المقاسات الجزئية شبه الاولية التامة والمقاسات شبه الاولية التامة

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الخلاصة

لتكن R حلقة ابدالية ذا عنصر محايد وليكن M مقاساً احادياً . في هذا البحث درسنا مفهومي المقاسات الجزئية شبه الاولية التامة والمقاسات شبه الاولية التامة إذ يقال عن المقاس الجزئي الفعلي المتغير التام W انه مقاس جزئي شبه اولي تام اذا كان $X*X\subseteq W$ لكل X مقاس جزئي متغير تام يؤدي الى ان $X\subseteq W$ ويسمى M مقاساً شبه اولي تام اذا كان المقاس الجزئي (0) مقاس شبه اولي تام. اعطينا الخواص الاساسية لهذين المفهومين وكذلك درسنا العلاقات بين المقاسات الجزئية شبه الاولية التامة (المقاسات شبه الاولية التامة) مع انواع اخرى من المقاسات الجزئية (المقاسات) ذات العلاقة معهما.

الكلمات المفتاحية: المقاسات الجزئية شبه الاولية التامة– المقاسات شبه الاولية التامة المقاسات الجزئية من النمط invarian التامة– المقاسات الاولية التامة

On Fully Semiprime Submodules and Fully Semiprime Modules

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Abstract

Let R be a commutative ring with unity and let M be a unitary R-module. In this paper we study fully semiprime submodules and fully semiprime modules, where a proper fully invariant R-submodule W of M is called fully semiprime in M if whenever $X*X\subseteq W$ for all fully invariant R-submodule X of M, implies $X\subseteq W$.

M is called fully semiprime if (0) is a fully semiprime submodule of M. We give basic properties of these concepts. Also we study the relationships between fully semiprime submodules (modules) and other related submodules (modules) respectively.

Key words: Fully semiprime submodule, fully semiprime modules, fully invariant submodule, fully prime modules.

Introduction

J.Abuihlail in [1], suggested the definition of fully semiprime submodule and fully semiprime module as projects, where a proper fully invarianr R-module $W\subsetneq M$ is fully semiprime in M, if whenever $X*X\subseteq W$ for all fully invariant R-submodules $X\subseteq M$, it follows that $X\subseteq W$.

An R-module M is called fully semiprime if whenever $X*X=0$ for all fully invariant R-submodule X of M, it follows that X=0; that is M is a fully semiprime module if $0\subsetneq M$ is fully semiprime.

Also for R-submodules X, Y $\subseteq$ M, the internal product $X*Y$ is defined by $\sum\{f(X):f\in Hom(M,Y)\}$.

Notice that, if $Y\subseteq M$ is fully invariant, then $X*Y\subseteq M$ is also fully invariant, and if $X\subseteq M$ is fully invariant, then $X*Y\subseteq X\cap Y$.

The internal product of submodules of a given module over an associative not necessarily commutative ring was first introduced by Bican et.al, [2] to present the notion of prime modules. The definition is modified in [3], where arbitrary submodules are replaced by fully invariant ones. To avoid any possible confusion, such modules are referred to as fully prime modules, where a proper fully invariant submodule $W\subsetneq M$ is fully prime, if whenever $X*Y\subseteq W$, for all fully invariant R-submodule $X\subseteq M$, $Y\subseteq M$, it follows that $X\subseteq W$ or $Y\subseteq W$. An R-module is called fully prime if $(0) \subsetneq M$ is a fully prime submodule; that is whenever $X*Y=(0)$ for all fully invariant R-submodules $X\subseteq M$, $Y\subseteq M$, it follows that X=(0) or Y=(0).

In this paper we give a comprehensive study of the concepts fully semiprime submodules and fully semiprime modules, where this paper consists of two sections. In section one, we give the basic properties of fully semiprime submodules and fully semiprime modules. Section two is devoted to study the relationships between fully semiprime modules and other modules such as uniform module, chained module, Z-regular module, quasi-Dedekind module, multiplication module and retractable module.

Next throughout this paper, R is commutative ring with unity and M be a unitary R-module.

1- Fully Semiprime Submodules and Fully Semiprime Modules-Basic Results

In this section we study the concepts of fully semiprime submodules and fully semiprime modules which are introduced in [1] as projects. The concepts are generalizations of fully prime submodules and fully prime modules which are studied in [3].

We give characterizations about theses concepts and establishe some basic properties about them.

We begin with the following definition.

1.1 Definition, [2]:

Let K, L be two fully invariant submodules of R-module M. Then

$K*L=\sum\{f(K): f:M\longrightarrow L\}$

A proper submodule N of an R-module M is called invariant if for each $f\in \underset{R}{End}(M)$, $f(N)\subseteq N$. M is called fully invariant if every submodule of M is invariant, see [4].

Invariant submodule is called fully invariant submodule by some authors, see [3,p.14].

1.2 Definition, [3]:

A fully invariant submodule N of an R-module M is called fully prime if for all fully invariant submodules K and L of M such that $K*L\subseteq N$, implies $K\subseteq N$ or $L\subseteq N$.

Now, we give the following concept.

1.3 Definition, [1]:

A fully invariant submodule N of an R-module M is called fully semiprime if for all fully invariant submodules K of M such that $K*K\subseteq N$, implies $K\subseteq N$.

We call M fully prime (fully semiprime) module if (0) is fully prime (fully semiprime) submodule, see [1].

Recall that:An R-module M is said to be a prime module if $ann_R M=ann_R N$ for every non-zero submodule N of M, where $ann_R M=\{r\in R:rx=0 \text{ for each } x\in M\}$, see [5].

An R-module M is called semiprime if and only if $ann_R N$ is a semiprime ideal of R for each non-zero R-submodule N of M, see [6].

Next, we give some remarks and examples.

1.4 Note:

Consider R as a left R-module, let I, J be two ideals of R. Then $I*J=IJ$, since every ideal of R is a fully invariant R-submodule. Thus I is a fully semiprime ideal if and only if I is a semiprime ideal.

1.5 Remarks and Examples:

1. Let N be a submodule of an R-module M. If N is a fully prime submodule, then N is a fully semiprime submodule.

2. If an R-module M is fully prime module, then M is prime module.

3. A submodule N of an R-module M is semiprime, if N is fully semiprime submodule.

proof: Suppose that $r\in R$, $x\in M$ such that $r^2x\in N$. Let $K=\langle rx\rangle$, K is a fully invariant submodule, then $K*K=\sum\{f(K): f:M\longrightarrow K=\langle rx\rangle\}$.

Now, $f(K)=f\langle rx\rangle=r\langle f(x)\rangle\subseteq\langle r^2x\rangle\subseteq N$. Thus $K*K\subseteq N$, implies $K\subseteq N$, so $rx\in N$.

4. If an R-module M is a fully semiprime module, then M is a semiprime module.

5. Z_6 as a Z-module is fully semiprime, since for all submodule N, $N\neq(0)$, then $N*N\neq(0)$. Thus Z_6 is a semiprime Z-module. But it is not a fully prime because it is not prime.

6. Z_4 as a Z-module is not semiprime module, since $ann_Z Z_4=4Z$ is not a semiprime ideal of Z. Hence Z_4 is not fully semiprime.

7. 6Z as a Z-submodule of Z is semiprime, so it is fully semiprime.

8. Let R be an integral domain and K be the quotient field of R. Then K is an R-module and the zero R-submodule of K is the only semiprime in K. That is (0) is the only fully semiprime submodule in K, because if $\exists$ $N<K$, $N\neq(0)$, N is fully semiprime submodule, then N is semiprime, which is a contradiction.

9. Z_{p^∞} as a Z-module has no fully semiprime submodule.

10. The homomorphic image of a fully semiprime module need not fully semiprime module, for example: Z as a Z-module is a fully prime. Then Z is fully semiprime. Now, let $\pi: Z \longrightarrow Z/(4) \cong Z_4$. Z_4 as a Z-module is not a fully semiprime.

Now, we have the following proposition.

1.6 Proposition:

If N is fully semiprime R-submodule of M, then $[N \underset{R}{:} K]$ is a semiprime ideal of R for all $N \underset{\neq}{\subset} K$.

proof: We have N is fully semiprime submodule, then N is semiprime submodule (by remarks and examples (1.4),3). Then it is easy to show that $[N \underset{R}{:} K]$ is semiprime ideal for all $K \supseteq N$.

The following result is a consequence of proposition (1.6).

1.7 Corollary:

If N is a fully semiprime submodule of an R-module M, then $[N \underset{R}{:} M]$ is a semiprime ideal.

The following result is a characterization of fully semiprime submodule, but first the following lemma is needed.

1.8 Lemma:

Let K be a fully invariant submodule of an R-module M. Then $I(K*K) \supseteq IK*IK$ for every ideal I of R.

proof: $IK*IK = \sum\{f(IK) : f: M \longrightarrow IK\}$

$= I\{\sum f(K) : f: M \longrightarrow IK \subseteq K\}$

Since $K*K == \sum\{g(K) : g: M \longrightarrow K\}$. It follows that $IK*IK \subseteq I(K*K)$

1.9 Proposition:

Let N be a submodule of an R-module M. Then N is a fully semiprime submodule if and only if $[N \underset{M}{:} I]$ is a fully semiprime submodule of M for every ideal I of R.

proof: ($\Rightarrow$) suppose that N is a fully semiprime submodule of M, let K be a fully invariant submodule of M such that $K*K \subseteq [N \underset{M}{:} I]$, implies $I(K*K) \subseteq N$, then by lemma (1.8), $IK*IK \subseteq N$, but N is a fully semiprime submodule. Thus $IK \subseteq N$. Therefore $K \subseteq [N \underset{M}{:} I]$. Hence $[N \underset{M}{:} I]$ is a fully semiprime submodule.

The converse follows by taking I=R, because $[N \underset{M}{:} R] = N$.

Next, we have the following proposition.

1.10 Proposition:

Let N be a fully invariant submodule of M. If N is a fully semiprime submodule, then M/N is fully semiprime module.

proof: Let K/N be a fully invariant submodule of M/N such that $K/N*K/N = N = O_{M/N}$. Then K is a fully invariant submodules of M, with $K \underset{M}{*} K \subseteq N$ [3, corollary (1.1.21)]. Hence $K \subseteq N$ (since N is a fully semiprime submodule). That is $\frac{K}{N} = O_{M/N}$.

The converse of proposition (1.10) holds under the condition M is self projective.

1.11 Proposition:

Let M be a self projective module and let N be a fully invariant submodule of M. If M/N is fully semiprime, then N is a fully semiprime submodule in M.

proof: Let K be a fully invariant submodule of N such that $K*K\subseteq N$. Then $K'=\frac{K+N}{N}$ is a fully invariant submodule of $\frac{M}{N}$ [3, lemma (1.1.20)(ii)] and $K' \underset{M/N}{*} K'=0$. Hence K'=0, that is $K+N\subseteq N$ and so $K\subseteq N$. Thus N is a fully invariant semiprime submodule.

As an application of proposition (1.10) we give the following corollary.

1.12 Corollary:

Let $\psi:M\longrightarrow M'$ be an epimorphisim. If ker ψ is a fully semiprime submodule, then M' is a fully semiprime module.

proof: Since ψ is an epimorphisim, then $M/\ker\psi\cong M'$. But $\ker\psi$ is a fully semiprime submodule, so by proposition (1.10), $M/\ker\psi$ is a fully semiprime module. This completes the proof.

Before we give the next result, we introduce the following lemma.

1.13 Lemma:

Let $\theta:M\longrightarrow M'$ be an isomorphism, where M, M' be two R-modules, let K be a fully invariant submodule of M. Then $\theta(K*K)\subseteq\theta(K)*\theta(K)$.

proof: $\theta(K*K) = \theta(\sum\{f(K):f:M\longrightarrow K\})$

$= \sum\{\theta f(K):f:M\longrightarrow K\}$

$= \sum\{(\theta\circ f)(K):f:M\longrightarrow K\}$.

But $\theta^{-1}:M'\longrightarrow M$ is an isomorphism and $K\subseteq M$, so there exists $K'\subseteq M'$ such that $\theta^{-1}(K')=K$; that is $\theta(K)=K'$. Hence $\theta(K*K)=\sum\{\theta\circ f\circ\theta^{-1}(K'):f:M\longrightarrow K\}$.

But $M'\xrightarrow{\theta^{-1}} M\xrightarrow{f} K\xrightarrow{\theta}\theta(K)=K'$, hence $(\theta\circ f\circ\theta^{-1}):M'\longrightarrow K'=\theta(K)$. Hence $\theta(K*K)=\sum\{\theta\circ f\circ\theta^{-1}(K'):\ \theta\circ f\circ\theta^{-1}:M'\longrightarrow\theta(K)=K'\}$.

Now $\theta(K)*\theta(K)=K'*K'=\sum\{h(K'):h:M'\longrightarrow K'=\theta(K)\}$. It follows that $\theta(K*K)\subseteq\theta(K)*\theta(K)$.

However, we get the following proposition.

1.14 Proposition:

If M and M' are two isomorphic R-modules, then M' is fully semiprime module if and only if M is fully semiprime module.

proof: Let $\theta:M\longrightarrow M'$, θ is an isomorphism, and let L be a fully invariant submodule of M' such that $L*L=0$. To prove L=0. Let $K=\theta^{-1}(L)$, that is $\theta(K)=L$. Then $L*L=\theta(K)*\theta(K)\supseteq\theta(K*K)$ (by lemma (1.13). But $\theta(K)*\theta(K)=0$ (since $L*L=0$), implies $\theta(K*K)=0$. Since θ is one to one, we have $K*K=0$ and since M is fully semiprime, we get K=0. This implies $\theta(K)=\theta(0)=0$. Therefore L=(0) and hence M' is a fully semiprime module.

1.15 Proposition:

Let N and K be two fully semiprime submodules of an R-module M. Then $N\cap K$ is a fully semiprime submodule of M.

proof: Let L be a fully invariant submodule of M such that $L*L\subseteq N\cap K$. But $N\cap K\subseteq K$ and $N\cap K\subseteq N$. $L*L\subseteq K$ and $L*L\subseteq N$. Thus $L\subseteq K$ and $L\subseteq N$ (since K and L are fully semiprime). Therefore $L\leq N\cap K$. Hence $N\cap K$ is a fully semiprime submodule of M.

By using the mathematical induction, we obtain the following result.

1.16 Corollary:

The intersection of a finite collection of fully semiprime submodules of an R-module is a fully semiprime submodule.

1.17 Proposition:

Let M be an R-module. Then M is fully semiprime module if and only if for all $m\in M$, $(m)*(m)=(0)$, implies m=0.

proof: $(\Rightarrow)$ It is clear.

To prove the other side, let $N*N=(0)$. Suppose $N\neq(0)$, so there exists $m\in N$, $m\neq(0)$ such that $(m)\subseteq N$, implies $(m)*(m)\subseteq N*N=(0)$. Then $(m)*(m)=(0)$. Hence $m=0$, which is a contradiction. Thus $N=0$ and this completes the proof.

For our next proposition, the following lemma is needed.

1.18 Lemma:

If M_1, M_2 are two R-modules and N_1, N_2 are two fully invariant submodules of M_1 and M_2 respectively, then $(N_1*N_1)\oplus(N_2*N_2)\subseteq(N_1\oplus N_2) * (N_1\oplus N_2)$.

proof: $(N_1*N_1)\oplus(N_2*N_2) = \sum\{(f(N_1),g(N_2)): f:M_1\longrightarrow N_1, g:M_2\longrightarrow N_2\}$. For any $f:M_1\longrightarrow N_1$, $g:M_2\longrightarrow N_2$, define $h: M_1\oplus M_2\longrightarrow N_1\oplus N_2$ by $h(x,y)=(f(x),g(y))$ for all $(x,y)\in M_1\oplus M_2$. It is clear that h is well defined homomorphism. Now, $h(N_1\oplus N_2)=(f(N_1),g(N_2))$. But $(N_1\oplus N_2) * (N_1\oplus N_2)=\sum\{\theta(N_1\oplus N_2):\theta: M_1\oplus M_2\longrightarrow N_1\oplus N_2\}$.It follows that $\sum h(N_1\oplus N_2)=\sum (f(N_1),g(N_2))$ be in $(N_1\oplus N_2) * (N_1\oplus N_2)$. Thus we have $(N_1*N_1)\oplus(N_2*N_2)\subseteq(N_1\oplus N_2)*(N_1\oplus N_2)$.

1.19 Proposition:

Let M_1, M_2 be two R-modules and $M=M_1\oplus M_2$ such that $annM_1+annM_2=R$. If N_1 and N_2 are fully semiprime R-submodules of M_1 and M_2 respectively, then $N_1\oplus N_2$ is also fully semiprime.

proof: to prove $N_1\oplus N_2$ is fully semiprime, let N be a fully invariant submodule of M such that $N*N\subseteq N_1\oplus N_2$. But $N=K\oplus L$ for some $K\leq M_1$, $L\leq M_2$, by [4,theorem (4.2),ch.1]. Then $(K\oplus L)* K\oplus L\subseteq N_1\oplus N_2$. Thus by lemma (1.15), we get $(K*K) \oplus (L*L) \subseteq N_1\oplus N_2$ so $K*K\subseteq N_1$, $L*L\subseteq N_2$. But N_1 and N_2 are fully semiprime, then $K\subseteq N_1$, $L\subseteq N_2$. Thus $K\oplus L\subseteq N_1\oplus N_2$ and hence $N_1\oplus N_2$ is fully semiprime.

The converse of proposition (1.16) holds if $(N_1\oplus N_2) * (N_1\oplus N_2)= (N_1*N_1)\oplus(N_2*N_2)$.

1.20 Proposition:

Let M_1, M_2 be two R-modules, let $(N_1\oplus N_2) * (N_1\oplus N_2)= (N_1*N_1)\oplus(N_2*N_2)$ for each $N_1\leq M_1$, $N_2\leq M_2$. Then $N_1\oplus N_2$ is fully semiprime implies, N_1 and N_2 are fully semiprime.

proof: Let $K\leq M_1$, $L\leq M_2$ such that $K*K\subseteq N_1$ and $L*L\subseteq N_2$. Hence $(K*K)\oplus(L*L)\subseteq N_1\oplus N_2$. It follows that $(K\oplus L)*(K\oplus L)\subseteq N_1\oplus N_2$. Since $N_1\oplus N_2$ is fully semiprime, $K\oplus L\subseteq N_1\oplus N_2$. Hence $K\subseteq N_1$ and $L\subseteq N_2$. Thus K and L are fully semiprime.

1.21 Proposition:

Let M_1, M_2 be two R-modules such that $annM_1+annM_2=R$. Then $M_1\oplus M_2$ is fully semiprime module if and only if M_1 and M_2 are fully semiprime modules.

proof: Let N be a fully invariant submodule of $M_1\oplus M_2$.If $N*N=0$, to prove $N=0$. Since $N= N_1\oplus N_2$ [4,theorem (2.4)], then $N*N=(N_1\oplus N_2) * (N_1\oplus N_2)$. By lemma (1.15). $(N_1*N_1)\oplus(N_2*N_2)\subseteq(N_1\oplus N_2) * (N_1\oplus N_2)=(0)\oplus(0)$. Then $(N_1*N_1)\oplus(N_2*N_2)=(0)\oplus(0)$, implies $N_1*N_1=(0)$ and $N_2*N_2=(0)$. Therefore $N_1=(0)$ and $N_2=(0)$ (because M_1, M_2 are fully semiprime). Thus $N_1\oplus N_2=(0)\oplus(0)$ and hence M is fully semiprime.

Conversely, suppose that $M_1\oplus M_2$ is a fully semiprime module, let N_1 be a fully invariant submodule of M_1 such that $N_1*N_1=(0)$. We can show that there is one to one correspondence between N_1*N_1 and $(N_1\oplus(0))*(N_1\oplus(0))$ as follows. For any $f:M_1\longrightarrow N_1$, $f(N_1)\in N_1*N_1$, f can be extended to $\tilde{f}: M_1\oplus M_2\longrightarrow N_1\oplus(0)$ by $\tilde{f}(m_1,m_2)=(f(m_1),0)$ for all $(m_1,m_2)\in M_1\oplus M_2$ it is clear that $\tilde{f}(N_1\oplus(0))=f(N_1)\oplus(0)$, hence if $f(N_1)=0$, then $\tilde{f}(N_1\oplus(0))=(0)$.

Similarly, if $g: M_1\oplus M_2\longrightarrow N_1\oplus(0)$, $g(N_1\oplus(0))\in(N_1\oplus(0))*(N_1\oplus(0))$, then we define $\tilde{g}:M_1\longrightarrow N_1$ by $\tilde{g}(m_1)=g(m_1,0)$, $\forall\, m\in M_1$, hence $\tilde{g}(N_1)=g(N_1\oplus(0))$ and so $\tilde{g}(N_1)\in N_1*N_1$. Thus $N_1*N_1=0 \Leftrightarrow (N_1\oplus(0))*(N_1\oplus(0))=0$. It follows that $(N_1\oplus(0))*(N_1\oplus(0))=0$, and hence $N_1\oplus(0)=0$. Thus $N_1=0$.

Similarly if N_2 is an invariant submodule of M_2 such that, $N_2*N_2=0$, implies $N_2=(0)$ and hence M_2 is fully semiprime.

Next, we prove the following.

1.22 Proposition:

Let $M=M_1\oplus M_2$ be a direct sum of two R-modules M_1 and M_2 such that $annM_1+annM_2=R$. If L_1 is a fully semiprime submodule of M_1. Then $L_1\oplus M_2$ is a fully semiprime submodule of M.

proof: Let N be a fully invariant submodule of $M= M_1\oplus M_2$ such that $N*N\subseteq L\oplus M_2$. By [4,theorem (4.2),ch.1], there exists $N_1\leq M_1$, $N_2\leq M_2$ such that $N=N_1\oplus N_2$. Hence $(N_1\oplus N_2) * (N_1\oplus N_2)\subseteq L_1\oplus M_2$. But $(N_1*N_1)\oplus(N_2*N_2)\subseteq(N_1\oplus N_2) * (N_1\oplus N_2)$ by lemma (1.15). Therefore $(N_1*N_1)\oplus(N_2*N_2)\subseteq L_1\oplus M_2$. It is clear that $N_1*N_1\subseteq L_1$, but L_1 is fully semiprime submodule, then $N_1\subseteq L_1$. Thus $N_1\oplus N_2\subseteq L_1\oplus M_2$. Therefore $L_1\oplus M_2$ is fully semiprime submodule of M.

2- The Relationships Between Fully Semiprime Modules and Certain Types of Modules

In this section, we establishe some relationships between fully semiprime submodules and some type of modules.

Recall that an R-module M is said to be uniform module if every non-zero submodule of M is essential see [7], where a submodule N of an R-module M is essential provided that $N\cap K\neq 0$ for every non-zero submodule K of M, see [7].

Hence, we have the following proposition.

2.1 Proposition:

Let M be a uniform R-module. Then M is a fully semiprime module if and only if M is a fully prime module.

proof: Suppose that M is a fully semiprime R-module, let K, L be two fully invariant submodules of M such that $K*L=(0)$. Assume that $K\neq(0)$, $L\neq(0)$. Then $K\cap L\neq(0)$ (since M is uniform). On the other hand $K\cap L\subseteq K$, $K\cap L\subseteq L$. Also, note that K, L are fully invariant, implies $K\cap L$ is fully invariant. Then $(K\cap L)*(K\cap L)\subseteq K*L=(0)$. Thus $(K\cap L)*(K\cap L)\subseteq(0)$. But M is fully semiprime so $K\cap L=(0)$ which is a contradiction. Thus either $K=(0)$ or $L=(0)$. Then M is fully prime.

The converse is obvious.

The following results are consequences of proposition (2.1), but first we need to recall the following definition.

An R-module M is said to be chained module if and only if every non-empty set of submodule of M is ordered by inclusion, [8].

Hence, we have the following consequence of (2.1).

2.2 Corollary:

Let M be a chained R-module. Then M is fully semiprime if and only if M is fully prime.

proof: It is known that every chained R-module M is uniform, then the result follows from proposition (2.1).

An R-module M is called quasi-Dedekind if every submodule N of M is quasi-invertible [9,definition (1.1), ch.2], where a submodule N of M is called quasi-invertible if $Hom(M/N,M)=0$ [9,definition (1.1),ch.1].

2.3 Corollary:

If M is a uniform fully semiprime R-module, then M is a quasi-Dedekind R-module.

proof: From proposition (2.1) and remark (1.4), we get M is a prime module. Thus by [6,theorem (3.11),ch.3], M is quasi-Dedekind.

As an application of corollary (2.3), we give the following example.

2.4 Example:

Z as a Z-module is uniform and fully prime module, also it is quasi-Dedekind.

Recall that an R-module M is called Z-regular if and only if each cyclic submodule of M is projective direct summand of M. Equivalently if for each $a\in M$, $\exists f\in M^*=Hom(M,R)$ such that $a=f(a)a$, [10].

By using this concept, we give the following proposition.

2.5 Proposition:

If M is a Z-regular R-module, then M is fully semiprime module.

proof: Let K be a fully invariant submodule of M such that $K*K=(0)$. Suppose $K\neq 0$. Then there exists $x\in K$, $x\neq 0$. Since M is Z-regular, then there exists $f:M\longrightarrow R$ such that $x=f(x)x$. Define $g:R\longrightarrow K$ by $g(r)=rx$ for each $r\in R$. Then $M\xrightarrow{f}R\xrightarrow{g}K$, so $g\circ f:M\longrightarrow K$ such that $(g\circ f)(x)=g(f(x))=f(x)x\neq 0$. Thus $0\neq g\circ f$ and hence $K*K\neq 0$ which is a contradiction. Thus $K=0$.

Now, we have the following proposition.

2.6 Proposition:

Let I be an ideal of R. Then the following statements are equivalent:

1. I is a fully semiprime submodule in R.
2. I is a semiprime submodule of R.
3. R/I is a semiprime ring.

proof: $1\Rightarrow 2$, let J be an ideal of R such that $J^2\subseteq I$. Then $J*J\subseteq I$ (by note (1.4)). Thus $J\subseteq I$, by (1).

$2\Rightarrow 1$, let $J*J\subseteq I$. Then $J^2\subseteq I$ (by note (1.4)) implies $J\subseteq I$.

$2\Leftrightarrow 3$, it is obvious.

2.7 Notes, [11]:

1. For any R-module M and for any ideals I, J of R. Then $(IM)*(JM)\subseteq(IJ)M$. And the reverse inclusion is also easily established provided M is self generator, that is Trac(M,JM)=JM.
2. The multiplication modules over commutative ring are self generator whose submodules are fully invariant.

Recall that an R-module M is called multiplication if for every submodule N of M, there exists an ideal I of R such that N=IM, equivalently for every submodule N of M, $N=[N \underset{R}{:} M]M$, [7].

2.9 Corollary:

Let M be a multiplication R-module. Then $(IM)*(JM)=(IJ)M$.

2.10 Proposition:

Let M be a faithful multiplication R-module M. Then M is fully semiprime if and only if R is semiprime ring.

proof: Let I be a proper ideal of R such that $I^2=0$, let N=IM, N is a fully invariant submodule of M. Then $N*N=I^2M=0$ by corollary (2.9), so $N*N=0$. But M is fully semiprime, implies N=0 and so IM=0. Then $I\subseteq ann(M)=0$, since M is faithful. Thus I=0.

Conversely, let N be a fully invariant submodule and $N*N=0$, since, N=IM for some ideal I of R. Implies that $N*N=I^2M$, so $I^2M=0$. Thus $I^2\subseteq ann(M)=0$. Therefore I=0 (since R is semiprime), and hence N=IM=(0).

Now, we have the following proposition.

2.11 Proposition:

Let M be a multiplication R-module with $ann_R(M)$ is semiprime. Then M is fully semiprime module.

proof: Let N be a fully invariant submodule of M such that $N*N=0$. But N=IM for some ideal I of M, since M is multiplication module. Hence $IM*IM=0$ which implies that $I^2M=0$. Thus $I^2\subseteq ann_R(M)$ and hence $I\subseteq ann_R(M)$.Then IM=0. Therefore N=(0). This completes the proof.

The following is an immediate consequence of proposition (2.11).

2.12 Corollary:

Let M be a multiplication R-module. Then M is a semiprime R-module if and only if $ann_R(M)$ is semiprime ideal.

proof: Assume that M is semiprime R-module. Then (0) is a semiprime submodule, so $(O:M)=ann_R(M)$ is semiprime ideal.

Conversely, let $ann_R(M)$ be a semiprime ideal. Then by proposition (2.11), we get M is fully semiprime and hence M is semiprime by remarks (1.4), see [4].

By proposition (2.11), we have the following.

2.13 Corollary:

Let M be a faithful multiplication R-module. The following are equivalent:

1. M is fully semiprime.
2. R is semiprime ring.
3. M is semiprime module.

proof: $1\Leftrightarrow 2$ it is obvious.

$2\Leftrightarrow 3$ R is semiprime ring $\Leftrightarrow$ (0) is semiprime ideal $\Leftrightarrow$ $ann_R(M)$ is semiprime ideal $\Leftrightarrow$ M is semiprime module (by proposition (2.11)).

An R-module M is called coprime if for every proper submodule N of M, $ann_R(M)=ann_R(M/N)$, [2].

Recall that an R-module M is called a scalar module if for all $f\in End_R(M);f\neq 0$, there exists $r\in R$, $r\neq 0$ such that $f(m)=rm$, see [12].

By using these concepts, we can prove the following.

2.14 Proposition:

Let M be a coprime scalar and fully semiprime R-module. Then M is simple.

proof: Assume that N be a proper R-submodule of M, let $f:M\longrightarrow N$. Then there exists $r\in R$ such that $f(m)=rm$ for all $m\in M$ (since M is a scalar module). Therefore $f(m)=rm\subseteq N$, implies $r\in[N \underset{R}{:} M]$. But M is a coprime module, so $ann_R M=[N \underset{R}{:} M]$. Thus $r\in ann_R M$. Then $rM=0$. Thus $f(N)=rN=0$. Hence $\sum\{f(N):f:M\longrightarrow N\}=0$. Then $N*N=0$ and since N is a fully semiprime, we get $N=0$.

Recall that an R-module M is called retractable if $Hom(M,N)\neq 0$, for every non-zero submodule N of M, see [13].

Now, we state and prove the following result.

2.15 Theorem:

Let M be an R-module, if M is fully semiprime, then M is retractable and the converse is true if $End_R(M)$ is semiprime.

proof: Assume that M is fully semiprime and let N be a submodule of M, $N\neq 0$. Assume M is not retractable, that is $Hom(M,N)=0$, then $N*N=0$ and so $N=0$ which is a contradiction. Hence M is retractable.

Conversely, if M is retractable and $End_R(M)$ is semiprime , let N be a fully invariant submodule of M such that $N*N=0$. Suppose $N\neq 0$. Since M is retractable, then there exists $f:M\longrightarrow N$, $f\neq 0$. But $0= N*N=\sum\{f(N):f:M\longrightarrow N\}=0$, hence $f(N)=0$. Then for any $m\in M$, $f^2(m)=f(f(m))=0$. Then $f^2=0$ and hence $f=0$ which is a contradiction. Thus $N=0$. Therefore M is a fully semiprime R-module.

We end this section by the following corollary.

2.16 Corollary:

Let M be a finitely generated multiplication R-module. The following are equivalent:

1. M is retractable and $End_R(M)$ is semiprime.
2. M is fully semiprime.
3. M is semiprime.
4. $ann_R(M)$ is semiprime.

proof: It is obvious.

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