Repeated Corrected Simpson's 3/8 Quadrature Method for Solving Fredholm Linear Integral Equations of the Second Kind

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Abstract

In this paper, we use the repeated corrected Simpson's 3/8 quadrature method for obtaining the numerical solutions of Fredholm linear integral equations of the second kind. This method is more accurately than the repeated corrected Trapezoidal method and the repeated Simpson's 3/8 method. To illustrate the accuracy of this method, we give a numerical example.

Keywords : Fredholm integral equations, repeated corrected Simpson's 3/8 method, numerical approximation.

Introduction

Recall that the following linear integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_a^{b(x)} k(x,y)u(y)dy, \qquad \ldots(1.1)$$

where $a \le x \le b$, λ is areal number, f, k are given continuous functions and u is the unknown function that must be determined. If b(x) = b then equation (1.1) is called Fredholm linear integral equation of the second kind. Also, if b(x) = x then equation (1.1) is called Volterra linear integral equation of the second kind.

The mathematical modeling for many problems in different disciplines, such as engineering, chemistry, physics and biology leads to the above form of integral equations. Most differential equations can be expressed as integral equations, [1], [2]. There are some analytical methods for solving integral equations, [3], [4].

There are several numerical methods for solving the integral equations given by (1.1). For example, Repeated modified Trapezoid quadrature method [5], Runge-Kutta method [6], Taylor-series method [7], expansion methods [8] and decompostion method [9], [10].

In this paper, we use the repeated corrected Simpson's 3/8 method to find the numerical solution for (1.1) in case b(x) = b, where b is a known constant. To do this, recall that the repeated corrected Simpson's 3/8 method, [11]:

$$\int_a^b f(x)\,dx = \frac{3h\lambda}{224}[31f_0 + 81(f_1+f_2)+31f_3] + \frac{3h^2\lambda}{1120}[19f_0' + 27(f_2'-f_1') - 19f_3'] + \frac{-81(b-a)h^8}{4480} f^{(8)}(\xi), a \le \xi \le b \quad \ldots(1.2)$$

where $f_0 = f(a)$, $f_1 = f(a+h)$, $f_2 = f(a+2h)$, $f_3 = f(b)$, $f_0' = f'(a)$, $f_1' = f'(a+h)$, $f_2' = f'(a+2h)$ and $f_3' = f'(b)$.

Solution of Fredholm Linear Integral Equations of the second kind

In this section we use repeated corrected Simpson's 3/8 method to find the numerical solutions of the following integral equation:

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(y)dy, \qquad \ldots(2.1)$$

To do this, we approximate the integral part that appeared in the right-hand side of equation (2.1) by the repeated corrected Simpson's 3/8 rule to get

$$u(x) = f(x) + \frac{3h\lambda}{224}\Bigg[31k(x,x_0)u_0 + 81\sum_{j=1,4,7,\ldots}^{n-2}\left[k(x,x_j)u_j + k(x,x_{j+1})u_{j+1}\right] + 62\sum_{j=3,6,\ldots}^{n-3}\left[k(x,x_j)u_j\right] + 31k(x,x_n)u_n\Bigg] + \frac{3h^2\lambda}{1120}\Bigg[19k(x,x_0)u_0' + 19J(x,x_0)u_0 + 27\sum_{j=1,4,7}^{n-2}\big[k(x,x_{j+1})u_{j+1}' + J(x,x_{j+1})u_{j+1} - k(x,x_j)u_j' - J(x,x_j)u_j\big] - 19k(x,x_n)u_n' - 19J(x,x_n)u_n\Bigg] \qquad \ldots(2.2)$$

where $J(x,y) = \dfrac{\partial k(x,y)}{\partial y}$. Hence for $x = x_0, x_1, \ldots, x_n$, we get the following system equations:

$$u_i = f_i + \frac{3h\lambda}{224}\left[31k_{i,0}u_0 + 81\sum_{j=1,4,7,\dots}^{n-2}\left[k_{i,j}u_j + k_{i,j+1}u_{j+1}\right] + 62\sum_{j=3,6,\dots}^{n-3}\left[k_{i,j}u_j\right] + 31k_{i,n}u_n\right] +$$
$$\frac{3h^2\lambda}{1120}\left[19k_{i,0}u'_0 + 19J_{i,0}u_0 + 27\sum_{j=1,4,7}^{n-2}\left[k_{i,j+1}u'_{j+1} + J_{i,j+1}u_{j+1} - k_{i,j}u'_j - J_{i,j}u_j\right] - 19k_{i,n}u'_n - 19J_{i,n}u_n\right], \quad i = 0,1,\dots,n \qquad \dots(2.3)$$

where $u_i = u(x_i), u'_i = u'(x_i), f_i = f(x_i),\ k_{i,j} = k(x_i, y_j),\ J_{i,p} = J(x_i, x_p),\ i, j = 0,1,\dots,n,\ p = 0, n.$

From equation (2.1) and setting $H(x, y) = \frac{\partial k(x, y)}{\partial x}$ we obtain

$$u'(x) = f'(x) + \lambda\int_a^b H(x, y)u(y)dy, a \le x \le b \qquad \dots(2.4)$$

We note that if u is the solution of (2.1) it is a solution of (2.4) too. Now, for solving equation (2.4), we must consider two cases.

Case (1):

The partial derivative $L(x,y) = \frac{\partial k(x,y)}{\partial x \partial y}$ does not exists, in this case, we approximate the integral part that appeared in the right hand side of equation (2.4) with the repeated Simpson's 3/8 rule to obtain:

$$u'(x) = f'(x) + \frac{3h\lambda}{8}\left[H(x, x_0)u_0 + 3\sum_{j=1}^{\frac{n}{3}}\left[H(x, x_{3j-1})u_{3j-1} + H(x, x_{3j-2})u_{3j-2}\right] + 2\sum_{j=1}^{\frac{n}{3}-1}\left[H(x_i, x_{3j})u_{3j}\right] + H(x, x_n)u_n\right] \qquad \dots(2.5)$$

By setting $x = x_0, x_1, \dots, x_n$, in (2.5) one can get:

$$u'_i = f'_i + \frac{3h\lambda}{8}\left[H_{i,0}u_0 + 3\sum_{j=1}^{\frac{n}{3}}\left[H_{i,3j-1}u_{3j-1} + H_{i,3j-2}u_{3j-2}\right] + 2\sum_{j=1}^{\frac{n}{3}-1}\left[H_{i,3j}u_{3j}\right] + H_{i,n}u_n\right] \qquad \dots(2.6)$$

From equations (2.6) and (2.3) one can get the following system which consists of $2n - \frac{n}{3} + 3$ equations with $2n - \frac{n}{3} + 3$ unknowns u_i, $i = 0,1,\dots,n, u'_0, u'_p$ and for i=0, p, n,(p = 1,2,...,n-1)

$$\begin{cases} u_i = f_i + \frac{3h\lambda}{224}\left[31k_{i,0}u_0 + 81\sum_{j=1,4,7,...}^{n-2}\left[k_{i,j}u_j + k_{i,j+1}u_{j+1}\right] + 62\sum_{j=3,6,...}^{n-3}\left[k_{i,j}u_j\right] + 31k_{i,n}u_n\right] + \frac{3h^2\lambda}{1120}\Big[19k_{i,0}u'_0 + 19J_{i,0}u_0 + \\ \quad 27\sum_{j=1,4,7}^{n-2}\left[k_{i,j+1}u'_{j+1} + J_{i,j+1}u_{j+1} - k_{i,j}u'_j - J_{i,j}u_j\right] - 19k_{i,n}u'_n - 19J_{i,n}u_n\Big], i = 0,1,...,n, \\ u'_0 = f'_0 + \frac{3h\lambda}{8}\left[H_{0,0}u_0 + 3\sum_{j=1}^{\frac{n}{3}}\left[H_{0,3j-1}u_{3j-1} + H_{0,3j-2}u_{3j-2}\right] + 2\sum_{j=1}^{\frac{n}{3}-1}\left[H_{0,3j}u_{3j}\right] + H_{0,n}u_n\right], \\ u'_p = f'_p + \frac{3h\lambda}{8}\left[H_{p,0}u_0 + 3\sum_{j=1}^{\frac{n}{3}}\left[H_{p,3j-1}u_{3j-1} + H_{p,3j-2}u_{3j-2}\right] + 2\sum_{j=1}^{\frac{n}{3}-1}\left[H_{p,3j}u_{3j}\right] + H_{p,n}u_n\right], p = 1,2,...,n-1, \\ u'_n = f'_n + \frac{3h\lambda}{8}\left[H_{n,0}u_0 + 3\sum_{j=1}^{\frac{n}{3}}\left[H_{n,3j-1}u_{3j-1} + H_{n,3j-2}u_{3j-2}\right] + 2\sum_{j=1}^{\frac{n}{3}-1}\left[H_{n,3j}u_{3j}\right] + H_{n,n}u_n\right]. \end{cases} \quad ...(2.7)$$

By solving the above system given by equation (2.7) the numerical solutions of equation (2.1), are obtained.

Case (2):

The partial derivative $L(x,y) = \frac{\partial k(x,y)}{\partial x \partial y}$ exists, in this case, we approximate the integral part that appeared in the right hand side of equation (2.4) with the repeated corrected Simpson's 3/8 rule to obtain:

$$u'(x) = f'(x) + \frac{3h\lambda}{224}\Big[31H(x,x_0)u_0 + 81\sum_{j=1,4,7,...}^{n-2}\left[H(x,x_j)u_j + H(x,x_{j+1})u_{j+1}\right] + 62\sum_{j=3,6,...}^{n-3}\left[H(x,x_j)u_j\right] + 31H(x,x_n)u_n\Big] + \frac{3h^2\lambda}{1120}\Big[19H(x,x_0)u'_0 + 19L(x,x_0)u_0 + 27\sum_{j=1,4,7,...}^{n-3}\left[H(x,x_{j+1})u'_{j+1} + L(x,x_{j+1})u_{j+1} - H(x,x_j)u'_j - L(x,x_j)u_j\right] - 19H(x,x_n)u'_n - 19L(x,x_n)u_n\Big] \quad ...(2.8)$$

By setting $x = x_0, x_1, ..., x_n$, in (2.8) one can get:

$$u'_i = f'_i + \frac{3h\lambda}{224}\left[31H_{i,0}u_0 + 81\sum_{j=1,4,7,...}^{n-2}\left[H_{i,j}u_j + H_{i,j+1}u_{j+1}\right] + 62\sum_{j=3,6,...}^{n-3}\left[H_{i,j}u_j\right] + 31H_{i,n}u_n\right] + \frac{3h^2\lambda}{1120}\Big[19H_{i,0}u'_0 + 19L_{i,0}u_0 + 27\sum_{j=1,4,7,...}^{n-3}\left[H_{i,j+1}u'_{j+1} + L_{i,j+1}u_{j+1} - H_{i,j}u'_j - L_{i,j}u_j\right] - 19H_{i,n}u'_n - 19L_{i,n}u_n\Big] \quad ...(2.9)$$

From equations (2.9) and (2.3) one can get the following system which consists of $2n - \frac{n}{3} + 3$ equations with $2n - \frac{n}{3} + 3$ unknowns u_i, $i = 0,1,...,n$, u'_0, u'_p and u'_n For i=0, p, n, (p = 1,2,…,n-1).

$$\begin{cases} u_i = f_i + \dfrac{3h\lambda}{224}\left[31k_{i,0}u_0 + 81\sum\limits_{j=1,4,7,...}^{n-2}\left[k_{i,j}u_j + k_{i,j+1}u_{j+1}\right] + 62\sum\limits_{j=3,6,...}^{n-3}\left[k_{i,j}u_j\right] + 31k_{i,n}u_n\right] + \dfrac{3h^2\lambda}{1120}\left[19k_{i,0}u'_0 + 19J_{i,0}u_0 + \right. \\ \left. 27\sum\limits_{j=1,4,7}^{n-2}\left[k_{i,j+1}u'_{j+1} + J_{i,j+1}u_{j+1} - k_{i,j}u'_j - J_{i,j}u_j\right] - 19k_{i,n}u'_n - 19J_{i,n}u_n\right], i = 0,1,...,n \\ u'_0 = f'_0 + \dfrac{3h\lambda}{224}\left[31H_{0,0}u_0 + 81\sum\limits_{j=1,4,7,...}^{n-2}\left[H_{0,j}u_j + H_{0,j+1}u_{j+1}\right] + 62\sum\limits_{j=3,6,...}^{n-3}H_{0,j}u_j + 31H_{0,n}u_n\right] + \dfrac{3h^2\lambda}{1120}\left[19H_{0,0}u'_0 + 19L_{0,0}u_0 + \right. \\ \left. 27\sum\limits_{j=1,4,7,...}^{n-2}\left[H_{0,j+1}u'_{j+1} + L_{0,j+1}u_{j+1} - H_{0,j}u'_j - L_{0,j}u_j\right] - 19H_{0,n}u'_n - 19L_{0,n}u_n\right] \\ u'_p = f'_p + \dfrac{3h\lambda}{224}\left[31H_{p,0}u_0 + 81\sum\limits_{j=1,4,7,...}^{n-2}\left[H_{p,j}u_j + H_{p,j+1}u_{j+1}\right] + 62\sum\limits_{j=3,6,...}^{n-3}H_{k,j}u_j + 31H_{p,n}u_n\right] + \dfrac{3h^2\lambda}{1120}\left[19H_{p,0}u'_0 + 19L_{p,0}u_0 + \right. \\ \left. 27\sum\limits_{j=1,4,7,...}^{n-2}\left[H_{p,j+1}u'_{j+1} + L_{p,j+1}u_{j+1} - H_{p,j}u'_j - L_{p,j}u_j\right] - 19H_{p,n}u'_n - 19L_{p,n}u_n\right], \text{where } p = j, j+1 \\ u'_n = f'_n + \dfrac{3h\lambda}{224}\left[31H_{n,0}u_0 + 81\sum\limits_{j=1,4,7,...}^{n-2}\left[H_{n,j}u_j + H_{n,j+1}u_{j+1}\right] + 62\sum\limits_{j=3,6,...}^{n-3}H_{n,j}u_j + 31H_{n,n}u_n\right] + \dfrac{3h^2\lambda}{1120}\left[19H_{n0}u'_0 + 19L_{n0}u_0 + \right. \\ \left. 27\sum\limits_{j=1,4,7,...}^{n-2}\left[H_{nj+1}u'_{j+1} + L_{nj+1}u_{j+1} - H_{nj}u'_j - L_{nj}u_j\right] - 19H_{nn}u'_n - 19L_{nn}u_n\right] \end{cases} \quad ...(2.10)$$

By solving the above system given by equation (2.10) the numerical solutions of equation (2.1), are obtained.

Numerical Example

In this section, we give some numerical examples to illustrate the above method for solving Fredholm linear integral equation of the second kind. Also, we compare this method with other methods such as repeated Simpson's 3/8 method and repeated corrected Trapezoidal method for solving Fredholm linear integral equation of the second kind.

Consider the following Fredholm linear integral equation of the second kind:

$$u(x) = x - \frac{2}{7}(x+1)^{\frac{7}{2}} + \frac{2}{5}(x+1)^{\frac{5}{2}}x - \frac{4}{35}x^{\frac{7}{2}} + \int_0^1 (x+y)^{\frac{3}{2}} u(y)dy,\ 0 \le x \le 1 \qquad ...(3.1)$$

where $f(x) = x - \frac{2}{7}(x+1)^{\frac{7}{2}} + \frac{2}{5}(x+1)^{\frac{5}{2}}x - \frac{4}{35}x^{\frac{7}{2}}$, $k(x,y) = (x+y)^{\frac{3}{2}}$. This example is constructed such that the exact solution is u(x) = x. It is clear that $\frac{\partial^2 k(x,y)}{\partial x \partial y} = \frac{3}{4\sqrt{(x+y)}}$ does not exist at x = y = [0, 1]. To do this, first, we divide the interval [0,1] into 9 intervals such that $x_i = \frac{i}{9}$, $i = 0,1,...,9$ Therefore, the numerical solution of system (3.1) can be obtained by equation (2.7), and illustrated in Table 1، Which contains the exact solutions and the numerical solutions by using the repeated Simpson's 3/8, repeated corrected trapezoidal and repeated corrected Simpson's 3/8 method are given for $h = 0.112$. Also, the least square error denoted by LSER and absolute error denoted by ABSER are presented in Table (1).

Second, we divide the interval [0,1] into 18 subintervals that $x_i = \frac{i}{18}$, $i = 0,1,...,18$ Therefore, the numerical solution of system (3.1) can be obtained by equation (2.7), the results are presented in Table 2. The exact solutions and the numerical solutions using the repeated Simpson's 3/8, repeated corrected trapezoidal and repeated corrected Simpson's 3/8 method for $h = 0.056$. Also, in this table, the least square error denoted by LSER and absolute error denoted by ABSER are presented.

Third, we divide the interval [0,1] into 36 subintervals that $x_i = \frac{i}{36}$, $i = 0,1,...,36$ Therefore, the numerical solution of system (3.1) can be obtained by equation (2.7), the results of which being presented in Table 3. In that table, the exact solutions and the numerical solutions by using the repeated Simpson's 3/8, repeated corrected trapezoidal and repeated corrected Simpson's 3/8 method are given for $h = 0.028$. Also, the least square error denoted by LSER and absolute error denoted by ABSER are also included in table 3.

Discussion

In this paper we proposed the repeated corrected Simpson's 3/8 method for solving Fredholm linear integral equation of the second kind. This method can be easily applied and it is efficient and accurate to estimate the solution of Fredholm linear integral equation when the function $\frac{\partial k(x,y)}{\partial x}$, $\frac{\partial k(x,y)}{\partial y}$ and $f'(x)$ exist.

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Table (1): Comparison between the solution via corrected Trapezoidal, repeated Simpson's 3/8 and corrected Simpson's 3/8 with exact solution with N = 9

Nodes	Exact Solutions	Numerical solutions u_i, N=9		
x_i	u_i	corrected Trapezoidal	repeated Simpson's 3/8	corrected Simpson's 3/8
0	0	0.00000422427391	-0.00000804493901	0.00000074026838
0.1	0.1	0.11111322328397	0.11111173431260	0.11111125469004
0.2	0.2	0.22222443564682	0.22222707233601	0.22222240727548
0.3	0.3	0.33333593424825	0.33334133788125	0.33333358046105
0.4	0.4	0.44444756925228	0.44445521086227	0.44444476277767
0.56	0.56	0.55555929602997	0.55556892483339	0.55555595228195
0.67	0.67	0.66667109471999	0.66668258127157	0.66666714815178
0.78	0.78	0.77778295432453	0.77779623077884	0.77777834986280
0.89	0.89	0.88889486785395	0.88890990091898	0.88888955702285
1	1	1.00000683043123	1.00002360760331	1.00000076931735
LSER		1.8653e-010	2.0399e-009	2.5200e-012
ABSER		4.04300e-005	1.2464e-004	4.522109e-006

Table (2): Comparison between the solution via corrected Trapezoidal, repeated Simpson's 3/8 and corrected Simpson's 3/8 with exact solution with N = 18

Nodes	Exact Solutions	Numerical solutions u_i, N=18		
x_i	u_i	corrected Trapezoidal	repeated Simpson's 3/8	corrected Simpson's 3/8
0	0	0.00000036160574	-0.00000094631322	0.00000006419735
0.1	0.1	0.11111124066062	0.11111112823436	0.11111112071068
0.2	0.2	0.22222235617522	0.22222255247321	0.22222223531051
0.3	0.3	0.33333349057657	0.33333387923624	0.33333335065382
0.4	0.4	0.44444463362786	0.44444517497656	0.44444446649763
0.56	0.56	0.55555578243406	0.55555645888978	0.55555558277799
0.67	0.67	0.66666693572924	0.66666773862347	0.66666669945537
0.78	0.78	0.77777809281810	0.77777901780483	0.77777781650046
0.89	0.89	0.88888925326072	0.88889029834788	0.88888893389011
1	1	1.00000041675099	1.00000158133359	1.00000005160523
LSER		1.30200e-012	1.75118e-011	2.0854e-014
ABSER		4.6354e-006	1.61007e-005	5.5708e-007

Table (3): Comparison between the solution via corrected Trapezoidal, repeated Simpson's 3/8 and corrected Simpson's 3/8 with exact solution with N = 36

Nodes	Exact Solutions	Numerical solutions u_i, N=36		
x_i	u_i	corrected Trapezoidal	repeated Simpson's 3/8	corrected Simpson's 3/8
0	0	0.00000003136368	– 0.00000009987493	0.00000000555524
0.1	0.1	0.11111111905472	0.11111111207258	0.11111111176558
0.2	0.2	0.22222223038488	0.22222224437474	0.22222222311676
0.3	0.3	0.33333334291492	0.33333336952570	0.33333333450532
0.4	0.4	0.44444445598728	0.44444449257726	0.44444444592461
0.56	0.56	0.55555556941910	0.55555561485323	0.55555555737127
0.67	0.67	0.66666668313083	0.66666673686136	0.66666666884296
0.78	0.78	0.77777779707885	0.77777785883792	0.77777778033788
0.89	0.89	0.88888891123562	0.88888898090618	0.88888889185461
1	1	1.00000002558200	1.00000010313524	1.00000000339195
LSER		9.53366e-015	1.4855e-013	1.71916e-016
ABSER		5.488e-007	2.0724e-006	6.96503e-008

الطريقة التربيعة سمبسون 8/3 المعدلة المتكررة لحل معادلات تكاملية خطية من نوع فريدهولم من النوع الثاني

وفاء فيصل منصور
شيماء مخلف شريدة
قسم علوم الرياضيات / كلية التربية للعلوم الصرفة (ابن الهيثم) / جامعة بغداد

استلم البحث في:28 ايار 2012 ، قبل البحث في:9 كانون الاول 2012

الخلاصة

في هذا البحث نستخدم سمبسون 8/3 المطورة لايجاد الحل العددي للمعادلات التكاملية الخطية من النوع الثاني. هذة الطريقة تحل المعادلات التكاملية الخطية بدقة أكثر من طريقة شبة المنحرف المطورة وطريقة سمبسون 8/3. لتوضيح دقة هذه الطريقة تم اعطاء مثال عددي.

الكلمات المفتاحية : معادلات فريد هولم التكاملية، طريقة سمبسون (8/3) المطورة، التقريب العددي