# Essentially Quasi-Invertible Submodules and Essentially Quasi-Dedekind Modules

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## Abstract

Let R be a commutative ring with identity . In this paper we study the concepts of essentially quasi-invertible submodules and essentially quasi-Dedekind modules as a generalization of quasi-invertible submodules and quasi-Dedekind modules . Among the results that we obtain is the following : M is an essentially quasi-Dedekind module if and only if M is aK-nonsingular module,where a module M is K-nonsingular if, for each $f \in End_R(M)$ , Kerf $\leq_e$ M implies f = 0 .

**Kew words** : Essentially quasi-invertible submodules , Essentially quasi-Dedekind Modules .

## Introduction

The concepts of a quasi-invertible submodule of an R-module and quasi-Dedekind module were introduced in [5] .Where a submodule N of an R-module M is called quasi-invertible if $Hom(M/N, M) = 0$ , and an R-module M is called quasi-Dedekind if each nonzero submodule of M is quasi-invertible . As a generalizations to these concepts we introduce the following concepts : We call a submodule N of M is essentially quasi-invertible if , N $\leq_e$ M and N is quasi-invertible .And an R-module M is called essentially quasi-Dedekind if every essential submodule N of M is quasi-invertible ; ( i.e $Hom(M/N, M) = 0$ ) . This paper consists of two sections , §1 is devoted to study essentially quasi-invertible submodules , in §2 we study and give the basic properties of essentially quasi-Dedekind modules .

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## 1 . Essentially Quasi-Invertible Submodules

In this section we introduce the concept of essentially quasi-invertible submodules. We develop basic properties of essentially quasi-invertible submodule .

We start with the following definition :

**Definition (1.1)**

Let M be an R-module and $N \leqslant_e M$ , then N is called an essentially quasi-invertible submodule of M if , $Hom(M/N, M) = 0$ ; that is N is essentially quasi-invertible if , $N \leqslant_e M$ and N is quasi-invertible . An ideal J in a ring R is called an essentially quasi-invertible ideal of R if , J is an essentially quasi-invertible R-submodule of R .

**Remarks and Examples (1.2)**

1) It is clear that every essentially quasi-invertible submodule is quasi-invertible submodule .

Recall that an R-module M is called a semisimple if every submodule of M is a direct summand of M , [3, p.189] .

2) If M is a semisimple R-module , then M is the only essentially quasi-invertible submodule of M .

3) Consider $Z_4$ as a Z-module , $N = (\bar{2}) \leqslant_e Z_4$ , but $Hom(Z_4/(\bar{2}), Z_4) \cong Z_2 \neq 0$ , so $N = (\bar{2})$ is not essentially quasi-invertible submodule of $Z_4$ , similarly in the Z-module $Z_{20}$ , $N = (\bar{2}) \leqslant_e Z_{20}$ , but it is not quasi-invertible .

4) If N is an essentially quasi-invertible R-submodule of an R-module M , then $ann_R M = ann_R N$ .

**Proof :** It is clear . □

The converse of (Rem.and.Ex. 1.2(4) ) is not true in general , for example : Let $M = Z \oplus Z$ , considered as a Z-module and let $N = Z \oplus (0) \leq M$ , then it is clear that $ann_R M = ann_R N = (0)$ , but N is not essentially quasi-invertible submodule of M , since $N \nleqslant_e M$ and also N is not quasi-invertible .

5) Let J be an ideal of a ring R . Then J is an essentially quasi- invertible if and only if $ann_R(J) = 0$ .

<u>**Proof :**</u> It is easy .

6) Let J be an ideal of a ring R . The following statements are equivalent :

a) J is an essentially quasi- invertible ideal of R .
b) J is a quasi-invertible ideal of R .
c) $ann_R(J) = 0$ .

**Proof :**

$(a) \Leftrightarrow (c)$ : It follows by ( Rem.and.Ex. 1.2(5) ) .

$(b) \Leftrightarrow (c)$ : It follows by [5 , prop. 2.2] . □

7) Let R be a ring . The following statements are equivalent :

a) R is an integral domain .
b) R is quasi-Dedekind .

**Proof :** It follows by (Rem.and.Ex. 1.2(6)) . □

8) If $M = M_1 \oplus M_2$ is an R-module , and K be an essentially quasi- invertible submodule in $M_i$ for some i= 1,2 , then it is not necessarily that K is an essentially quasi-invertible submodule of M , for example :

Let $M = Z \oplus Z_2$ as Z-module , then K = $Z_2$ is an essentially quasi- invertible submodule of $Z_2$ as Z-module , but $Z_2 \cong (0) \oplus Z_2$ which is not essentially quasi-invertible of $M = Z \oplus Z_2$ , since $(0) \oplus Z_2 \not\leqslant_e Z \oplus Z_2$.

**Proposition (1.3)**

Let M be an R-module , and let $N_1$ , $N_2$ be an essentially quasi- invertible R-submodules of M , then $N_1 \cap N_2$ is an essentially quasi-invertible R-submodule of M.

**Proof :**

Since $N_1 \leqslant_e M$ , $N_2 \leqslant_e M$ then $Hom(M/N_1, M) = 0$ and $Hom(M/N_2, M) = 0$. Also $N_1 \leqslant_e M$ , $N_2 \leqslant_e M$ imply $N_1 \cap N_2 \leqslant_e M$ . But $Hom(M/N_1 \cap N_2, M) \subseteq Hom(M/N_1, M) + Hom(M/N_2, M)$ .Hence $Hom(M/N_1 \cap N_2, M) = 0$ and so that $N_1 \cap N_2$ is an essentially quasi- invertible R-submodule of M . □

The following lemma is needed for the next proposition .

**Lemma (1.4 )**

Let M be an R-module such that for each nonzero submodule K of M , $0_p \neq K_P \leq M_P$ for each maximal ideal P of R . If $N_P \leqslant_e M_p$ implies $N \leqslant_e M$ .

**Proof :**

Suppose that there exists $0 \neq U \leq M$ such that $U \cap N = 0$ .Hence $(U \cap N)_P = 0_P$ which implies that $U_P \cap N_P = 0_P$ , but $0_p \neq U_P \leq M_P$ by hypothesis , so that $N_P \not\leqslant_e M_p$ which is a contradiction . □

**Proposition (1.5)**

Let M be an R-module , $N \leqslant M$ . If $N_P$ is an essentially quasi-invertible $R_P$ - submodule of $R_P$-module $M_P$ (for each maximal ideal P of R ) , then N is an essentially quasi-invertible submodule of an R-module M.

**Proof :**

Since $N_P$ is an essentially quasi-invertible $R_P$-submodule of $M_P$ , $Hom(M_P/N_P, M_P) = 0$ . But by [4 , Ex.3 , p.75] , $(Hom(M/N, M))_P \subseteq Hom(M_P/N_P, M_P) = 0$ , thus $(Hom(M/N, M))_P = 0$ and by [4, Prop.3.13 , p.70] , $Hom(M/N, M) = 0$ ; that is N is a quasi-invertible submodule of M . Beside this , by (Lemma (1.4 )) , $N \leqslant_e M$ . Thus N is an essentially quasi-invertible submodule of M . □



Recall that an R-submodule N of an R-module M is called a SQI-submodule if , for each $f \in Hom(M/N, M)$ , f(M/N) is a small submodule in M , [6 , p.44] . And an R-submodule N of an R-module M is called a small submodule of M ( $N \ll M$ , for short ) if , for all $K \leqslant M$ with N+K = M implies K = M, [3, P.106] .

**Remark (1.6 )**

It is clear that every quasi-invertible submodule is an SQI-submodule and hence every essentially quasi-invertible submodule is an SQI-submodule .

The converse of (Remark 1.6 ) is not true in general , consider the following example .

**Example (1.7 )**

Consider the Z-module $Z_4$ , $N = (\bar{2})$ , then N is an SQI-submodule of $Z_4$, since for all $f \in Hom(Z_4/(\bar{2}), Z_4)$ , then $f(Z_4/(\bar{2}) \lneqq Z_4$ , and every proper submodule of $Z_4$ is a small in $Z_4$ , so $f(Z_4/(\bar{2}) \ll Z_4$ , but it is known that $N = (\bar{2})$ is not essentially quasi-invertible in $Z_4$ ,( see Rem.and.Ex. 1.2(3)) .

## 2 . Essentially Quasi-Dedekind Modules

In this section we give the definition of essentially quasi-Dedekind module with some examples . We prove that essentially quasi-Dedekind module and K-nonsingular module which is introduced by [8] are equivalent .We give conditions under which submodule (resp. quotient module) of essentially quasi-Dedekind is essentially quasi-Dedekind .

**Definition (2.1)**

An R-module M is called essentially quasi-Dedekind if , $Hom(M/N, M) = 0$ for all $N \leq_e M$ . A ring R is essentially quasi-Dedekind if R is an essentially quasi-Dedekind R-module .

**Remarks and Examples (2.2)**

1) It is clear that every quasi-Dedekind module is an essentially quasi- Dedekind module, but the converse is not true in general , for example :
   Each of $Z_{10}$ , $Z_{15}$ are essentially quasi-Dedekind as a Z-module , but it is not quasi-Dedekind .
2) Every integral domain R is an essentially quasi-Dedekind R-module, by [5 ,Ex 1.4 , p.24] and (Rem.and.Ex 2.2(1)) .
3) $Z_4$ as a Z-module is not essentially quasi-Dedekind , since $(\bar{2}) \leq_e Z_4$ ,
   but $Hom(Z_4/(\bar{2}), Z_4) \cong Z_2 \neq 0$ .
4) Let $M = Z_{p^\infty}$ as a Z-module . Then M is not essentially quasi- Dedekind , but $End_Z(M)$ ( is the ring of P-adic integers) is a commutative domain [see Ex 4.1.2 ,8] , so $End_Z(M)$ is essentially quasi-Dedekind , by (Rem.and.Ex 2.2(2)) .
5) Let M be a uniform R-module . Then M is a quasi-Dedekind R-module if and only if M is an essentially quasi-Dedekind R-module .

**Proof :** It is clear . □

Roman C.S in [8] , introduce the following : " An R-module M is called K-nonsingular if , for each $f \in End_R(M)$, Kerf $\leq_e$ M implies f = 0 " . However we prove the following:

**Theorem (2.3 )**

Let M be an R-module . Then M is an essentially quasi-Dedekind R-module if and only if M is a K-nonsingular R-module .

**Proof:** ⇒) Let $f \in End_R(M)$ , $f \neq 0$ . Suppose that Kerf $\leq_e$ M , defined $g : M/Kerf \longrightarrow M$ by g (m+Kerf) = f (m) for all $m \in M$ . It is easy to see that g is well-defined and g is a nonzero homomorphism . Thus $Hom(M/Kerf, M) \neq 0$ which is a contradiction , since M is an essentially quasi-Dedekind R-module .

⇐) N $\leq_e$ M . Suppose that there exists $f : M/N \longrightarrow M$ and $f \neq 0$ . we have $M \xrightarrow{\pi} M/N \xrightarrow{f} M$ ,where π is the canonical projection .Let $\psi = fo\pi \in End_R(M)$. $N \subseteq Ker\psi$ and N$\leq_e$M implies $Ker\psi \leq_e$M , $\psi(M) = fo\pi(M) = f(M/N) \neq 0$ which is a contradiction with M is a K-nonsingular R-module . □

Although the concepts of essentially quasi-Dedekind module and K-nonsingular module are equivalent ,but we see that it is convenient to use the notion essentially quasi-Dedekind in this paper .

**Proposition (2.4 )**

Every semisimple R-module is an essentially quasi-Dedekind R-module.

**Proof :** It is easy . □

The converse of (Prop 2.4) is not true in general, consider the following example .

**Example (2.5 )**

It is known that Z as a Z-module is essentially quasi-Dedekind , but it is not semisimple .

Recall that an ideal I of a ring R is semiprime if , for all $r \in R$ with $r^2 \in I$ implies $r \in I$ [or , for all ideal A of R with $A^2 \subseteq I$ implies $A \subseteq I$ ] .And a ring R is called semiprime if (0) is a semiprime ideal of R ; i.e R does not contain nonzero nilpotent ideals , [2] .

**Proposition (2.6 )**

Let R be a ring . The following statements are equivalent :

1) R is an essentially quasi-Dedekind ring .
2) R is a semiprime ring .
3) Z(R) = 0 ( R is a nonsingular ring ) .

**Proof :**

$(2) \Leftrightarrow (3)$ : It is follows by [2 , Prop 1.27, p.35]

$(2) \Rightarrow (1)$ : Let $f \in End_R(R)$ such that Kerf $\leq_e$ R . To prove f = 0 .

Suppose that $f \neq 0$ , there exists $0 \neq r \in R$ such that f(a) = ra for all $a \in R$ . Since Kerf $\leq_e$ R and $0 \neq r \in R$ , then there exists $0 \neq t \in R$ such that $0 \neq rt \in Kerf$ , hence 0 =

f(rt) = rf(t) = $r^2t$ . This implies $(rt)^2 = 0$ and since R is semiprime , rt = 0 which is a contradiction . Thus f = 0 and R is essentially quasi-Dedekind .



(1) $\Rightarrow$ (3) : Suppose that $Z(R) \neq 0$ . Then there exists $0 \neq a \in Z(R)$ and hence $ann_R(a) \leq_e R$ , this implies $ann_R(a)$ is a quasi-invertible ideal and so that by (5 , Prop 2.2) , $ann_R(ann_R(a)) = 0$ , but $(a) \subseteq ann_R(ann_R(a))$ , hence a = 0 which is a contradiction . □

**Proposition (2.7 )**

Let R be a ring . Then R is essentially quasi-Dedekind if and only if R[x] is essentially quasi-Dedekind , where R[x] is the ring of polynomials with one indeterminate x .

**Proof :**

$\Rightarrow$) Suppose that R is essentially quasi-Dedekind , so by (Prop 2.6) R is a nonsingular ring, and hence by [2 , Ex. 13, p.37] , R[x] is a nonsingular ring . Thus R[x] is essentially quasi-Dedekind , by ( Prop 2.6) .

$\Leftarrow$) Suppose that R is not essentially quasi-Dedekind , so by (Prop 2.6) , R is not a semiprime ring ; that is there exists $a \in L(R)$ and $a \neq o$ , where $L(R) = \{x \in R : x^n = 0$ , for some $n \in N\}$ , then $a^n = 0$ , for some $n \in N$ . Define $f(x) = a \neq 0$ , so $f(x) \in R[x]$ , and R[x] is a semiprime ring, by (Prop 2.6) . On the other hand $[f(x)]^n = a^n = 0$ , implies $f(x) \in L(R[X]) = 0$. It follows that f = 0 which is a contradiction . Thus R is essentially quasi-Dedekind . □

**Proposition (2.8 )**

Let M be a faithful R-module . Then R is essentially quasi- Dedekind if and only if $N \oplus \frac{M}{N}$ is a faithful R-module , for all $N \leq M$ .

**Proof :**

$\Rightarrow$) Suppose that R is essentially quasi-Dedekind , so by ((Prop 2.6), R is semiprime . Let $r \in ann_R(N \oplus \frac{M}{N})$ , then $r \in ann_R(N) \cap ann_R(\frac{M}{N})$ ; that is rN = 0 and $rM \subseteq N$ , so $r^2M \subseteq rN = 0$ implies $r^2 \in ann_R(M) = 0$ then $r^2 = 0$ , thus r = 0 , since R is a semiprime ring . Therefore $N \oplus \frac{M}{N}$ is a faithful R-module for all $N \leq M$ .

$\Leftarrow$) Suppose that $N \oplus \frac{M}{N}$ is a faithful R-module , for all $N \leq M$ . To prove that R is essentially quasi- Dedekind . We shall prove that R is a semiprime ring. Let $r \in R$ with $r^2 = 0$ , suppose that $r \neq 0$ , so $r \notin ann_R(M)$ , since M is a faithful R-module, then $rM \neq 0$.Let $N = rM \leq M$ , hence rN = $r^2M$ = 0 , so $r \in ann_R(N)$ , but $r \in ann_R(\frac{M}{N})$ ( since $rM \subseteq rM = N$ ) , so

$r \in ann_R(N) \cap ann_R(\frac{M}{N}) = ann_R(N \oplus \frac{M}{N}) = 0$, thus r = 0 which is a contradiction. Hence R is essentially quasi-Dedekind . □



**Proposition (2.9 )**

Let M be an R-module and let $\overline{R} = R/J$ , where J is an ideal of R such that $J \subseteq ann_R(M)$ . Then M is an essentially quasi-Dedekind R-module if and only if M is an essentially quasi-Dedekind $\overline{R}$ -module.

**Proof :**

By [3, p.51] , we have $Hom_R(M/N, M) = Hom_{\overline{R}}(M/N, M)$ for all $N \leq M$ . Suppose that M is an essentially quasi-Dedekind R-module , then $Hom_{\overline{R}}(M/N, M) = Hom_R(M/N, M) = 0$ for all $N \leq_e M$ , implies M is an essentially quasi-Dedekind $\overline{R}$ -module .

The converse follows similarly . □

Let R be an integral domain , and let M be an R-module . An element $x \in M$ is called a torsion element of M if , $ann_R(x) \neq 0$ . The set of all torsion elements of M denoted by T(M) and it is a submodule of M . If T(M) = 0 the R-module M is said to be torsion-free , [1, p.45] .

The following result shows that essentially quasi-Dedekind preserves under isomorphism .

**Proposition (2.10 )**

Let $M_1$ , $M_2$ be R-modules such that $M_1 \cong M_2$ . Then $M_1$ is an essentially quasi-Dedekind R-module if and only if $M_2$ is an essentially quasi-Dedekind R-module .

**Proof :**

$\Rightarrow$) Suppose that $M_1$ is an essentially quasi-Dedekind R-module . Let $\phi : M_1 \longrightarrow M_2$ , $\phi$ is an isomorphism . To prove that $M_2$ is an essentially quasi-Dedekind R-module . Let $f \in End_R(M_2)$, $f \neq 0$ . We have $M_1 \xrightarrow{\phi} M_2 \xrightarrow{f} M_2 \xrightarrow{\phi^{-1}} M_1$ , let $h = \phi^{-1} o f o \phi \in End_R(M_1)$ , and hence $h \neq 0$ , then Kerh $\not\leq_e$ $M_1$ . To prove Kerf $\not\leq_e$ $M_2$ , we cliam that $Kerf = \{y \in M_2 : \phi^{-1}(y) \in Kerh\}$ , to prove our a sseration . Let $y \in Kerf$, $f(y) = 0$, $h(\phi^{-1}(y)) = (\phi^{-1} o f o \phi)(\phi^{-1}(y)) = (\phi^{-1} o f)(y) = \phi^{-1}(f(y)) = \phi^{-1}(0) = 0$ .Then for all $y \in Kerf$ ,$\phi^{-1}(y) \in Kerh$, so $\phi^{-1}(Kerf) \subseteq Kerh \not\leq_e M_1$ which implies $\phi^{-1}(Kerf) \not\leq_e M_1$ , so Kerf $\not\leq_e M_2$ . Thus $M_2$ is an essentially quasi-Dedekind R-module .

$\Leftarrow$) The proof is similarly . □

**Remark (2.11 )**

Let M be an R-module and let $N \leq M$ . If $M/N$ is an essentially quasi- Dedekind R-module . Then M is not necessarily an essentially quasi-Dedekind R-module , as we can see by the following example .

**Example (2.12)**

Let M = $Z_4$ as a Z-module , and $N = (\bar{2}) \le Z_4$ , then $Z_4/(\bar{2}) \cong Z_2$ is an essentially quasi-Dedekind Z-module ,but M = $Z_4$ is not an essentially quasi-Dedekind Z-module .

Now, we turn our attention to a submodule of essentially quasi-Dedekind. First consider the following remark :



**Remark (2.13 )**

Let M be an essentially quasi- Dedekind R-module , $N \le M$ . Then it is not necessarily that N be an essentially quasi-Dedekind R-module .To show this , consider the following example which appeared in [7] .

Let $M = Q \oplus Z_2$ as a Z-module is essentially quasi-Dedekind .

Take $N = Z \oplus Z_2 \le Q \oplus Z_2$ as a Z-module , then N is not essentially quasi-Dedekind as a Z-module , since if $f : N \longrightarrow N$ define by $f(x,\bar{y}) = (0,\bar{x})$ , $x \in Z$, $\bar{y} \in Z_2$ , then $f \ne 0$ and

$Kerf = \{(x,\bar{y}) \in N : f(x,\bar{y}) = (0,\bar{0})\} = \{(x,\bar{y}) \in N : \bar{x} = \bar{0}\} = 2Z \oplus Z_2$ . Hence Kerf $\le_e$ N . Thus $N = Z \oplus Z_2$ is not an essentially quasi-Dedekind as a Z-module.

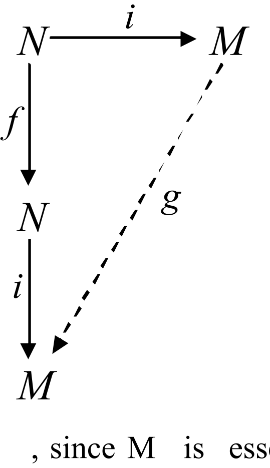
Now , in the next proposition we give a condition which makes R-submodule of an essentially quasi-Dedekind R-module is essentially quasi-Dedekind .

**Proposition (2.14 )**

Let M be an essentially quasi-Dedekind R-module , and M is quasi – injective. If N $\le_e$ M then N is an essentially quasi-Dedekind R-module.

**Proof :**

Let $f \in End_R(N)$ , $f \ne 0$ , to prove that Kerf $\not\le_e$ N . Assume that Kerf $\le_e$ N . Since M is quasi–injective , then there exists $g \in End_R(M)$ such that goi = iof , ( where i is the inclusion mapping) .




It follows that $g \ne 0$ , and this implies Kerg $\not\le_e$ M , since M is essentially quasi-Dedekind . But $Kerf \subseteq Kerg$ , so Kerf $\not\le_e$ M . On the other hand N $\le_e$ M and by assumption Kerf $\le_e$ N imply Kerf $\le_e$ M . To show this , since N $\le_e$ M then for all $U \le M$ , $U \ne 0$ then $N \cap U \ne 0$ and $N \cap U \le N$ .But Kerf $\le_e$ N , hence $Kerf \cap (N \cap U) \ne 0$; that is $(Kerf \cap U) \cap N \ne 0$ which implies that $Kerf \cap U \ne 0$ which is a contradiction . Thus Kerf $\not\le_e$ N and hence N is an essentially quasi-Dedekind R-module. □

**Corollary (2.15)**

Let M be an R-module . If $\overline{M}$ is an essentially quasi-Dedekind R-module then M is an essentially quasi-Dedekind R-module .

**Proof :** Suppose that $\overline{M}$ is an essentially quasi-Dedekind R-module , and since $\overline{M}$ is a quasi –injective R-module and M $\leq_e \overline{M}$ , so by (Prop 2.14 ) , M is an essentially quasi-Dedekind R-module . □

**Corollary (2.16)**

Let M be an R-module . If E(M) is an essentially quasi-Dedekind R-module then M is an essentially quasi-Dedekind R-module .

**Proof :** It is clear . □

The converse of (Coro2.16) is not true in general, consider the following example .

**Example (2.17)**

Let M = $Z_2$ as a Z-module . M is an essentially quasi-Dedekind Z-module. But E($Z_2$) = $Z_{2^\infty}$ is not an essentially quasi-Dedekind Z-module , (see Rem.and.Ex 2.2(4)) .

Now we prove the following proposition :

**Proposition (2.18 )**

Let M be an R-module such that ,for each $f \in Hom(M, E(M))$, $f \neq 0$ implies Kerf $\not\leq_e$ M . Then M is essentially quasi-Dedekind .

**Proof** : Let $g \in End_R(M)$ , $g \neq 0$ . Then $iog \in Hom(M, E(M))$ , and $iog \neq 0$ , where i is the inclusion mapping. Hence Ker(iog) $\not\leq_e$ M . But Kerg = Ker(iog) . Thus Kerg $\not\leq_e$ M and M is essentially quasi-Dedekind . □

Next we study the behavior of the quotient module of essentially quasi-Dedekind module . First we have the following .

**Remark (2.19 )**

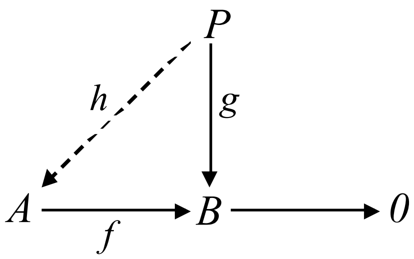
Let M be an R-module , $N \leq M$ . If M is an essentially quasi- Dedekind R-module , then $M/N$ is not necessarily essentially quasi- Dedekind R-module , consider the following example .

**Example(2.20)**

It is well-known that Z as a Z-module is essentially quasi- Dedekind .

Let $N = (4) \leq Z$ , $Z/N = Z/(4) \cong Z_4$ is not essentially quasi-Dedekind as a Z-module , ( see Rem.and.Ex 2.2(3) ) .

We need to recall that an R-module P is projective if and only if , for any R-modules A , B and for any epimorphism $f : A \longrightarrow B$ and for any homomorphism $g : P \longrightarrow B$ , there exists a homomorphism $h : P \longrightarrow A$ such that foh = g (i.e the following diagram is a commutative) , [3 , p.117] .



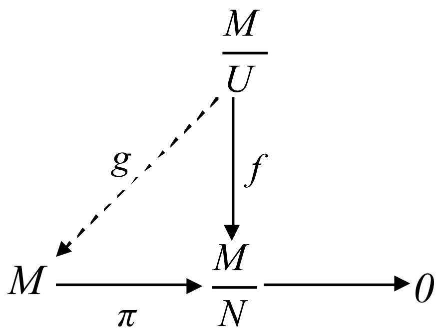

Now , in the next proposition we give a condition under which the (Remark 2.19) is true .

**Proposition (2.21 )**

Let M be an R-module such that $M/K$ is a projective R-module for all K $\leq_e$ M. If M is an essentially quasi-Dedekind R-module , then $M/N$ is an essentially quasi-Dedekind R-module for all $N \leq M$ .

**proof :**

Let $U/N \leq_e M/N$ . Then U $\leq_e$ M and hence by hypothesis $M/U$ is a projective R-module . Suppose that there exists $f \in Hom(\frac{M/N}{U/N}, \frac{M}{N})$, $f \neq 0$ . But $Hom(\frac{M/N}{U/N}, \frac{M}{N}) \cong Hom(\frac{M}{U}, \frac{M}{N})$ and since $M/U$ is projective , so there exists $g : \frac{M}{U} \longrightarrow M$ such that $\pi o g = f$ , where $\pi$ is the canonical projection mapping.




Since $f \neq 0$ then $g \neq 0$ , thus $Hom(\frac{M}{U}, M) \neq 0$ , U $\leq_e$ M ; that is M is not an essentially quasi-Dedekind R-module ,which is a contradiction. Thus $M/N$ is an essentially quasi-Dedekind R-module for all $N \leq M$ . □

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# المقاسات الجزئية شبه–معكوسة الواسعة و المقاسات شبه – ديديكاندية الواسعة

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## الخلاصة

لتكن R حلقة أبدالية ذا عنصر محايد . في هذا البحث درسنا مفهومي المقاسات الجزئية شبه–معكوسة الواسعة والمقاسات شبه – ديديكاندية الواسعة أعمام إلى المقاسات الجزئية شبه–معكوسة و المقاسات شبه – ديديكاندية. ومن بين النتائج التي حصلنا عليها النتيجة الاتية " M مقاس شبه– ديديكاندي واسع اذا كان M مقاس غير منفرد من النمط – K " ، اذ المقاس M هو مقاس غير منفرد من النمط – K اذا كان لكل تشاكل f من M إلى M على الحلقة R بحيث $Kerf \leqslant_e M$ يؤدي إلى أن $f = 0$ .