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Fractional Pantograph Delay Equations Solving by the Meshless Methods

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Abstract

This work describes two efficient and useful methods for solving fractional pantograph delay equations (FPDEs) with initial and boundary conditions. These two methods depend mainly on orthogonal polynomials, which are the method of the operational matrix of the fractional derivative that depends on Bernstein polynomials and the operational matrix of the fractional derivative with Shifted Legendre polynomials. The basic procedure of this method is to convert the pantograph delay equation to a system of linear equations, and by using, the operational matrices, we get rid of the integration and differentiation operations, which makes solving the problem easier. The concept of Caputo has been used to describe fractional derivatives. Finally, some numerical examples are identified to show the utility and capability of the two proposed approaches. The Mathematica® 12 program has been relied upon in the calculations.

Keywords: Bernstein polynomials, Legendre polynomials, Pantograph Delay Equations.

1. Introduction

Fractional calculus has recently been increasingly used due to its wide applications in real-world problems. It has been used as a powerful tool in various fields such as chemistry, physics, engineering, and applied sciences, where it can be seen in thermal modeling [1], networks [2], optics [3], optimal control [4], elasticity [5], fluid mechanics [6], and many other applications. The fractional nature of this class of problems makes the solution very difficult. As a result, many researchers and authors try to generalize and develop the current methods to simply apply them numerically or analytically and find approximate solutions to them. These methods include the Collocation method [7, 8], Adomian's decomposition method [9–11], homotopy perturbation method [12, 13], and other methods [14–16].

Fractional delay differential equations (FDDEs) are used in a variety of technical systems, including characterizing propagation, transport development, or population gestures [17, 18]. The pantograph equation is one of the most important types of delay differential equations, and it is used to describe a wide range of phenomena. Various applications of these equations in practical disciplines such as biology, physics, economics, and electrodynamics have been researched by many scholars [19–21]. Researchers have been interested in finding strategies to solve these kinds of problems and have presented a number of papers on the subject, where Rabiei and Ordok [22] introduce fractional-order Boubaker polynomials related to the Boubaker polynomials to achieve the numerical result for pantograph differential equations of fractional order in any arbitrary interval. Yuttanan et al. [23] presented a new class of functions called fractional-order generalized Taylor wavelets (FOGTW) for the nonlinear fractional delay and nonlinear fractional pantograph differential equations. In [24], a method utilizing the Chebyshev cardinal functions (CCFs) is formulated to find an accurate result for fractional pantograph delay differential equations; see more works [25–27]. Operational matrices are one of the most widely used numerical methods for solving fractional differential equations (FDEs) because they are based on polynomials, which are one of the simplest functions in terms of structure. As a result, the benefit of this method is that it transforms the problem into a set of algebraic equations. Using the operational matrix, we get rid of the derivations and integrals, making the solution easier and simpler. Studies that relied on operational matrices methods can be seen in [28–32].

The main objective of this work is to apply the operational matrix method for derivatives that depend on Bernstein polynomials, as well as the operational matrix method, which depends on shifted Legendre polynomials, for solving fractional pantograph delay equations (FPDEs) with initial and boundary conditions. The rest of this paper is organized as follows: In Section 2, we provide some basic definitions of fractional calculus. In Section 3, Bernstein polynomials and their operational matrices (BOM) are introduced. Also, in Section 4, Shifted Legendre polynomials and their operational matrices (LOM) are considered. In Section 5, some numerical examples of pantograph equations with prime and boundary conditions will be solved by the proposed methods. Finally, we discuss our findings and conclusions.

2. Basic Definitions

The Riemann-Liouville and Caputo operators [33] are the two most commonly used definitions in fractional calculus. This section will give the definitions and some of their properties.

Definition 1: The Riemann–Liouville fractional integration of order α is defined as [33]

$$I^{\alpha}y(x) = \{\frac{1}{\Gamma(\alpha)}\int_0^x \frac{y(s)}{(x-s)^{1-\alpha}}ds, \quad \alpha > 0, x > 0, y(x), \quad \alpha = 0 \quad . \tag{1}$$

For the Riemann–Liouville integral

1. $I^{\alpha_1}I^{\alpha_2}y(x) = I^{\alpha_1+\alpha_2}y(x)$,
2. $I^{\alpha}(\lambda_1 y(x) + \lambda_2 g(x)) = \lambda_1 I^{\alpha}y(x) + \lambda_2 I^{\alpha}g(x)$,
3. $I^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\alpha+\beta}, \quad \beta > -1$,

where $\alpha_1, \alpha_2, \lambda_1$ and λ_2 are constants.

Definition 2: Caputo's fractional derivative operator of order α is defined as [33]

$$D^{\alpha}y(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{y^{(n)}(s)}{(x-s)^{\alpha+1-n}}ds, \qquad n-1 < \alpha \leq n, n \in N. \tag{2}$$

The characteristics of Caputo derivative are

1. $D^{\alpha} I^{\alpha} y(x) = y(x)$,
2. $I^{\alpha} D^{\alpha} y(x) = y(x) - \sum_{i=0}^{n-1} \quad y^{(i)}(0) \frac{x^i}{i!}$,
3. $D^{\alpha} \lambda = 0$,

Where λ is constant.

In this study, the concept of Caputo was relied upon to define the fractional derivative.

3. Bernstein polynomials and their operational matrix [34,35]

In this section, we will go over Bernstein's polynomial and some of its fundamental properties, as well as a description of an operational matrix with integer and fractional derivatives, and explain how to apply the approach (BOM) for solving the pantograph equation.

3. 1. Bernstein polynomials

The Bernstein polynomials of degree n on the interval [0,1] are defined by

$$B_{i,n}(x) = (n\ i\ ) x^i (1-x)^{n-i},$$

for $i = 0, 1, \dots, n,$ where

$$(n\ i\ ) = \frac{n!}{i!\,(n-i)!}.$$

Bernstein basis polynomials in a linear combination $B_n(x) = \sum_{i=0}^{n} \quad c_i B_{i,n}(x)$, are called a Bernstein polynomial or polynomial in Bernstein form of degree , and c_i are called Bernstein coefficients.

Now we can approximate any polynomial of degree n to the form of linear combination, as given below [34]

$$y(x) = \sum_{i=1}^{n} \quad c_i B_{i,n} = C^T \Phi(x), \tag{3}$$

where $C^T = [c_0, c_1, \dots, c_n]$,

$\Phi(x) = [B_{0,n}, B_{1,n}, \dots, B_{n,n}],^T$

or in the form of a matrix resulting from multiplying a square matrix $(n+1) \times (n+1)$ and vector $(n + 1) \times 1$:

$\Phi(x) = AX$,

$A = [(-1)^0 (n\ 0\ )\ (-1)^1 (n\ 0\ )(n-0\ 1\ )\ \dots\ (-1)^{n-0} (n\ 0\ )(n-0\ n-0\ )\ 0\ (-1)^0 (n\ i\ )\ \dots\ (-1)^{n-i} (n\ i\ )(n-i\ n-i\ )\ \vdots \vdots \ddots \vdots\ 0\ 0\ \dots\ (-1)^0 (n\ n\ )\ ]_{(n+1)\times(n+1)},\ X = [1\ x\ x^2\ \vdots\ x^n\ ]_{(n+1)\times 1}$.

We note that the $\|A\| \neq 0$, which indicates that the matrix A is invertible.

On the other hand, when using the Bernstein polynomials in the least-squares approximation, the fact that they are not orthogonal becomes a disadvantage. According to [35], one method for direct least-squares approximation by polynomials in Bernstein form relies on the construction of the basis $\{d_0^n(x), d_1^n(x), \dots, d_n^n(x)\}$ that is "dual" to the Bernstein basis of degree n on to [0,1]. The attribute that distinguishes this dual basis is

$$\int_0^1 \quad b_i^n(x) d_j^n(x) dx = \{1 \quad for\ i = j, 0 \quad for\ i \neq j,$$

for $i, j = 0,1, \dots\dots, n,$

where

$$d_j^n(x) = \sum_{k=0}^{n} \quad \lambda_{jk} B_k^n(x), j = 0,1, \dots, n,$$

$$\lambda_{jk} = \frac{(-1)^{j+k}}{(n\ j\ )(n\ k\ )} \sum_{i=0}^{min(j,k)} \quad (2i+1)(n+i+1\ n-j\ )(n-i\ n-j\ )(n+i+1\ n-k\ )(n-i\ n-k\ ), \qquad (4)$$

for $j, k = 0,1, \dots, n$.

3. 2 Operational Matrix of Fractional Derivative for Bernstein Polynomial (BOM)

In the beginning, we will present the operational matrix D of derivatives for Bernstein polynomials, which is a square matrix of degree $(n+1) \times (n+1)$, then the derivative of the vector $\Phi(x)$ is

$$\frac{d\Phi(x)}{dx} = D\Phi(x), \qquad (5)$$

where D is given in [34], $D = A\sigma B^*$,

where σ is $(n+1) \times (n)$ matrix given in [35] as $\sigma = [0\ 0\ 0\ \dots\ 0\ 1\ 0\ 0\ \dots\ 0\ 0\ 2\ 0\ \dots\ 0\ \vdots \vdots \vdots \ddots\ 0\ 0\ 0\ 0\ \dots\ n\ ]$

and, B^* is $(n) \times (n+1)$ matrix define as $B^* = \left[A_{[1]}^{-1}\ A_{[2]}^{-1}\ \vdots\ A_{[n]}^{-1}\ \right]$,

$A_{[k]}^{-1}\ is\ k^{th}\ \ row\ of\ A^{-1}, k = 1,2, \cdots, n.$

We can generalize Eq. (5) as follows

$$\frac{d^n}{dx^n} \Phi(x) = (D)^n \Phi(x), where\ n = 1, 2, \dots. \qquad (6)$$

Let us now introduce the operational matrix of the fractional derivatives $D^{(\alpha)}$, $\alpha > 0$, of size $(n+1) \times (n+1)$ is defined by the concept of Caputo in [35]

$$D^{(\alpha)} = \left[\sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{0,j,0}\ \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{0,j,1}\ \dots \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{0,j,n}\ \vdots \vdots \vdots\ \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{i,j,0}\ \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{i,j,1}\ \dots \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{i,j,n}\ \vdots \vdots \vdots\ \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{n,j,0}\ \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{n,j,1}\ \dots \sum_{j=\lceil\alpha\rceil}^{n} \quad \omega_{n,j,n}\ \right]$$

here $\omega_{i,j,l}$ is given by:

$$\omega_{i,j,l} = (-1)^{j-i}(n\ i\ )(n-i\ j-i\ ) \frac{\Gamma(j+1)}{\Gamma(j+1-q)} \sum_{k=0}^{n} \quad \lambda_{lk}\mu_{kj},$$

where λ_{lk} is given in Eq. (4) and

$$\mu_{kj} = \sum_{s=k}^{n} (-1)^{s-k}(n\ k\ )(n-k\ s - k\ )\frac{1}{j-\alpha+s+1}.$$

3. 3 The steps of applying the BOM method for solving the Delay-Pantograph equation

In this paper, we consider the fractional neutral pantograph differential equation

$$D^{\alpha}y(x) = a(x)y(px) + b(x)D^{\gamma}y(px) + d(x)y(x) + g(x), \quad (7)$$

where $x \in [0,1]$, $\quad 0 < \gamma \leq \alpha \leq 2$, $\quad 0 < p < 1$.

$y(x)$ is the unknown function,

$a, b, d, g \in C[0,1]$, with initial and boundary conditions.

Now to apply the BOM method, we follow the steps below:

I- We will approximate unknown function $y\ (x)$ by Bernstein polynomials as (3), and we have

$Dy(x) = C^T D\ \Phi(x), Dy(px) = C^T D\ \Phi(px),\ldots,$

$D^n y(x) = C^T D^n\ \Phi(x), D^n y(px) C^T D^n\ \Phi(px),$

$D^{\alpha}y(x) = C^T D^{(\alpha)}\Phi(x), D^{\alpha}y(px) = C^T D^{(\alpha)}\Phi(px),$

$where\ \alpha\ is\ order\ of\ the\ fractional\ derivative.$

II- We will substitute all initial or boundary conditions given in the problem.

III- We use the roots of Chebyshev polynomials to help reduce interpolation errors as the collocation node,

$$x_i = \frac{1}{2} + \frac{1}{2cos\left((2i+1)\frac{\pi}{2n}\right)}, \quad i = 0,1,\ldots,n-1.$$

IV- We substitute the approximate polynomial $y(x)$ and its derivatives in Eq. (7) to get the system of algebraic equations that we can solve using computer programs like Mathematica and thus get the vector values C^T. As a result, we will get an approximate solution to the problem.

4. The shifted Legendre polynomials and their operational matrix [36]

Shifted Legendre polynomials will be discussed in this Section. Then we will describe the method (LOM) and how to use it to solve the problem.

4. 1. Legendre polynomials

Legendre polynomials are known for the interval [-1,1] and can be calculated using the regression relationship as following

$$L_{i+1}(z) = \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z), \quad i = 1,2,\ldots \quad (8)$$

where $L_0(z) = 1\ and\ L_1(z) = z.$

Substitute the variable $z = 2x - 1$ for these polynomials in the interval [0,1], and create the so-called shifted Legendre polynomial, let $P_i\ (x)$ be a symbol for the shifted Legendre polynomial. $P_i\ (x)$ is calculated as follows:

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{(i+1)} P_i(x) - \frac{i}{i+1} P_{i-1}(x)\,, \qquad i = 1,2,\ldots, \tag{9}$$

here $P_0(x) = 1$ and $P_1(x) = x$.

The following formulations of the shifted Legendre polynomial of degree i analytic form are found in the source [36]:

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{x^k}{(k!)^2}. \tag{10}$$

Whereas: $P_i(1) = 1$ and $P_i(0) = (-1)^i$. The requirement of orthogonality is:

$\int_0^1 P_i(x)P_j(x)dx = \{\frac{1}{2i+1} \quad for\ i = j, 0 \quad for\ i \neq j.$

The shifted Legendre polynomials' power series is written as follows:

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k}(i+k\ k\ )(i\ k\ )x^k.$$

Now we can approximate any function based on shifted Legendre polynomial as follows:

$$y(x) = \sum_{j=0}^{\infty} c_j P_j(x),$$

where

$$c_j = (2j+1)\int_0^1 y(x)P_j(x)dx, \quad j = 1,2,\ldots$$

Only the first $(n + 1)$ terms of shifted Legendre polynomials are considered in practice. Next, there is

$$y(x) = \sum_{j=0}^{n} c_j P_j(x) = C^T \Phi(x), \tag{11}$$

where

$C^T = [c_0, \ldots, c_n], the\ shifted\ Legendre\ coefficient\ vector\ , \Phi(x) = [P_0(x), P_1(x), \ldots, P_n(x)]^T\ \ the\ shifted\ Legendre\ vector.$

4. 2. Operational Matrix of Fractional Derivative for Shifted Legendre Polynomials (LOM)

The operational matrix of derivatives defined based on shifted Legendre polynomials is denoted by $D^{(1)}$ of degree $(n+1) \times (n+1)$ and can be expressed in the following form [36]

$D^{(1)} = (d_{ij}) = \{2(2j+1), \quad for\ j = i-k, 0, \quad otherwise\ \{k = 1,3,\dots,n, \quad if\ n\ odd.\ k = 1,3,\dots,n-1, \quad if\ n\ even.$

The vector's derivative $\Phi(x)$ can be written as follows

$$\frac{d\Phi(x)}{dx} = D^{(1)}\Phi. \tag{12}$$

We can generalize Equation (12) as

$$\frac{d^n\Phi(x)}{dx^n} = \left(D^{(1)}\right)^n\Phi(x), \qquad n = 1, 2, \dots\dots \tag{13}$$

Let us now represent the fractional derivative operational matrix $D^{(\alpha)}$, $\alpha > 0$, of size $(n+1)\times(n+1)$ defined by the Caputo concept in the source [36] as follows:

$$D^{\alpha} = \left[0\ 0\ \cdots\ 0\ \vdots\ \vdots\ \cdots\ \vdots\ 0\ 0\ \cdots\ 0\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k}\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,1,k} \cdots \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,n,k}\ \vdots\ \vdots\ \cdots\ \vdots\ \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,0,k}\ \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,1,k} \cdots \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,n,k}\ \vdots\ \vdots\ \cdots\ \vdots\ \sum_{k=\lceil\alpha\rceil}^{n} \theta_{n,0,k}\ \sum_{k=\lceil\alpha\rceil}^{n} \theta_{n,1,k} \cdots \sum_{k=\lceil\alpha\rceil}^{n} \theta_{n,n,k}\right],$$

where $\theta_{i,j,k}$ $is\ defined\ by$

$$\theta_{i,j,k} = (2j+1)\sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!\,(l+j)!}{(i-k)!\,k!\,\Gamma(k-\alpha+1)(j-l)!\,(l!)^2(k+l-\alpha+1)}.$$

It is worth noting that the initial $\lceil\alpha\rceil$ rows in $D^{(\alpha)}$are all zero.

4.3 The steps of applying the LOM method for solving the Delay-Pantograph equation

I- We will approximate unknown function $y(x)$ by Legendre polynomials as Eq. (11), and we have

$Dy(x) = C^T D\ \Phi(x), Dy(px) = C^T D\ \Phi(px),\dots,$

$D^n y(x) = C^T D^n\ \Phi(x), D^n y(px) C^T D^n\ \Phi(px),$

$D^{\alpha} y(x) = C^T D^{(\alpha)}\Phi(x), D^{\alpha} y(px) = C^T D^{(\alpha)}\Phi(px),$

$where\ \alpha\ is\ order\ of\ the\ fractional\ derivative.$

II- We will substitute all initial or boundary conditions given in the problem.

III- We use the roots of Chebyshev polynomials can help reduce interpolation errors as the collocation node,

$$x_i = \frac{1}{2} + \frac{1}{2cos\left((2i+1)\frac{\pi}{2n}\right)}, \quad i = 0,1,\dots,n-1.$$

IV- We substitute the approximate polynomial $y(x)$ and its derivatives in Eq. (7) to get the system of algebraic equations that we can solve using computer programs like Mathematica and thus get the vector values C^T as a result, we will get an approximate solution to the problem.

5. Illustrative examples

In this Section, some numerical examples are given to demonstrate the applicability and accuracy of the proposed method. $Mathematica^{®}$ 12 is used for all numerical calculations.

Example 1: - Consider the following fractional neutral pantograph differential equation

$$D^{\alpha}y(x) = -y(x) + 0.1y(0.8x) + 0.5D^{\alpha}y(0.8x) + g(x), x \in (0,1], \quad (14)$$

$$y(0) = 0, with\ 0 < \alpha \leq 1.$$

Now to solve Eq. (14) by (BOM), the exact solution to this problem for $\alpha = 0.8$ is $y(x) = x^{3.8}$ with $(x) = 2.211894885744887x^3 + 0.9571706039258614x^{3.8}$, and with $n = 2$, by applying the suggested method in Section (3-2), we have

$$D^{0.8} = [-1.14334\ -1.55\ -0.276996\ 1.17281\ 0.872243\ -1.55\ -0.0294677\ 0.677756\ 1.82699\ ]$$

By applying the condition given in the problem and following the solution steps mentioned in Section)3-3), a system of algebraic equations will be produced, and by solving this system, we get the values of $c_0 = 0, c_1 = 0.0118614, c_2 = 0.472083$, the approximate solution is $y(x) = 0.0237227x + 0.44836x^2$.

If $n = 15$, we have the following approximate solution: $y(x) = 9.543602968653811 \times 10^{-17} + 0.0000065116763974070\textit{56}x - 0.0007034719961872769x^2 + 0.03467000244609836x^3 + 1.4120584194534938x^4 - 1.2900408484650545x^5 + 2.1428840263535394x^6 - 2.118319283516632x^7 - 0.6752474254279832x^8 + 6.081788398525532x^9 - 10.489702105988727x^{10} + 10.517845113990916x^{11} - 7.003103640066911x^{12} + 3.1968135804363556x^{13} - 0.9498166132520964x^{14} + 0.14086738703346668x^{15}$.

Table 1 shows the absolute error$|y_{exact} - y_{app.}|$ value for the some $x \in (0,1]$ at n= 9, 11, 13, and 15.

Table 1: The absolute error of Ex. 1 for n=9,11,13,15 by (BOM)

x	$n = 9$	$n = 11$	$n = 13$	$n = 15$
0.1	1.21756×10^{-7}	2.92165×10^{-8}	6.77509×10^{-8}	3.19953×10^{-8}
0.2	1.07484×10^{-7}	3.66589×10^{-8}	7.85818×10^{-8}	3.32075×10^{-8}
0.3	1.40511×10^{-7}	5.90616×10^{-8}	5.80438×10^{-8}	2.87904×10^{-8}
0.4	3.05621×10^{-7}	6.14799×10^{-8}	2.5095×10^{-8}	1.82177×10^{-8}
0.5	5.54154×10^{-8}	2.54537×10^{-8}	9.48675×10^{-9}	4.38442×10^{-9}
0.6	2.92554×10^{-7}	3.27859×10^{-8}	2.82086×10^{-8}	6.08513×10^{-10}
0.7	3.1252×10^{-7}	8.2099×10^{-8}	3.73298×10^{-8}	1.17502×10^{-8}
0.8	6.18788×10^{-8}	1.2187×10^{-7}	3.22221×10^{-8}	1.2558×10^{-9}
0.9	2.07441×10^{-7}	1.90066×10^{-7}	1.81549×10^{-8}	1.60439×10^{-8}

In order to use the LOM approach to solve example 1, we must follow the steps outlined in section (4-3). If we assume $n = 2$, we get

$$D^{0.8} = [0\ 0\ 0\ 1.81521\ 0.495057\ -0.206274\ -0.495057\ 4.08422\ 1.06084\ ]$$

The approximate solution $y(x) = 0.0237227x + 0.44836x^2$, and the value of $c_0 = 0.161315, c_1 = 0.236041, c_2 = 0.0747266$. If $n = 15$, the approximate solution

$y(x) = -5.917880389286895 \times 10^{-17} + 0.0000049412555727930\textit{77}x - 0.0005604595883997804x^2 + 0.030640435190110164x^3 + 1.4662615402598584x^4 - 1.7082026330161135x^5 + 4.183434690591156x^6 - 8.78814674633918x^7 +$

$14.437646065525456x^8 - 18.130978138638397x^9 + 17.216360479207815x^{10} - 12.200969372917305x^{11} + 6.30791535582312015x^{12} - 2.277429378098255x^{13} + 0.5224877022689723x^{14} - 0.058464614185574325x^{15}$.

We present **Table 2**, in which we will show the absolute error value of n=9,11,13, and 15

Table 2: The absolute error of Ex. 1 for n=9,11,13,15 by (LOM)

x	$n = 9$	$n = 11$	$n = 13$	$n = 15$
0.1	1.21719×10^{-7}	2.88811×10^{-8}	2.34826×10^{-9}	1.72445×10^{-8}
0.2	1.07453×10^{-7}	3.60137×10^{-8}	1.13909×10^{-8}	1.44977×10^{-8}
0.3	1.40539×10^{-7}	5.78547×10^{-8}	9.88362×10^{-9}	1.44738×10^{-8}
0.4	3.05492×10^{-7}	6.08407×10^{-8}	9.05138×10^{-9}	1.08162×10^{-8}
0.5	5.55965×10^{-8}	2.58773×10^{-8}	3.71809×10^{-8}	1.03386×10^{-9}
0.6	2.92938×10^{-7}	3.09694×10^{-8}	6.09679×10^{-8}	8.16747×10^{-9}
0.7	3.1172×10^{-7}	8.21428×10^{-8}	6.37435×10^{-8}	1.17671×10^{-8}
0.8	6.3123×10^{-8}	1.21236×10^{-7}	5.67603×10^{-8}	5.21886×10^{-9}
0.9	2.09417×10^{-7}	1.93688×10^{-7}	6.72274×10^{-8}	5.09303×10^{-9}

In **Figure 1**, we show a plot of both the exact solution and the approximate solution of the two proposed methods for n=15.

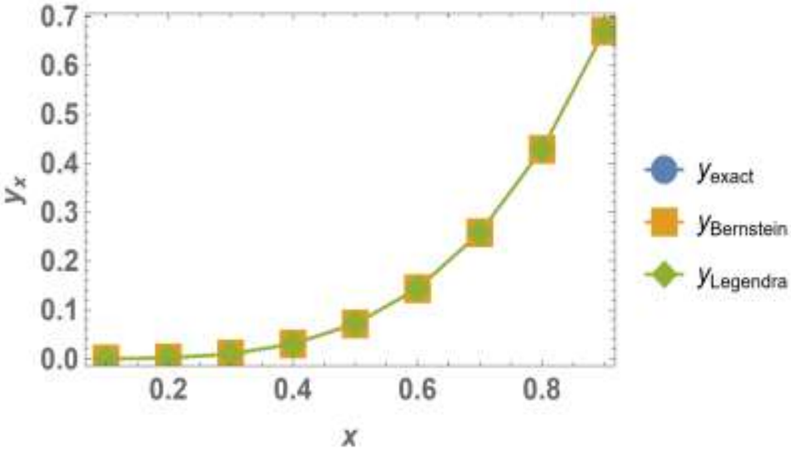

Figure 1: The comparison of the exact solution and the approximate solution of (BOM, LOM) methods for Ex. 1 at n=15

Also, in **Figure 2**, logarithmic plots of the greatest absolute error of the approximate solution are given using Bernstein or Legendre polynomials with operational matrices to solve example 1 from n = 2 to n = 15.

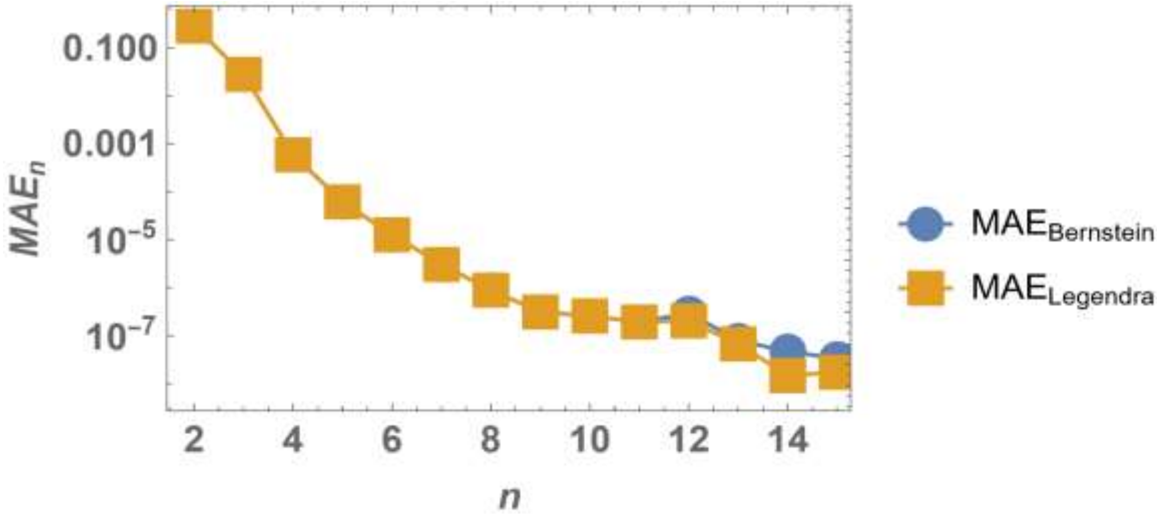

Figure 2: Logarithmic plots for the MAE_n of Ex. 1 form n=2 to n=15

We note from the above results that the two proposed methods are efficient because their results are highly accurate compared to the results of the exact solution, as shown in **Figure 1**. Also, we can see the efficiency of the two proposed methods in **Figure 2,** which shows the decrease in absolute error when the value of n is increased**.**

In order to show which of the two methods is more efficient, we present **Table 3**, in which we will show the absolute error values of the two proposed methods at n = 15

Table 3: The absolute error $|y_{exact} - y_{app.}|$ of (BOM, LOM) for Ex. 1 at n=15

x	Absolute error (B)	Absolute error (L)
0.1	3.19953×10^{-8}	1.72445×10^{-8}
0.2	3.32075×10^{-8}	1.44977×10^{-8}
0.3	2.87904×10^{-8}	1.44738×10^{-8}
0.4	1.82177×10^{-8}	1.08162×10^{-8}
0.5	4.38442×10^{-9}	1.03386×10^{-9}
0.6	6.08513×10^{-10}	8.16747×10^{-9}
0.7	1.17502×10^{-8}	1.17671×10^{-8}
0.8	1.2558×10^{-9}	5.21886×10^{-9}
0.9	1.60439×10^{-8}	5.09303×10^{-9}

We notice from **Table 3** that the values of Legendre's absolute error are less than those of Bernstein's absolute error, which indicates that (LOM) method is more efficient than (BOM) method in solving example 1.

Example 2: Consider the following fractional pantograph equation

$$D^2y(x) + D^{\frac{3}{2}}y(x) + y(x) = y\left(\frac{x}{2}\right) + \frac{3x^2}{4} + 4\sqrt{\frac{x}{\pi}} + 2, x \in [0,1], \quad (15)$$

subject to $y(0) = 0,\ y(1) = 1$, and the exact solution to this problem is $y(x) = x^2$.

When applying the (BOM) method for solving Eq. (15) at $n = 2$, we have

$$D^{\frac{3}{2}} = \begin{bmatrix} \frac{24}{35\sqrt{\pi}} & \frac{24}{7\sqrt{\pi}} & \frac{136}{35\sqrt{\pi}} \\ -\frac{48}{35\sqrt{\pi}} & -\frac{48}{7\sqrt{\pi}} & -\frac{272}{35\sqrt{\pi}} \\ \frac{24}{35\sqrt{\pi}} & \frac{24}{7\sqrt{\pi}} & \frac{136}{35\sqrt{\pi}} \end{bmatrix},$$

We also obtained the values of the unknown vector C^T, which are as follows $c_0 = 1.18599 \times 10^{-17}, c_1 = 0.00228219, c_2 = 1$.

We also come up with the following approximate solution:

$y(x) = 1.18599 \times 10^{-17} + 0.00456438x + 0.995436x^2$. If $n = 20$, we have the following approximate solution:

$y(x) = -2.786853970684164 \times 10^{-15} - 0.0000300305318542959950x + 1.006670875743893x^2 - 0.31622314969631304x^3 + 7.39417733179431x^4 - 101.22636028238139x^5 + 882.5852080762097x^6 - 5099.047105444493x^7 + 19465.37892066044x^8 - 44677.52886989678x^9 + 27582.502303316025x^{10} + 211915.94892108513x^{11} - 939140.5398020139x^{12} + 2196019.891017601x^{13} - 3463946.68783369x^{14} + 3900531.650722769x^{15} - 3162032.689661729x^{16} + 1810684.418812781x^{17} - 696899.7151342098x^{18} + 162020.35868317497x^{19} - 17212.384417226098x^{20}$.

Table 4 shows the absolute error$\left|y_{exact} - y_{app.}\right|$ value for the some $x \in [0,1]$ at n= 14,16,18, and 20

Table 4: The absolute error of Ex. 2 for n=14,16,18,20 by (BOM)

x	$n = 14$	$n = 16$	$n = 18$	$n = 20$
0.1	3.3016×10^{-6}	2.0304×10^{-6}	1.679×10^{-6}	1.42028×10^{-6}
0.2	5.64738×10^{-6}	3.96901×10^{-6}	2.44675×10^{-6}	1.92033×10^{-6}
0.3	5.90691×10^{-6}	4.01465×10^{-6}	3.34437×10^{-6}	2.143×10^{-6}
0.4	6.08868×10^{-6}	4.68942×10^{-6}	2.963×10^{-6}	2.48202×10^{-6}
0.5	6.6422×10^{-6}	3.86708×10^{-6}	3.25554×10^{-6}	2.06817×10^{-6}
0.6	4.61383×10^{-6}	4.11001×10^{-6}	2.54331×10^{-6}	1.97996×10^{-6}
0.7	4.97604×10^{-6}	2.68729×10^{-6}	2.29425×10^{-6}	1.58232×10^{-6}
0.8	2.62815×10^{-6}	2.43508×10^{-6}	1.62869×10^{-6}	9.01401×10^{-7}
0.9	2.04652×10^{-6}	1.1369×10^{-6}	6.75106×10^{-7}	3.45435×10^{-7}

To solve example 2 using the (LOM) technique, we must follow the steps indicated in section (4-3). If we take n=2, we get

$D^{\frac{3}{2}} = \left[0\ 0\ 0\ 0\ 0\ 0\ \frac{16}{\sqrt{\pi}}\ \frac{48}{5\sqrt{\pi}}\ -\frac{16}{7\sqrt{\pi}}\right]$,

We also obtain the values of the unknown vector C^T, which are as follows $c_0 = 0.334094, c_1 = 0.5, c_2 = 0.165906$. and their approximate solution is

$y(x) = 2.77555756 \times 10^{-17} + 0.00456438x + 0.995436x^2$.

If $n = 20$, the approximate solution will be as follows:

$y(x) = -7.827619401012162 \times 10^{-17} - 0.000030068562737359008x + 1.0066701273460272x^2 - 0.3161559593254199x^3 + 7.3915208390024448x^4 - 101.1675850252221x^5 + 881.7639994208571x^6 - 5091.2533573008905x^7 + 19412.68134451585x^8 - 44415.30443366349x^9 + 26600.722236690082x^{10} + 214722.2039331961x^{11} - 945315.3904772103x^{12} + 2206507.388919022x^{13} - 3477652.419776158x^{14} + 3914177.755634204x^{15} - 3172189.258528861x^{16} + 1816152.725780014x^{17} - 698910.3552130745x^{18} + 162471.92620954153x^{19} - 17259.100690250787x^{20}$.

Table 5 shows the absolute error $|y_{exact} - y_{app.}|$ value for the some $x \in [0,1]$ at n= 14,16,18, and 20

Table 5: The absolute error of Ex. 2 for n=14,16,18,20 by (LOM)

x	$n = 14$	$n = 16$	$n = 18$	$n = 20$
0.1	3.30159×10^{-6}	2.03041×10^{-6}	1.67929×10^{-6}	1.41633×10^{-6}
0.2	5.64737×10^{-6}	3.96904×10^{-6}	2.44733×10^{-6}	1.91244×10^{-6}
0.3	5.9069×10^{-6}	4.0147×10^{-6}	3.34525×10^{-6}	2.13118×10^{-6}
0.4	6.08866×10^{-6}	4.68948×10^{-6}	2.96416×10^{-6}	2.4665×10^{-6}
0.5	6.64218×10^{-6}	3.86716×10^{-6}	3.25698×10^{-6}	2.04886×10^{-6}
0.6	4.6138×10^{-6}	4.11011×10^{-6}	2.54504×10^{-6}	1.95697×10^{-6}
0.7	4.97601×10^{-6}	2.6874×10^{-6}	2.29617×10^{-6}	1.55857×10^{-6}
0.8	2.62812×10^{-6}	2.43526×10^{-6}	1.63045×10^{-6}	8.94486×10^{-7}
0.9	2.04649×10^{-6}	1.13714×10^{-6}	6.78982×10^{-7}	4.05584×10^{-7}

In **Figure 3**, we show a plot of both the exact solution and the approximate solution of the two proposed methods of example 2 for n=20.

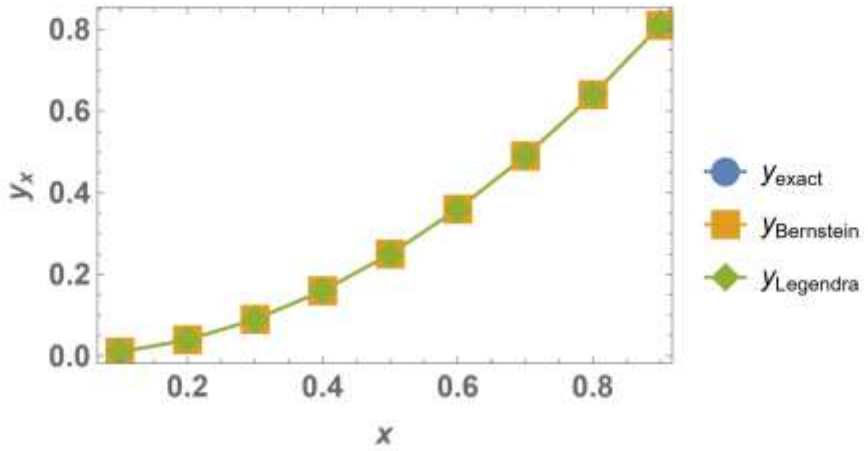

Figure 3: The comparison of the exact solution and the approximate solution of (BOM, LOM) methods for Ex. 2 at n=20

In **Figure 4**, logarithmic plots of the greatest absolute error of the approximate solution are given using Bernstein or Legendre polynomials with operational matrices to solve example 2 from n = 2 to n = 20.

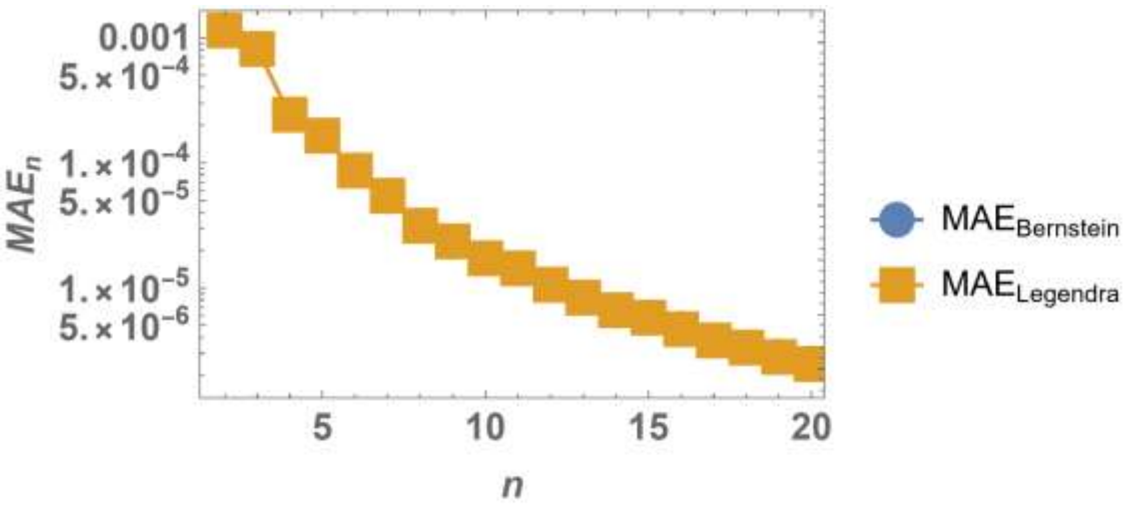

Figure 4: Logarithmic plots for the MAE_n of Ex. 2 form n=2 to n=20

The two proposed approaches are efficient, as seen by the above results because their results are accurate when compared to the precise solution's results, as shown in **Figure 1**. **Figure 2** demonstrates the decrease in absolute error when the value of n is increased, demonstrating the efficacy of the two proposed techniques.

Table 6 illustrates the absolute error numbers for (BOM, LOM) methods at n = 20 to determine which of the two strategies is superior for solving example 2.

Table 6: The absolute error $|y_{exact} - y_{app.}|$ of (BOM, LOM) for Ex. 2 at n=20

x	Absolute error (B)	Absolute error (L)
0.1	1.42028×10^{-6}	1.41633×10^{-6}
0.2	1.92033×10^{-6}	1.91244×10^{-6}
0.3	2.143×10^{-6}	2.13118×10^{-6}
0.4	2.48202×10^{-6}	2.4665×10^{-6}
0.5	2.06817×10^{-6}	2.04886×10^{-6}
0.6	1.97996×10^{-6}	1.95697×10^{-6}
0.7	1.58232×10^{-6}	1.55857×10^{-6}
0.8	9.01401×10^{-7}	8.94486×10^{-7}
0.9	3.45435×10^{-7}	4.05584×10^{-7}

We notice from the above table that the greatest absolute error we acquired in the method (BOM) is 2.48202×10^{-6}, but in the method (LOM) is 2.4665×10^{-6}, indicating that the method (LOM) is the best in solving Ex. 2.

6. Conclusion

In this study, we propose the methods of operational matrices that depend on Bernstein polynomials and Shifted Legendre polynomials for fractional derivatives in solving fractional delay pantograph equations. The numerical results of the approximate solutions in examples 1 and 2 were given accuracy when compared with the exact solutions, proving the effectiveness and efficiency of the proposed methods. **Figures 2** and **4** showed the absolute error of the two examples for more than n, where when the values of n increase, the value of the absolute error decreases, which shows the accuracy and success of the proposed methods. **Tables 3** and **6** show that the method (LOM) is the best solution for the two examples. *Mathematica*®12 program was used to find the numerical results.

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