Groups Effect of Types D_5 and A_5 on The Points of Projective Plane Over F_q, $q = 29,31$

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Abstract

The purpose of this paper is to find an arc of degree five in $PG(2,q)$, $q = 29,31$, with stabilizer group of type dihedral group of degree five D_5 and arcs of degree six and ten with stabilizer groups of type alternating group of degree five A_5, then study the effect of D_5 and A_5 on the points of projective plane. Also, find a pentastigm which has collinear diagonal points.

Key words: Projective plane, arc.

Introduction

Let F_q denotes the Galois field of q elements and $V(3,q)$ be the vector space of row vectors of length three with entries in F_q. Let $PG(2,q)$ be the corresponding projective plane. The *points* $[x_0, x_1, x_2]$ of $PG(2,q)$ are the 1-dimensional subspaces of $V(3,q)$. Subspaces of dimension two of the form $V(aX_0 + bX_1 + cX_2)$ are called *lines*. The number of points and the number of lines in $PG(2,q)$ is q^2+q+1. There are $q+1$ points on every line and $q+1$ lines through every point.

The lines of $PG(2,q)$ are constructed by the following way:
$P_{i_1}, P_{i_2}, \ldots, P_{i_{q^2+q+1}}$ be the points lie on the line $\ell_1 = V(X_2)$ where the index i_j refers to the position of the point P_{i_j} in the plane. Fixed the line $\ell_1 = V(X_2)$ as a first line. The second line is constructed by adding 1 to the index i_j and so on.

Definition 1.1[1]: An $(n;r)$ arc K or arc of degree r in $PG(k,q)$ with $n \geq r+1$ is a set of n points with property that every hyperplane meets K in at most r points of K and there is some hyperplane meeting K in exactly r points. An $(n;2)$-arc is also called an n-*arc*.

Definition 1.2[1]: An $(n;r)$-arc is *complete* if it is maximal with respect to inclusion; that is, it is not contained in an $(n+1;r)$-arc.

Definition 1.3[1]: A line of $PG(k,q)$, $k > 1$ is an i-*secant* of an $(n;r)$ arc K if $|\ell \cap K| = i$. A 2-secant is called a *bisecant*, a 1-secant a *unisecant (tangent)* and a 0-secant is an *external line*.

Let c_i be the number of points of $PG(2,q)\setminus K$ with index exactly i. So, the parameters c_0 is the number of points through which no bisecant of K passes and c_3 is the number of points where three bisecants meet.

Theorem 1.4[1]: (The Fundamental Theorem of Projective Geometry)
If $\{P_0, \ldots, P_{n+1}\}$ and $\{P'_0, \ldots, P'_{n+1}\}$ are both subsets of $PG(n,q)$ of cardinality $n+2$ such that no $n+1$ points chosen from the same set lie in a hyperplane, then there exists a unique projectivity $\Im$ such that $P'_i = P_i \Im$ for $i = 0,1,\ldots,n+1$.

Definition 1.5[1]: Let $U_0 = [1,0,0], U_1 = [0,1,0], U_2 = [0,0,1], U = [1,1,1]$. The set $\Gamma_q = \{U_0, U_1, U_2, U\}$ is called the *standard frame*.

Definition 1.6[3]: Let χ_1 and χ_2 be two projective spaces of dimension n. A projectivity $\Im : \chi_1 \rightarrow \chi_2$ is a bijection given by a non-singular $(n+1)\times(n+1)$ matrix A such that $P(X') = P(X)\Im$ if and only if $tX' = XA$, where $t \in Fq \setminus \{0\}$.

Two projective spaces χ_1 and χ_2 are projectively equivalent if there is a projectivity between them.

Definition 1.7[1]: A point of index three is called a *Brianchon point* or B-*point* for short. Write $j \cdot kl \cdot mn = P_i P_j \cap P_k P_l \cap P_m P_n$ for B-point.

Remark 1.8[1]: (i) An $(n+1;r)$ arc is constructing from an $(n;r)$-arc, K by adding one point of index zero to K.
(ii) Two arcs K and K' are projectively equivalent if there is a projectivity between them.
(ii) With parameters c_0, K is complete if and only if $c_0 = 0$.

Some groups that occur in this paper are listed below. For more details see [2].
C_n = cyclic group of order n;
S_n = symmetric group of degree n;
A_n = alternating group of degree n;
D_n = dihedral group of order $2n$ $=\langle r,s \mid r^n = s^2 = (rs)^2 = 1\rangle$.
During this research the primitive element $\upsilon = 2$ in F_{29} and $\omega = 3$ in F_{31} are used.

Hirschfeld stated in [1] that in $PG(2,9)$ there is a unique arc of degree five with stabilizer group isomorphic to a dihedral group of degree five D_5. Also, there is a unique arc of degree six with stabilizer group isomorphic to alternating group of degree five A_5. In 1984, Sadeh [3] and in 2011 Al-Zangana [4] proved the same results in $PG(2,11)$ and $PG(2,19)$ also they proved that the same arc of degree six has ten B-points which is an arc of degree ten with stabilizer group of type A_5. In 1995, Storm and Maldeghem [5] proved in $PG(2,q)$ when $q \equiv \pm 1 \mod(10)$ they proved theoretically these results.

According to these previous results, the following question arises:

1- What is the arc of degree five which has stabilizer group of type dihedral group of degree five D_5 and what is the effect of D_5 on the points of projective plane.
2- What is the pentastigm which has collinear diagonal points.
3- What is the arcs of degree six and ten which have stabilizer groups of type alternating group of degree five A_5 and what is the effect of A_5 on the points of projective plane.

1- Inequivalent Arcs of Degree Five

From the fundamental theorem of Projective Geometry, there is a unique inequivalent arc of degree four in the projective plane with stabilizer group isomorphic to S_4. So, the standard frame $\Gamma_q = \{\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}\}$ formed a projectively unique 4-arc in the projective plane.

An arc of degree five is constructed by adding one point of index zero to Γ_q. And to find equivalents arcs, a mathematical programming language GAP has been used [6].

Theorem 2.1: (i) There are ten inequivalent 5-arcs through the frame Γ_{29} in $PG(2,29)$ with parameters $[c_0,c_1,c_2]=[601,250,15]$ as summarized in Table 1.
(ii) There are eleven inequivalent 5-arcs through the frame Γ_{31} in $PG(2,31)$ with parameters $[c_0,c_1,c_2]=[703,270,15]$ as summarized in Table 2.

Table 1: Inequivalent 5-arc and their stabilizer group in $PG(2,29)$

No.	5-Arc	Stabilizer Group
1	$A_1 = \Gamma_{29} \cup \{\mathrm{P}(v^{27}, v^{14}, 1)\}$	$\mathrm{C}_2 = \langle [[0, v^{15}, 0], [v^{13}, 0, 0], [v^{13}, 1, v^{14}]] \rangle$
2	$A_2 = \Gamma_{29} \cup \{\mathrm{P}(v^{24}, v^{11}, 1)\}$	I

3	$A_3=\Gamma_{29}\cup\{P(v^2,v^6,1)\}$	$C_2=\langle[[v^{17},0,0],[0,0,v^{14}],[0,v^{20},0]]\rangle$
4	$A_4=\Gamma_{29}\cup\{P(v^8,v^4,1)\}$	$C_2=\langle[[0,0,v^{14}],[0,v^{18},0],[v^{22},0,0]]\rangle$
5	$A_5=\Gamma_{29}\cup\{P(v^{10},v^{12},1)\}$	I
6	$A_6=\Gamma_{29}\cup\{P(v^{12},v^{16},1)\}$	$C_2=\langle[[0,v^2,0],[v^{26},0,0],[0,0,v^{14}]]\rangle$
7	$A_7=\Gamma_{29}\cup\{P(v^{26},v^{15},1)\}$	$C_2=\langle[[v^{12},v,v^{14}],[v^{26},v^{26},v^{26}],[0,0,v^3]]\rangle$
8	$A_8=\Gamma_{29}\cup\{P(v^{27},v^5,1)\}$	I
9	$A_9=\Gamma_{29}\cup\{P(v^{18},v^7,1)\}$	$C_4=\langle[[0,v^{14},0],[1,1,1],[v^{14},0,0]]\rangle$
10	$A_{10}=\Gamma_{29}\cup\{P(v^{22},v^{16},1)\}$	$D_5=\left\langle \begin{array}{l} r=[[v^8,0,0],[0,0,v^{14}],[0,v^2,0]], \\ s=[[0,v^8,0],[1,1,1],[v^2,0,0]] \end{array}\right\rangle$

Table 2: Inequivalent 5-arc and their stabilizer group in $PG(2,31)$

No.	5-Arc	Stabilizer Group
1	$A'_1=\Gamma_{31}\cup\{P(\omega^2,\omega^{10},1)\}$	I
2	$A'_2=\Gamma_{31}\cup\{P(\omega^6,\omega^4,1)\}$	I
3	$A'_3=\Gamma_{31}\cup\{P(\omega^{12},\omega^{27},1)\}$	$C_2=\langle[[\omega^{19},0,0],[\omega^{27},\omega^{18},\omega^{15}],[\omega^3,\omega^3,\omega^3]]\rangle$
4	$A'_4=\Gamma_{31}\cup\{P(\omega^{13},\omega^{29},1)\}$	I
5	$A'_5=\Gamma_{31}\cup\{P(\omega^{17},\omega^{11},1)\}$	I
6	$A'_6=\Gamma_{31}\cup\{P(\omega^{14},\omega^{20},1)\}$	$C_2=\langle[[\omega^{29},\omega^5,\omega^{15}],[\omega^{14},\omega^{14},\omega^{14}],[0,0,\omega^{17}]]\rangle$
7	$A'_7=\Gamma_{31}\cup\{P(\omega^{15},\omega^9,1)\}$	$C_2=\langle[[\omega^{15},\omega^{15},\omega^{15}],[0,0,1],[0,1,0]]\rangle$
8	$A'_8=\Gamma_{31}\cup\{P(\omega^8,\omega^{20},1)\}$	$C_2=\langle[[\omega^{23},\omega^5,\omega^{15}],[0,\omega^6,0],[\omega^{15},\omega^{15},\omega^{15}]]\rangle$
9	$A'_9=\Gamma_{31}\cup\{P(\omega^{26},\omega^9,1)\}$	$C_4=\langle[[\omega^9,\omega^9,\omega^9],[\omega^{11},\omega^{24},\omega^{15}],[0,0,\omega]]\rangle$
10	$A'_{10}=\Gamma_{31}\cup\{P(\omega^5,\omega^{25},1)\}$	$S_3=\left\langle \begin{array}{l} [[0,\omega^{10},0],[\omega^{20},0,0],[0,0,\omega^{15}]], \\ [[\omega^{10},0,0],[0,\omega^{20},0],[1,1,1]] \end{array}\right\rangle$
11	$A'_{11}=\Gamma_{31}\cup\{P(\omega^4,\omega^{27},1)\}$	$D_5=\left\langle \begin{array}{l} r'=[[\omega^{26},0,0],[1,1,1],[\omega^{19},\omega^{11},\omega^{15}]], \\ s'=[[0,\omega^7,0],[\omega^{19},\omega^{11},\omega^{15}],[1,1,1]] \end{array}\right\rangle$

Collinearities of the Diagonal Points of Pentastigm.

Definition 3.1[3]: An n-stigm in $PG(2,q)$ is a set of n points, no three of which are collinear, together with the $\frac{1}{2}n(n-1)$ lines that are joins of pairs of the points. The points and lines are called vertices and sides of the n-stigm. The vertices form an n-arc. A 5-stigm is also called pentastigm.

The diagonal points of an n-stigm are the intersections of two sides which do not pass through the same vertex.

In general, any 5-arc has 15 diagonal points since

$$\frac{1}{2}\binom{5}{2}\binom{3}{2}=15,$$

and these points are exactly the fifteen points of index two.
Write $ij\cdot kl$ for $P_iP_j\cap P_kP_l$.

Let $P_0=\mathrm{U}_0$, $P_1=\mathrm{U}_1$, $P_2=\mathrm{U}_2$, $P_3=\mathrm{U}$, $P_4=\mathrm{P}(a_0,a_1,a_2)$ be the vertices of a pentastigm ρ. So, the following equation is satisfied:

$$a_0a_1a_2(a_0-a_1)(a_0-a_2)(a_1-a_2)\neq 0.$$

In this section, the inequivalent 5-arc that has a five diagonal points is found in $PG(2,q)$, $q=29,\ 31$.

Lemma 3.2[1]: The condition that five diagonal points of a pentastigm ρ are collinear in $PG(2,q)$ is that $x^2-x-1=0$ has solution in F_q.

Corollary 3.3:
(i) If $q=29$, the equation $x^2-x-1=0$ has two solutions $6,\ -5$.

(ii) If $q=31$, the equation $x^2-x-1=0$ has two solutions $13,\ -12$.

So, there is a Pentastigm with five collinear points in $PG(2,29)$ and $PG(2,31)$.

Theorem 3.4: (i) In $PG(2,29)$, the pentastigm which has the 5-arc A_{10} as vertices has five diagonal points which are collinear on the line $\ell_{627}=\mathrm{V}(X_0-X_1-5X_2)$ as shown below.

(1) $01\cdot 23=\mathrm{P}(1,1,0)$, (6) $02\cdot 34=\mathrm{P}(v^{16},0,1)$, (11) $04\cdot 13=\mathrm{P}(1,v^{16},1)$,

(2) $01\cdot 24=\mathrm{P}(v^{6},1,0)$, (7) $03\cdot 12=\mathrm{P}(0,1,1)$, (12) $04\cdot 23=\mathrm{P}(v^{16},v^{16},1)$,

(3) $01\cdot 34=\mathrm{P}(v^{22},1,0)$, (8) $03\cdot 14=\mathrm{P}(v^{22},1,1)$, (13) $12\cdot 34=\mathrm{P}(0,v^{8},1)$,

(4) $02\cdot 13=\mathrm{P}(1,0,1)$, (9) $03\cdot 24=\mathrm{P}(v^{6},1,1)$, (14) $13\cdot 24=\mathrm{P}(1,v^{22},1)$,

(5) $02\cdot 14=\mathrm{P}(v^{22},0,1)$, (10) $04\cdot 12=\mathrm{P}(0,v^{16},1)$, (15) $14\cdot 23=\mathrm{P}(v^{22},v^{22},1)$.

Amongst these diagonal points, the five diagonal points

$01\cdot 23=\mathrm{P}(1,1,0)$,
$02\cdot 14=\mathrm{P}(v^{22},0,1)$,
$03\cdot 24=\mathrm{P}(v^{6},1,1)$,
$04\cdot 13=\mathrm{P}(1,v^{16},1)$,
$12\cdot 34=\mathrm{P}(0,v^{8},1)$,

lie on the line $\ell_{627} = \mathrm{V}(X_0 - X_1 - 5X_2)$.

(ii) In $PG(2,31)$, the pentastigm which has the 5-arc A'_{11} as vertices has five diagonal points which are collinear on the line $\ell_{379} = \mathrm{V}(X_0 + 19X_1 + 12X_2)$ as shown below.

(1) $01\cdot 23 = \mathrm{P}(1,1,0)$, (6) $02\cdot 34 = \mathrm{P}(\omega^{25},0,1)$, (11) $04\cdot 13 = \mathrm{P}(1,\omega^{26},1)$,

(2) $01\cdot 24 = \mathrm{P}(\omega^{8},1,0)$, (7) $03\cdot 12 = \mathrm{P}(0,1,1)$, (12) $04\cdot 23 = \mathrm{P}(\omega^{26},\omega^{26},1)$,

(3) $01\cdot 34 = \mathrm{P}(\omega^{19},1,0)$, (8) $03\cdot 14 = \mathrm{P}(\omega^{4},1,1)$, (13) $12\cdot 34 = \mathrm{P}(0,\omega^{4},1)$,

(4) $02\cdot 13 = \mathrm{P}(1,0,1)$, (9) $03\cdot 24 = \mathrm{P}(\omega^{8},1,1)$, (14) $13\cdot 24 = \mathrm{P}(1,\omega^{22},1)$,

(5) $02\cdot 14 = \mathrm{P}(\omega^{4},0,1)$, (10) $04\cdot 12 = \mathrm{P}(0,\omega^{26},1)$, (15) $14\cdot 23 = \mathrm{P}(\omega^{4},\omega^{4},1)$.

Amongst these diagonal points, the five diagonal points

$01\cdot 34 = \mathrm{P}(\omega^{19},1,0)$,
$02\cdot 14 = \mathrm{P}(\omega^{4},0,1)$,
$03\cdot 12 = \mathrm{P}(0,0,1)$,
$04\cdot 23 = \mathrm{P}(\omega^{26},\omega^{26},1)$,
$13\cdot 24 = \mathrm{P}(1,\omega^{22},1)$,

lie on the line $\ell_{379} = \mathrm{V}(X_0 + 19X_1 + 12X_2)$.

The Group Action of D_5 on the 5-Arc

In this section, the group action of D_5 on the 5-arc A_{10} in $PG(2,29)$ and on the 5-arc A'_{11} in $PG(2,31)$ has been studied.

(I) When $q = 29$.

From Table 1, the Dihedral group D_5 generated by

$$r = \begin{bmatrix} v^8 & 0 & 0 \\ 0 & 0 & v^{14} \\ 0 & v^2 & 0 \end{bmatrix}, \quad s = \begin{bmatrix} 0 & v^8 & 0 \\ 1 & 1 & 1 \\ v^2 & 0 & 0 \end{bmatrix}$$

is the stabilizer group of the 5-arc A_{10}. The effects of the group D_5 on the projective plane $PG(2,29)$ are given below.

1. The group D_5 fixes the conic $C_{A_{10}}$.
2. The group D_5 acts transitively on A_{10} since

$$(\mathrm{U}_0, rs) \mapsto \mathrm{U}_1,$$
$$(\mathrm{U}_0, rs^4) \mapsto \mathrm{U}_2,$$

$$(\mathrm{U}_0, rs^2) \mapsto \mathrm{U},$$
$$(\mathrm{U}_0, rs^3) \mapsto \mathrm{P}(v^{22}, v^{16}, 1).$$

3. Each of the five projectivities r, rs, rs^2, rs^3, rs^4 fixes 25 points amongst the 601 points of index zero by transforming each point to itself.
4. Each of these 25 points lies on a line which is a unisecant to A_{10} and a bisecant of the conic

$C_{A_{10}} = X_0X_1 + 5X_0X_2 - 6X_1X_2$.

These lines are

$\ell_{244} = \mathrm{V}(5X_2 - X_1);$
$\ell_{790} = \mathrm{V}(X_0 + X_1 - X_2);$
$\ell_{659} = \mathrm{V}(X_0 + 4X_2);$
$\ell_{72} = \mathrm{V}(X_0 - 5X_1);$
$\ell_{422} = \mathrm{V}(X_0 - 6X_1 + 5X_2).$

5. Each of the five projectivities r, rs, rs^2, rs^3, rs^4 fixes 31 points of $PG(2,29)$ by transforming each point to itself. These points are exactly the following:

$\ell_{244} \cup \{\mathrm{P}(0, v^8, 1)\};$
$\ell_{790} \cup \{\mathrm{P}(1,1,0)\};$
$\ell_{659} \cup \{\mathrm{P}(v^6, 1, 1)\};$
$\ell_{72} \cup \{\mathrm{P}(1, v^{16}, 1)\};$
$\ell_{422} \cup \{\mathrm{P}(v^{22}, 0, 1)\}.$

Table 3: Points of index zero fixed by elements of D_5 in $PG(2,29)$

		$\ell_i \setminus \{P_1, P_2, P_3, P_4, P_5\}$
1	r	$\ell_{244} \setminus \{\mathrm{P}(1,0,0), \mathrm{P}(0,v^{22},1), \mathrm{P}(v^{17},v^{22},1), \mathrm{P}(0,v^{22},1), \mathrm{P}(v^{22},v^{22},1)\}$
2	rs	$\ell_{790} \setminus \{\mathrm{P}(v^{14},1,0), \mathrm{P}(\omega^{22},\omega^{16},1), \mathrm{P}(1,0,1), \mathrm{P}(v^{27},v^{27},1), \mathrm{P}(0,1,1)\}$
3	rs^2	$\ell_{659} \setminus \{\mathrm{P}(0,1,0), \mathrm{P}(v^{16},0,1), \mathrm{P}(v^{16},v^{16},1), \mathrm{P}(v^{16},1,1), \mathrm{P}(v^{16},v^{10},1)\}$
4	rs^3	$\ell_{72} \setminus \{\mathrm{P}(0,0,1), \mathrm{P}(v^{10},v^{16},1), \mathrm{P}(1,v^6,1), \mathrm{P}(v^{22},1,0), \mathrm{P}(v^{22},1,1)\}$
5	rs^4	$\ell_{422} \setminus \{\mathrm{P}(1,1,1), \mathrm{P}(0,v^{16},1), \mathrm{P}(v^6,1,0), \mathrm{P}(v^{22},v^{17},1), \mathrm{P}(v^8,0,1)\}$

6. These additional points to the lines are exactly the diagonal points of 5-arc A_{10}.
7. The other four projectivities fix only one point $\mathrm{P}(v^{16}, v^{22}, 1)$ which is the point intersection of the five lines $\ell_{244}, \ell_{790}, \ell_{659}, \ell_{72}, \ell_{422}$.

(II) When $q = 31$.

From Table 2, the Dihedral group D_5 generated by

$$r' = \begin{bmatrix} \omega^{26} & 0 & 0 \\ 1 & 1 & 1 \\ \omega^{19} & \omega^{11} & \omega^{15} \end{bmatrix}, \quad s' = \begin{bmatrix} 0 & \omega^{7} & 0 \\ \omega^{19} & \omega^{11} & \omega^{15} \\ 1 & 1 & 1 \end{bmatrix}$$

is the stabilizer group of the 5-arc A'_{11}. The effects of the group D_5 on the projective plane $PG(2,31)$ are given below.

1. The group D_5 fixes the conic $C_{A'_{11}}$.
2. The group D_5 acts transitively on A'_{11} since

$$(\mathrm{U}_0, r's') \mapsto \mathrm{U}_1,$$
$$(\mathrm{U}_0, r's'^3) \mapsto \mathrm{U}_2,$$
$$(\mathrm{U}_0, r's'^4) \mapsto \mathrm{U},$$
$$(\mathrm{U}_0, s'^2) \mapsto \mathrm{P}(\omega^4, \omega^{27}, 1).$$

3. Each of the five projectivities $r', r's', r's'^2, r's'^3, r's'^4$ fixes 27 points amongst the 703 points of index zero by transforming each point to itself.
4. Each of these 27 points lies on a line which is a unisecant to A'_{11} and a bisecant of the conic
$C_{A'_{11}} = X_0X_1 + 17X_0X_2 - 18X_1X_2 = \{\mathrm{P}(-7(t^2 - 7t), -5(1-9t), \mathrm{t}) \mid t \in \mathrm{F}^*_{31}\}$.
These lines are
$\ell'_{643} = \mathrm{V}(19X_2 - X_1);$
$\ell'_{927} = \mathrm{V}(5X_0 - 2X_1);$
$\ell'_{29} = \mathrm{V}(X_0 + 11X_2);$
$\ell'_{900} = \mathrm{V}(X_0 + 11X_1 + 19X_2);$
$\ell'_{757} = \mathrm{V}(X_0 - X_1 - X_2).$

Table 4: Points of index zero fixed by elements of D_5 in $PG(2,31)$

		$\ell'_i \setminus \{P_1, P_2, P_3, P_4, P_5\}$
1	r'	$\ell'_{643} \setminus \{\mathrm{P}(1,0,0), \mathrm{P}(0,\omega^4,1), \mathrm{P}(\omega^4,\omega^4,1), \mathrm{P}(\omega^{12},\omega^4,1), \mathrm{P}(1,\omega^4,1)\}$
2	$r's'$	$\ell'_{927} \setminus \{\mathrm{P}(0,0,1), \mathrm{P}(\omega^{14},\omega^{10},1), \mathrm{P}(\omega^4,1,0), \mathrm{P}(1,\omega^{26},1), \mathrm{P}(\omega^4,1,1)\}$
3	$r's'^2$	$\ell'_{29} \setminus \{\mathrm{P}(0,1,0), \mathrm{P}(\omega^8,\omega^8,1), \mathrm{P}(\omega^8,1,1), \mathrm{P}(\omega^8,\omega^{26},1), \mathrm{P}(\omega^8,0,1)\}$
4	$r's'^3$	$\ell'_{900} \setminus \{\mathrm{P}(\omega^8,1,0), \mathrm{P}(1,1,1), \mathrm{P}(\omega^4,\omega^{20},1), \mathrm{P}(\omega^{19},0,1), \mathrm{P}(0,\omega^{26},1)\}$
5	$r's'^4$	$\ell'_{757} \setminus \{\mathrm{P}(\omega^4,\omega^{26},1), \mathrm{P}(0,\omega^{15},1), \mathrm{P}(1,0,1), \mathrm{P}(1,1,0), \mathrm{P}(\omega^{24},1,1)\}$

5. Each of the five projectivities $r', r's', r's'^2, r's'^3, r's'^4$ fixes 33 points of $PG(2,31)$ by transforming each point to itself. These points are exactly the following:

$\ell'_{643} \cup \{\mathrm{P}(1,\omega^{22},1)\};$
$\ell'_{927} \cup \{\mathrm{P}(\omega^{19},1,0)\};$
$\ell'_{29} \cup \{\mathrm{P}(\omega^{26},\omega^{26},1)\};$
$\ell'_{900} \cup \{\mathrm{P}(\omega^4,0,1)\};$
$\ell'_{757} \cup \{\mathrm{P}(0,1,1)\}.$

6. These additional points to the lines are exactly the diagonal points of 5-arc A'_{11}.

7. The other four projectivities fix three points $P(\omega^{27},\omega^{29},1)$, $P(\omega^{3},\omega^{23},1)$, $P(\omega^{8},\omega^{4},1)$ which $P(\omega^{8},\omega^{4},1)$ is the intersection point of the five lines ℓ'_{643}, ℓ'_{927}, ℓ'_{29}, ℓ'_{900}, ℓ'_{757}.

Unique Inequivalent Arc of Degree Six With Stabilizer of Type A_5

In this section, the unique 6-arc K through the frame which has a stabilizer group $G(K)$ isomorphic to A_5 is found. Also, the effect of A_5 on the $PG(2,q)$, $q = 29,31$ is discussed.

Let $K = \{P_1 = U_0, P_2 = U_1, P_3 = U_2, P_4 = U, P_5 = P(a,b,1), P_6 = P(c,d,1)\}$. There are fifteen ways of choosing three bisecants no two of which intersect on K. These three bisecants form either a triangle or will intersect at a B-point.

(I) When $q = 29$.

From the 5-arc A_{10}, an arc of degree six β_{29} is constructed by adding $P(v^{14},v^{8},1)$ of index zero; that is, $\beta_{29} = A_{10} \cup \{P(v^{14},v^{8},1)\}$.

This arc has the following properties:

1- This arc has parameters $[c_0,c_1,c_2,c_3] = [480,360,15,10]$. Since $c_0 \neq 0$, so β_{29} is not complete arc.

2- The stabilizer group of β_{29} is of type A_5 as given below

$$G(\beta_{29}) = \langle g,h \mid g^2 = h^3 = (gh)^5 = 1\rangle,$$

where

$$g = \begin{bmatrix} v^{14} & 0 & 0 \\ 0 & v^{14} & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & v^{22} & 0 \\ 0 & 0 & 1 \\ v^{16} & 0 & 0 \end{bmatrix}.$$

3- The group $G(\beta_{29})$ has a subgroup of type D_5 generated by α_1,α_2 fixes the conic

$$C_{A_{10}} = X_0X_1 + 5X_0X_2 - 6X_1X_2 = \{P(4(t^2+t),-13(1+t),-9t) \mid t \in F_{29}^*\},$$

where

$$\alpha_1 = \begin{bmatrix} v^{8} & 0 & 0 \\ 0 & 0 & v^{14} \\ 0 & v^{2} & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & v^{8} & 0 \\ 1 & 1 & 1 \\ v^{2} & 0 & 0 \end{bmatrix}, \quad \alpha_1^{\,2} = \alpha_2^{\,5} = 1.$$

4-The calculation shows that the parameter $c_3 = 10$, so β_{29} has ten B-Points as given below.

(1) $12\cdot 35\cdot 46 = P(v^{6},1,0)$; (6) $14\cdot 25\cdot 36 = P(v^{22},1,1)$;

(2) $12\cdot 36\cdot 45 = P(v^{22},1,0)$; (7) $15\cdot 23\cdot 46 = P(0,v^{16},1)$;

(3) $13\cdot 24\cdot 56 = P(1,0,1)$; (8) $15\cdot 26\cdot 34 = P(v^{16},v^{16},1)$;

(4) $13\cdot 26\cdot 45 = P(v^{16},0,1)$; (9) $16\cdot 24\cdot 35 = P(1,v^{22},1)$;

(5) $14\cdot 23\cdot 56 = P(0,0,1)$; (10) $16\cdot 25\cdot 34 = P(v^{22},v^{22},1)$.

5-The set $K = \{P(v^{6},1,0), P(v^{22},1,0), P(1,0,1), P(v^{16},0,1), P(0,0,1), P(v^{22},1,1), P(0,v^{16},1), P(v^{16},v^{16},1), P(1,v^{22},1), P(v^{22},v^{22},1)\}$ of B-Points of β_{29} form a non complete 10-arc with parameters
$[c_0,c_1,c_2,c_3,c_4,c_5]=[60,480,210,90,15,6]$.
The stabilizer group of K is isomorphic to A_5.

8. The remaining five possibilities form triangles. In Table 5, the sides of these triangles and their vertices are given.

Table 5: Five triangles fixed by A_5 in $PG(2,29)$

(I)	(II)	(III)	(IV)	(V)
$P_1P_2 = V(X_2)$	$P_1P_3 = V(X_1)$	$P_1P_4 = V(X_1 - X_2)$	$P_1P_5 = V(X_1 + 4X_2)$	$P_1P_6 = V(5X_2 - X_1)$
$P_3P_4 = V(X_0 - X_1)$	$P_2P_5 = V(X_0 - 5X_2)$	$P_2P_6 = V(X_0 + 4X_2)$	$P_2P_4 = V(X_0 - X_2)$	$P_2P_3 = V(X_0)$
$P_5P_6 = V(X_0 + X_1 - X_2)$	$P_4P_6 = V(X_0 - 6X_1 + 5X_2)$	$P_3P_5 = V(5X_0 - X_1)$	$P_3P_6 = V(X_0 - 5X_1)$	$P_4P_5 = V(6X_0 - X_1 - 5X_2)$
$P(1,1,0)$	$P(v^{22},0,1)$	$P(v^{16},1,1)$	$P(1,v^{16},1)$	$P(0,v^{22},1)$
$P(v^{14},1,1)$	$P(v^{8},0,1)$	$P(v^{6},1,1)$	$P(v^{10},v^{16},1)$	$P(v^{17},v^{22},1)$
$P(v^{27},v^{27},1)$	$P(v^{22},v^{17},1)$	$P(v^{16},v^{10},1)$	$P(1,v^{6},1)$	$P(0,v^{8},1)$

7- Let $W = \{I, II, III, IV, V\}$ be the set of five triangles in Table 5. Each elements of the group $G(\beta_{29}) \cong A_5$ fixes the set W.

(II) When $q = 31$.

From the 5-arc A'_{11}, an arc of degree six β_{31} is constructed by adding $P(\omega^{8},\omega^{4},1)$ of index zero; that is, $\beta_{31} = A'_{11} \cup \{P(\omega^{8},\omega^{4},1)\}$.

This arc has the following properties:

1- β_{31} has parameters $[c_0,c_1,c_2,c_3]=[572,390,15,10]$. Since $c_0 \neq 0$, so β_{31} is not complete arc.

2- The stabilizer group of β_{31} is of type A_5 as given below

$G(\beta_{31}) = \langle g', h' \mid g'^2 = h'^3 = (g'h')^5 = 1 \rangle$,

where

$$g' = \begin{bmatrix} \omega^{16} & 0 & 0 \\ 0 & \omega^{16} & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad h' = \begin{bmatrix} 0 & \omega^{20} & 0 \\ 0 & 0 & 1 \\ \omega^{10} & 0 & 0 \end{bmatrix}.$$

3-The identity subgroup of the group $G(\beta_{31})$ fixes the conic

$C_{A'_{11}} = X_0X_1 + 17X_0X_2 - 18X_1X_2$.

4-The calculation show that the parameter $c_3 = 10$, so β_{31} has ten B-Points as given below.

(1) $12 \cdot 34 \cdot 56 = \mathrm{P}(1,1,0)$; (6) $14 \cdot 26 \cdot 35 = \mathrm{P}(\omega^8,1,1)$;

(2) $12 \cdot 35 \cdot 46 = \mathrm{P}(\omega^8,1,0)$; (7) $15 \cdot 23 \cdot 46 = \mathrm{P}(0,\omega^{26},1)$;

(3) $13 \cdot 24 \cdot 56 = \mathrm{P}(1,0,1)$; (8) $15 \cdot 24 \cdot 36 = \mathrm{P}(1,\omega^{26},1)$;

(4) $13 \cdot 26 \cdot 45 = \mathrm{P}(\omega^8,0,1)$; (9) $16 \cdot 23 \cdot 45 = \mathrm{P}(0,\omega^4,1)$;

(5) $14 \cdot 26 \cdot 36 = \mathrm{P}(\omega^4,1,1)$; (10) $16 \cdot 25 \cdot 34 = \mathrm{P}(\omega^4,\omega^4,1)$.

5-The set $\mathrm{K}' = \{\mathrm{P}(1,1,0), \mathrm{P}(\omega^8,1,0), \mathrm{P}(1,0,1), \mathrm{P}(\omega^8,0,1), \mathrm{P}(\omega^4,1,1), \mathrm{P}(\omega^8,1,1), \mathrm{P}(0,\omega^{26},1), \mathrm{P}(1,\omega^{26},1), \mathrm{P}(0,\omega^4,1), \mathrm{P}(\omega^4,\omega^4,1)\}$ of B-Points of β_{31} form a non complete 10-arc with parameters

$$[c_0, c_1, c_2, c_3, c_4, c_5] = [60, 480, 210, 90, 15, 6].$$

The stabilizer group of K' is isomorphic to A_5.

8. The remaining five possibilities form triangles. In Table 6, the sides of these triangles and their vertices are given.

Table 6: Five triangles fixed by A_5 in $PG(2,31)$

(I)	(II)	(III)	(IV)	(V)
$P_1P_2 = \mathrm{V}(X_2)$	$P_1P_3 = \mathrm{V}(X_1)$	$P_1P_4 = \mathrm{V}(X_1 - X_2)$	$P_1P_5 = \mathrm{V}(18X_2 - X_1)$	$P_1P_6 = \mathrm{V}(19X_2 - X_1)$
$P_3P_6 = \mathrm{V}(X_0 + 12X_1)$	$P_2P_5 = \mathrm{V}(X_0 - 19X_2)$	$P_3P_3 = \mathrm{V}(X_0)$	$P_2P_6 = \mathrm{V}(X_0 - 20X_2)$	$P_2P_4 = \mathrm{V}(X_0 - X_2)$
$P_4P_5 = \mathrm{V}(X_0 - 12X_1 + 11X_2)$	$P_4P_6 = \mathrm{V}(X_0 + 11X_1 + 19X_2)$	$P_5P_6 = \mathrm{V}(X_0 - X_1 - X_2)$	$P_3P_4 = \mathrm{V}(X_0 - X_1)$	$P_3P_5 = \mathrm{V}(X_0 + 11X_1)$

$P(\omega^4,1,0)$ $P(\omega^{19},0,1)$ $P(\omega^{14},\omega^{10},1)$	$P(\omega^4,0,1)$ $P(\omega^{19},0,1)$ $P(\omega^4,\omega^{20},1)$	$P(0,1,1)$ $P(\omega^{24},1,1)$ $P(0,\omega^{15},1)$	$P(\omega^8,\omega^{26},1)$ $P(\omega^{26},\omega^{26},1)$ $P(\omega^8,\omega^8,1)$	$P(1,\omega^4,1)$ $P(\omega^{12},\omega^4,1)$ $P(1,\omega^{22},1)$

9. Let $W'=\{I,II,III,IV,V\}$ be the set of five triangles in Table 6. Each elements of the group $G(\beta_{31})\cong A_5$ fixes the set W'.

Conclusion

1- There is an arc of degree five $\xi=\{P_1,P_2,P_3,P_4,P_5\}$ which has stabilizer group $G(\xi)$ of type D_5.

2- The pentastigm which has ξ as a vertex has collinear diagonal points.

3- The effect of the group $G(\xi)$ on points of $PG(2,q)$, $q=29,31$ depends on the order of its elements. Let G^2 be the set of five elements of $G(\xi)$ of order two and G^5 be the set of four elements of $G(\xi)$ of order five.

(i) Each element of G^2 fixes five a subset of the plane of length $q+2$ by sending it to itself. Each of this set, is a line ℓ_i^* with extra point P_i^*, $i=1,2,3,4,5$. The five extra points P_i' are exactly the diagonal points of ξ. Also, these lines are the bisecant to the conic C_ξ which passes through ξ and unisecants to ξ.

(ii) Each element of G^5 fixes a point $\mathbf{P}^*$ which is the intersection point of the five lines $\ell_i^*, i=1,2,3,4,5$.

4- The unique six arc with stabilizer group of type A_5 is constructed by adding the point $\mathbf{P}^*$ to ξ. So, the following figure is fixed by the group $G(\xi)$.

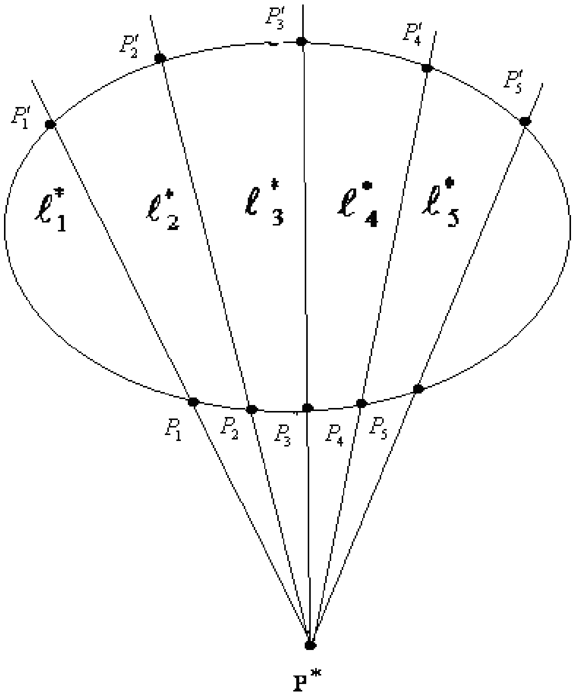

Figure 1.

According to these results and Storm and Maldeghem results [4, proposition 12] the following conjecture is deduced.

Conjecture: In $PG(2,q)$, when $q \equiv \pm 1 \mod(10)$ there is a unique arc of degree five ξ fixed by group $G(\xi)$ of type D_5 and there is a unique arc of degree six consists of ξ and a point P^* which is fixed by the elements of $G(\xi)$ of degree five. And the group $G(\xi)$ fixed the Figure 1.

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تاثير الزمر من الأنواع D_5 و A_5 في نقاط المستوي الأسقاطي في F_q ، $[q = 29,31]$

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الخلاصة

الغرض من هذا البحث أيجاد قوس من الدرجة الخامسة في المستوي الأسقاطي $PG(2,q)$, $q = 29,31$ ، ذات زمرة مثبتة من النوع زمرة داهيدرل من الدرجة الخامسة D_5 و اقواس من الدرجة السادسة والعاشرة ذات زمرة مثبتة من نوع الزمرة المتناوبة من الدرجة الخامسة A_5 ومن ثم دراسة تأثير D_5 و A_5 في نقاط المستوي الأسقاطي . وكذلك، ايجاد بنتاستام ذات نقاط قطرية على استقامة واحدة.

الكلمات المفتاحية: المستوي الاسقاطي، القوس.