and let $S \subset Q$ be a measurable set (MS) s.t.

$\vec{\vec{u}}(x,t)=\begin{cases}\overrightarrow{\overline{u}_k}(x,t), if\ (x,t)\in S\\ \vec{u}(x,t), if\ (x,t)\notin S\end{cases}.$

Which makes (56), gives

$\int_S \left(Z_1 f_{1u_1}+g_{1u_1}, Z_2 f_{2u_2}+g_{2u_2}, Z_3 f_{3u_3}+g_{3u_3}, Z_4 f_{4u_4}+g_{4u_4}\right).\left(\vec{\overline{u}}_k-\vec{u}\right)dxdt\geq 0$,

or

$\left(Z_1 f_{1u_1}+g_{1u_1}, Z_2 f_{2u_2}+g_{2u_2}, Z_3 f_{3u_3}+g_{3u_3}, Z_4 f_{4u_4}+g_{4u_4}\right).\left(\vec{\overline{u}}_k-\vec{u}\right)\geq 0, a.e.\,on\ Q,$

i.e. this inequality holds on $Q\backslash Q_k$ with $\mu(Q_k)=0,\forall k$, where μ is a LM, i.e. it is satisfies on $Q\backslash\cup_k Q_k$, with $\mu(\cup_k Q_k)=0$, but $\{\overrightarrow{\overline{u}_k}\}$ is a DSeq in $\overrightarrow{W}$, then there is $\vec{\overline{u}}\in\overrightarrow{W}$, s.t.

$\left(Z_1 f_{1u_1}+g_{1u_1}, Z_2 f_{2u_2}+g_{2u_2}, Z_3 f_{3u_3}+g_{3u_3}, Z_4 f_{4u_4}+g_{4u_4}\right).\left(\vec{\overline{u}}-\vec{u}\right)\geq 0, a.e.\,on\ Q,\forall\vec{\overline{u}}\in\overrightarrow{W}.$

i.e. (53) gives (56). The converse is clear.

4.2Theorem: (The SCSO)

In addition to the assums (A), (B) &(C). Suppose $\overrightarrow{W}$ is CO., f_i, g_i are affine w.r.t. (y_i,u_i) for each (x,t), g_{0i}, g_{2i} are CO. w.r.t. (y_i,u_i), $\forall(x,t), i=1,2,3,4$. Then the NCSO of Theorem 4.1, with $\lambda_0>0$ are also sufficient.

Proof: Assume $\vec{u}\in\overrightarrow{W}_A$, is satisfied the KTL condition ((53)- (54)). Let $G(\vec{u})=\sum_{l=0}^{2}\lambda_l G_l(\vec{u})$, then using Theorem 3.2, to get

$DG\left(\vec{u},\vec{\overline{u}}-\vec{u}\right)=\sum_{l=0}^{2}\lambda_l\int_Q\sum_{i=1}^{4}Z_{li}f_{liu_i}+g_{liu_i}\delta u_i dxdt\geq 0,$

Since

$f_i(x,t,y_i,u_i)=f_{i1}(x,t)y_i+f_{i2}(x,t)u_i+f_{i3}(x,t).$

Let $\vec{u}\&\vec{\overline{u}}$ are given QCVs, then $\vec{y}\&\ \vec{\overline{y}}$ are their corresponding QSVs. Substituting the pair $(\vec{u},\vec{y})$in ((1)-(6)) and MBS by $\alpha\in[0,1]$ once, and then substituting the pair $(\vec{\overline{u}},\vec{\overline{y}})$ in ((1)-(6)) and MBS by $(1-\alpha)$ once again, finally collecting each pair from the corresponding equations together one gets

$(\alpha y_1+(1-\alpha)\bar{y}_1)_{tt}-\Delta(\alpha y_1+(1-\alpha)\bar{y}_1)+(\alpha y_1+(1-\alpha)\bar{y}_1)-(\alpha y_2+(1-\alpha)\bar{y}_2)$
$+(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_1+(1-\alpha)\bar{y}_1)$
$=f_{11}(x,t)(\alpha y_1+(1-\alpha)\bar{y}_1)+f_{12}(x,t)(\alpha u_1+(1-\alpha)\bar{u}_1)+f_{13}(x,t),\qquad (57)$
$\alpha y_1(x,t)+(1-\alpha)\bar{y}_1(x,0)=0,\qquad (58)$
$\alpha y_1(x,0)+(1-\alpha)\bar{y}_1(x,0)=y_1^0(x), \alpha y_{1t}(x,0)+(1-\alpha)\bar{y}_{1t}(x,0)=y_1^1(x),\qquad (59)$
$(\alpha y_2+(1-\alpha)\bar{y}_2)_{tt}-\Delta(\alpha y_2+(1-\alpha)\bar{y}_2)+(\alpha y_1+(1-\alpha)\bar{y}_1)+(\alpha y_2+(1-\alpha)\bar{y}_2)$
$-(\alpha y_3+(1-\alpha)\bar{y}_3)-(\alpha y_4+(1-\alpha)\bar{y}_4)$
$=f_{21}(x,t)(\alpha y_2+(1-\alpha)\bar{y}_2)+f_{22}(x,t)(\alpha u_2+(1-\alpha)\bar{u}_2)+f_{23}(x,t),\qquad (60)$
$\alpha y_2(x,t)+(1-\alpha)\bar{y}_2(x,0)=0,\qquad (61)$
$\alpha y_2(x,0)+(1-\alpha)\bar{y}_2(x,0)=y_2^0(x), y_{2t}(x,0)+(1-\alpha)\bar{y}_{2t}(x,0)=y_2^1(x)\ ,\qquad (62)$
$(\alpha y_3+(1-\alpha)\bar{y}_3)_{tt}-\Delta(\alpha y_3+(1-\alpha)\bar{y}_3)-(\alpha y_1+(1-\alpha)\bar{y}_1)+(\alpha y_2+(1-\alpha)\bar{y}_2)$
$+(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_4+(1-\alpha)\bar{y}_4)$
$=f_{31}(x,t)(\alpha y_3+(1-\alpha)\bar{y}_3)+f_{32}(x,t)(\alpha u_3+(1-\alpha)\bar{u}_3)+f_{33}(x,t),\qquad (63)$
$\alpha y_3(x,t)+(1-\alpha)\bar{y}_3(x,0)=0,\qquad (64)$
$\alpha y_3(x,0)+(1-\alpha)\bar{y}_3(x,0)=y_3^0(x), \alpha y_{3t}(x,0)+(1-\alpha)\bar{y}_{3t}(x,0)=y_3^1(x),\qquad (65)$
$(\alpha y_4+(1-\alpha)\bar{y}_4)_{tt}-\Delta(\alpha y_4+(1-\alpha)\bar{y}_4)-(\alpha y_1+(1-\alpha)\bar{y}_1)+(\alpha y_2+(1-\alpha)\bar{y}_2)$
$-(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_4+(1-\alpha)\bar{y}_4)$

$= f_{41}(x,t)(\alpha y_4 + (1-\alpha)\bar{y}_4) + f_{42}(x,t)(\alpha u_4 + (1-\alpha)\bar{u}_4) + f_{43}(x,t),$ (66)

$\alpha y_4(x,t) + (1-\alpha)\bar{y}_4(x,0) = 0,$ (67)

$\alpha y_4(x,0) + (1-\alpha)\bar{y}_4(x,0) = y_4^0(x),\ \alpha y_{4t}(x,0) + (1-\alpha)\bar{y}_{4t}(x,0) = y_4^1(x),$ (68)

Equalities ((57)- (68)), show that if the QCV is $\vec{\bar{u}}$(with ($\vec{\bar{u}} = \alpha\vec{u} + (1-\alpha)\vec{\bar{u}}$)) has corresponding QSVs $\vec{\bar{y}}$ with ($\bar{y}_i = y_{i\bar{u}_i} = y_{i(\alpha u_i + (1-\alpha)\bar{u}_i)}$).

This means the operator $\vec{u} \to \vec{y}_{\vec{u}}$ is CO-linear (COL) w.r.t. $(\vec{u},\vec{y})$ in Q. Now, since $g_{1i}(x,t,y_i,u_i)$ is affine w.r.t. (y_i,u_i), in Q, then $G_1(\vec{u})$ is COL w.r.t. $(\vec{u},\vec{y})$, also, since g_{0i} &g_{2i} are CO w.r.t.(y_i,u_i) , in Q, $\forall i = 1,2,3,4$, then the funl. $G_0(\vec{u}), G_2(\vec{u})$ are CO. w.r.t. $(\vec{y},\vec{u})$ in Q (from the assum. on the funl g_{li} ($\forall l = 0,1,2,\&i = 1,2,3,4$) and from the sum of two integral of CO function is also CO), i.e. $G(\vec{u})$ is CO w.r.t. $(\vec{y},\vec{u})$, in Q in the CO set $\vec{W}$, and has a cont. DD satisfies

$DG(\vec{u},\vec{\bar{u}} - \vec{u}) \geq 0$, which means $G(\vec{u})$ has a minimum at $\vec{u}$, i.e.

$G(\vec{u}) \leq G(\vec{\bar{u}})$,$\forall\ \vec{\bar{u}} \in \vec{W}$, i.e.

$\lambda_0 G_0(\vec{u}) + \lambda_1 G_0(\vec{u}) + \lambda_2 G_2(\vec{u}) \leq \lambda_0 G_0(\vec{\bar{u}}) + \lambda_1 G_1(\vec{\bar{u}}) + \lambda_2 G_2(\vec{\bar{u}}), \forall \vec{\bar{u}} \in \vec{W}$

Let $\vec{\bar{u}} \in \vec{W}_A$, $\lambda_2 \geq 0$ and from (54), the above inequality becomes

$\lambda_0 G_0(\vec{u}) \leq \lambda_0 G_0(\vec{\bar{u}})$, $\forall \vec{\bar{u}} \in \vec{W}$, or $G_0(\vec{u}) \leq G_0(\vec{\bar{u}})$, $\forall \vec{\bar{u}} \in \vec{W}$, thus $\vec{u}$ ia a CQOCCCV.

5.Conclusions and Discussions:

In this work, the CQOCCCVP dominating by a QNLHBVP is studied. The existence of a CQOCCCV dominating by a QNLHBVP with EINQSCC is stated and demonstrated under appropriate HYP with using the ACTH. Moreover mathematical formulation of the QAEs related to QSEs is found so as its WF. The derivation of the DD for the Ham is attained. Lastly, both the NCSO and the SCSO "theorems" optimality of the proposed problem are stated and demonstrated.

The study of the proposed problem is considered very interesting in the field of applied mathematics since the proposed model represents a generalization for a wave equation; from a side, and from the other, these results are very important because they give the green light about the ability for solving such problems numerically.

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