(σ,τ)- Strongly Derivations Pairs on Rings

I. A. Saed
University of Technology
Received in : 30 November 2010
Accepted in : 27 February 2011

Abstract

Let R be an associative ring. In this paper we present the definition of (σ,τ)- Strongly derivation pair and Jordan (σ,τ)- strongly derivation pair on a ring R, and study the relation between them. Also, we study prime rings, semiprime rings, and rings that have commutator left nonzero divisior with (σ,τ)- strongly derivation pair, to obtain a (σ,τ)- derivation. Where σ,τ: R→R are two mappings of R.

Keywords
Prime ring, semiprime ring, (σ,τ)-derivation, (σ,τ)-Strongly derivation pair, Jordan (σ,τ)-Strongly derivation pair.

§1 Basic Concepts

Deinition 1.1: [1]

A nonempty set R is said to be associative ring if in R there are defined two operations, denoted by + and . respectively, such that for all a, b, c in R:

1- a + b is in R
2- a + b = b + a
3- (a+b) + c = a + (b+c)
4- There is an element 0 in R such that a+0 = a (for every a in R)
5- There exists an element –a in R such that a + (-a)=0.
6- a . b is in R.
7- a. (b.c) = (a.b).c
8- a. (b+c) = a.b + a.c and (b+c).a = b.a + c.a

Deinition 1.2: [1]

A ring R is called prime ring if for any a, b∈R, a R b = {0}, implies that either a=0 or b=0.

Definition 1.3:[1]

A ring R is called semiprime ring if for any a∈R, aRa = {0}, implies that a=0.

Remark 1.4:[1]

Every prime ring is semiprime ring, but the converse in general is not true.
The following example justifies this remark.

Example 1.5: [1]

$R = Z_6$ is a semiprime ring but is not prime.
Let $a \in R$ such that $aRa = \{0\}$, implies that $a^2 = 0$, hence a=0, therefore R is a semiprime ring.
But R is not prime, since $2 \neq 0$ and $3 \neq 0$ implies that $2R3 = \{0\}$.

Definition 1.6:[2]

A ring R is said to be n-torsion free, where $n \neq 0$ is an integer if whenever n a=0, with $a \in R$, then a=0.

Definition 1.7:[2]

Let R be a ring. A Lie product[,] on R is defined as$[x,y]=xy - yx$, for all $x,y \in R$.

Definition 1.8:[2]

Let R be a ring. An additive mapping $d:R \to R$ is called a derivation if $d(xy)= d(x)y + xd(y)$, for all $x,y \in R$ and we say that d is a Jordan derivation if $d(x^2)=d(x)x+xd(x)$, for all $x \in R$.

Definition 1.9:[3]

Let R be a ring. An additive mapping $d:R \to R$ is called a (σ,τ)-derivation, where $\sigma,\tau: R \to R$ are two mappings of R, if
$d(xy)= d(x)\sigma(y) + \tau(x) d(y)$, for all $x,y \in R$, and we say that d is a Jordan (σ,τ)-derivation if $d(x^2)=d(x) \sigma(x)+ \tau(x) d(x)$, for all $x \in R$.

Definition 1.10:[4]

Let R be a ring, additive mappings $d,g : R \to R$ is called S-derivation pair (d,g) if satisfies the following equations:
$d(xy) = d(x)y + xg(y)$, for all $x,y \in R$.
$g(xy)= g(x)y + xd(y)$, for all $x,y \in R$.
And is called Jordan S-derivation pair if:
$d(x^2)=d(x)x + xg(x)$, for all $x \in R$.
$g(x^2)=g(x)x + xd(x)$, for all $x \in R$.

Example 1.11:[4]

Let R be a non commutative ring and let $a,b \in R$, such that xa=xb=0, for all $x \in R$.
Define $d,g : R \to R$, as follows:
$d(x) = ax, g(x) = bx$
Then (d,g) is a S-derivation pair of R.

Remark 1.12:[4]

Every S-derivation pair is a Jordan S-derivation pair, but the converse is in general not true.
The following example illustrates this remark.

Example 1.13:[4]

Let R be a 2-torsion free non commutative ring, and let $a \in R$, such that $xax = 0$, for all $x \in R$, but $xay \neq 0$, for some $(x \neq y) \in R$.
An additive pair $d,g : R \to R$ is defined as
$d(x) = xa + ax$, $g(x) = [x,a]$
Then (d,g) is Jordan S-derivation pair, but not a S-derivation pair.

Definition 1.14:[5]

A ring R is said to be a commutator right (resp. left) nonzero divisior, if there exists elements a and b of R, such that $c[a,b] = 0$ (resp. $[a,b]c = 0$) implies $c=0$, for every $c \in R$.

§2 (σ,τ)-S-Derivation pairs

In this section, we will introduce the definition of (σ,τ)-Strongly derivation pair, and we denoted by (σ,τ)-S-derivation pair, and Jordan (σ,τ)-Strongly derivation pair and we denoted by Jordan (σ,τ)-S-derivation pair, also we will give the relation between them.
Where $\sigma,\tau : R \to R$ are two mappings on R.
Now, in this section we introduce the principle definition.

Definition 2.1

Let R be a ring, additive mappings $d,g : R \to R$ is called (σ,τ)-S-derivation pair (d,g) where $\sigma,\tau : R \to R$ are two mappings of R, if satisfy the following equations:
$d(xy)=d(x)\sigma(y) + \tau(x)g(y)$, for all $x,y \in R$.
$g(xy)=g(x)\sigma(y) + \tau(x)d(y)$, for all $x,y \in R$.
And is called Jordan (σ,τ)-S-derivation pair if:
$d(x^2)=d(x)\sigma(x) + \tau(x)g(x)$, for all $x \in R$.
$g(x^2)=g(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$.
The following example explains the principle definition:

Example 2.2

Let R be a non commutative ring and let $a,b \in R$, such that
$\tau(x)a = \tau(x)b = 0$, for all $x \in R$.
Define $d,g: R \to R$ as follows:
$d(x) = a\,\sigma(x)$, $g(x) = b\,\sigma(x)$, for all $x \in R$
where $\sigma, \tau: R \to R$ are two endomorphism mappings.
Then (d,g) is a (σ,τ)-S-derivation pair of R.
Let $x,y \in R$, so:

$$\begin{aligned} d(xy) &= a\sigma(xy) \\ &= a\sigma(x)\sigma(y) \\ &= a\sigma(x)\sigma(y) + \tau(x)\, b\sigma(y) \\ &= d(x)\sigma(y) + \tau(x)g(y) \end{aligned}$$

Also:

$g(xy) = b\sigma(xy)$
$= b\sigma(x)\sigma(y)$
$= b\sigma(x)\sigma(y) + \tau(x)\, a\sigma(y)$
$= g(x)\sigma(y) + \tau(x)d(y)$

Hence (d, g) is a (σ,τ)-S-derivation pair.

Remark 2.3

Every (σ,τ)-S-derivation pair is a Jordan (σ,τ)-S-derivation pair, but the Converse is in general not true.

The following example illustrates this:

Example 2.4

Let R be a 2-torsion free non commutative ring, and let $a \in R$, such that $\tau(x)\, a\, \sigma(x) = 0$, for all $x \in R$, but $\tau(x)\, a\, \sigma(y) \neq 0$, for some $(x \neq y) \in R$.

Define an additive pair d,g: $R \to R$, as follows:

$d(x) = \tau(x)\, a + a\, \sigma(x)$, $g(x) = \tau(x)a - a\sigma(x)$, for all $x \in R$.

where σ,τ: $R \to R$ are two endomorphism mappings.

Then (d,g) is a Jordan (σ,τ)-S-derivation pair, but not a (σ,τ)-S-derivation pair.

Let x,y $\in$ R, so:

$d(x^2)=\tau(x^2)\, a + a\sigma(x^2)$

$d(x)\sigma(x) + \tau(x)g(x) = (\tau(x)a + a\sigma(x))\ \sigma(x) + \tau(x)(\tau(x)a - a\sigma(x))$
$=\tau(x)a\sigma(x) + a\sigma(x)\sigma(x) + \tau(x)\tau(x)a - \tau(x)a\sigma(x)$
$=\tau(x^2)a + a\sigma(x^2)$

Hence $d(x^2) =d(x)\sigma(x) + \tau(x)\ g(x)$

Also:

$g(x^2) = \tau(x^2)a - a\sigma(x^2) = g(x)\sigma(x) + \tau(x)d(x)$

Thus, (d,g) is Jordan (σ,τ)-S-derivation pair.

Now, we show that (d,g) is not (σ,τ)-S-derivation pair.

$d(xy) = \tau(xy)a + a\sigma(xy)$

$d(x)\sigma(y) + \tau(x)g(y) = (\tau(x)a + a\sigma(x))\ \sigma(y) + \tau(x)(\tau(y)a - a\,\sigma(y))$
$=\tau(x)a\sigma(y) + a\sigma(x)\sigma(y) + \tau(x)\tau(y)a - \tau(x)a\sigma(y)$
$=\tau(xy)a + a\sigma(xy)$

Hence $d(xy)=d(x)\ \sigma(y) + \tau(x)g(y)$

But:

$g(xy)= g(x)\sigma(y) + \tau(x)d(y)$
$=(\tau(x)a - a\sigma(x))\sigma(y)+ \tau(x)(\tau(y)a + a\sigma(y))$
$=\tau(x)a\sigma(y) - a\sigma(x)\sigma(y) + \tau(x)\tau(y)a + \tau(x)a\sigma(y)$
$=\tau(xy)a - a\sigma(xy) +2\tau(x)a\sigma(y)$

On the other hand:

$g(xy)=\tau(xy)a - a\sigma(xy)$

Since $\tau(x)a\sigma(y)\neq 0$, for some $x\neq y \in R$, the two expressions are not equal, hence we get (d, g) is not (σ,τ)-S-derivation pair.

Proposition 2.5

Let R be a semiprime ring. Suppose that σ,τ are automorphisms of R. If R admits a (σ,τ)-S-derivation pair (d,g), such that $d(x)\ g(y)=0$
(resp. $g(x)\ d(y)=0$), for all $x,y \in R$, then d=0 (resp. g=0).

Proof

We have
$d(x)g(y)=0$, for all $x,y \in R$_____(1)
Replacing yx for y in (1) and using (1), we have:
$d(x)g(yx)=0$, for all $x,y \in R$.
$d(x)(g(y)\sigma(x) + \tau(y)d(x))=0$, for all $x,y \in R$.
$d(x)g(y)\sigma(x) + d(x)\tau(y)d(x)=0$, for all $x,y \in R$.
$d(x)\tau(y)d(x)=0$, for all $x,y \in R$._____(2)
By semiprimeness of R, (2) gives:
$d(x)=0$, for all $x \in R$.
If we have
$g(x)d(y)=0$, for all $x,y \in R$_____(3)
Replacing yx for y in (3) and using (3), we have:
$g(x)d(yx)=0$, for all $x,y \in R$.
$g(x)(d(y)\sigma(x) + \tau(y)g(x))=0$, for all $x,y \in R$.
$g(x)d(y)\sigma(x) + g(x)\tau(y)g(x)=0$, for all $x,y \in R$.
$g(x)\tau(y)g(x)=0$, for all $x,y \in R$_____(4)
By semiprimeness of R, (4) gives:
$g(x)=0$, for all $x \in R$.

Proposition 2.6

Let R be a semiprime ring. Suppose that σ,τ are automorphisms of R. If R admits a (σ,τ)-S-derivation pair (d,g), such that $d(x)=\pm\ \sigma(x)$ (resp. $g(x)=\pm\ \sigma(x)$), for all $x \in R$, then g=0 (resp. d=0).

Proof

We have
$d(x)=\sigma(x)$, for all $x \in R$_____(1)
Replacing x by xy in (1) and using (1), we get:
$d(xy) = \sigma(xy)$, for all $x,y \in R$.
$d(x)\sigma(y) + \tau(x)g(y)=\sigma(xy)$, for all $x,y \in R$.
$\sigma(x)\sigma(y)+\tau(x)g(y)= \sigma(x)\sigma(y)$, for all $x,y \in R$.
$\tau(x)g(y)=0$, for all $x,y \in R$ ____(2)
Left multiplication of (2) by g(y), leads to:
$g(y)\tau(x)g(y)=0$, for all $x,y \in R$ _____(3)
By semiprimeness of R, (3) gives:
$g(y)=0$, for all $y \in R$.

Similarly, we can show if $d(x)=-\sigma(x)$, for all $x \in R$, then $g=0$
In the same way, if $g(x)=\pm \sigma(x)$, for all $x\in R$, then $d=0$.

Proposition 2.7

Let R be any ring and σ,τ are two mappings on R. Then

1- If (d,g) is a (σ,τ)-S-derivation pair on R, then d+g is a (σ,τ)-derivation.
2- If (d,g) is a Jordan (σ,τ)-S-derivation pair on R, then d+g is a Jordan (σ,τ)-derivation.

Proof

1- We have
(d,g) is a (σ,τ)-S-derivation pair, so
$d(xy)= d(x)\,\sigma(y)+\tau(x)g(y)$, for all $x,y \in R$ _____(1)
$g(xy)= g(x)\,\sigma(y)+\tau(x)d(y)$, for all $x,y \in R$ _____(2)

By adding (1) and (2), we get
$(d+g)(xy)=(d+g)(x)\sigma(y)+\tau(x)(d+g)(y)$
Hence d+g is a (σ,τ)-derivation

2- We have
(d,g) is a Jordan (σ,τ)-S-derivation pair, so
$d(x^2)=d(x)\sigma(x) + \tau(x)g(x)$, for all $x \in R$ _____(3)
$g(x^2)=g(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$ _____(4)

By adding (3) and (2), we get
$(d+g)(x^2)=(d+g)(x)\sigma(x) + \tau(x)(d+g)(x)$, for all $x \in R$.
Hence d+g is a Jordan (σ,τ)-derivation.

§3 Relation Between (σ,τ)-S-Derivation pairs and (σ,τ)-Derivations

In this section, we study prime rings, semiprime rings, and rings that have a commutator left nonzero divisor with (σ,τ)-S-derivation pair, to obtain a (σ,τ)-derivation.

Theorem 3.1

Let R be a 2-torsion free semiprime ring, and (d,g) be a (σ,τ)-S-derivation pair on R, then d and g are (σ,τ)-derivations. Where σ,τ are automorphisms of R.

Proof

Suppose that (d,g) is (σ,τ)-S-derivation pair. Then:
$d(xyx)=d(x(yx)) = d(x)\sigma(yx) + \tau(x)g(yx)$, for all $x,y \in R$ _____(1)
That is:
$d(xyx)=d(x)\sigma(yx) + \tau(x)g(y)\sigma(x) + \tau(x)\tau(y)d(x)$, for all $x,y \in R$____(2)
Also:

$d(xyx)=d((xy)x)=d(xy)\sigma(x) + \tau(xy)g(x)$, for all $x,y \in R$_____(3)
That is:
$d(xyx)=d(x)\sigma(y)\sigma(x) + \tau(x)g(y)\sigma(x) + \tau(xy)g(x)$, for all $x,y \in R$___(4)
From (2) and (4), we get:
$\tau(xy)(d(x)-g(x))=0$, for all $x,y \in R$_____(5)
Replace $\tau(y)$ by $(d(x)-g(x))\ \tau(y)\ \tau(x)$ in (5), we get:
$\tau(x)(d(x)-g(x))\tau(y)\tau(x)(d(x)-g(x))=0$, for all $x,y \in R$_____(6)
Since R is semiprime, we get:
$\tau(x)d(x)=\tau(x)g(x)$, for all $x \in R$_____(7)
It follows that:
$d(x^2)=d(x)\sigma(x)+\tau(x)d(x)$, for all $x \in R$_____(8)
And:
$g(x^2)=g(x)\sigma(x)+\tau(x)g(x)$, for all $x \in R$_____(9)
Thus, by using [3, Theorem 2.3.7], we obtain that d and g are (σ,τ)-derivations on R.

Theorem 3.2

Let R be a prime, and (d,g) be a (σ,τ)-S-derivation pair on R, then d and g are (σ,τ)-derivations. Where σ,τ are automorphisms of R.

Proof

Since (d,g) is (σ,τ)-S-derivation pair, we have (see how relation (5) was obtained from relation (1) in the proof of Theorem 3.1)
$\tau(xy)(d(x)-g(x)=0$, for all $x,y \in R$_____(1)
And, by primeness of R, we get:
$d(x)=g(x)$, for all $x \in R$_____(2)
And hence d and g are (σ,τ)-derivations on R.

Theorem 3.3

Let R be a ring which has a commutator left nonzero divisor and (d,g) be a (σ,τ)-S-derivation pair on R, then d and g are (σ,τ)-derivations.Where σ,τ are automorphisms of R.

Proof

1. That is We have:
2. $d(yx^2)=d(y)\sigma(x^2) + \tau(y)g(x^2)$, for all $x,y \in R$____(1)
3. That is:
4. $d(yx^2)=d(y)\sigma(x^2)+ \tau(y)g(x)\sigma(x)+\tau(y)\tau(x)d(x)$, for all $x,y \in R$__(2)
5. On the other hand:
6. $d(yx^2)=d(yx)\sigma(x)+ \tau(yx)g(x)$, for all $x,y \in R$_____(3)
7.
8. $d(yx^2)=d(y)\sigma(x^2)+ \tau(y)g(x)\sigma(x) + \tau(y)\tau(x)g(x)$, for all $x,y \in R$_____(4)
9. From (2) and (4), we obtain:

$\tau(y)(\tau(x)d(x)-\tau(x)g(x))=0$, for all $x,y \in R$_____(5)
Replacing y by yr in (5), to get:
$\tau(yr)(\tau(x)d(x)-\tau(x)g(x))=0$, for all $x,y,r \in R$_____(6)
Again, left multiplying of (5) by $\tau(r)$, to get:
$\tau(r)\tau(y)(\tau(x)d(x)-\tau(x)g(x))=0$, for all $x,y,r \in R$_____(7)
Subtracting (7) from (6), we get:
$[\tau(y),\tau(r)](\tau(x)d(x)-\tau(x)g(x))=0$, for all $x,y,r \in R$_____(8)
Since R has a commutator left nonzero divisor, we get:
$\tau(x)d(x)=\tau(x)g(x)$, for all $x \in R$_____(9)
Linearizing (9), we get:
$\tau(x)d(y) + \tau(y)d(x)= \tau(x)g(y) + \tau(y)g(x)$, for all $x,y \in R$_____(10)
That is:
$\tau(x)(d-g)(y) + \tau(y)(d-g)(x)=0$, for all $x,y \in R$_____(11)
Replacing y by ry in (11), to get:
$\tau(x)(d-g)(ry) + \tau(ry)(d-g)(x)=0$, for all $x,y,r \in R$_____(12)
Again, left multiplying of (11) by $\tau(r)$, to get:
$\tau(r)\tau(x)(d-g)(y) + \tau(r)\tau(y)(d-g)(x)=0$, for all $x,y,r \in R$ _____(13)
Subtracting (12) from (13), we get:
$\tau(rx)(d-g)(y) - \tau(x)(d-g)(ry)=0$, for all $x,y,r \in R$ _____(14)
Replacing x by sx in (14), to get:
$\tau(rsx)(d-g)(y)-\tau(sx)(d-g)(ry)=0$, for all $x,y,r,s \in R$ _____(15)
Also, left multiplying of (14) by $\tau(s)$, to get:
$\tau(srx)(d-g)(y)-\tau(sx)(d-g)(ry)=0$, for all $x,y,r,s \in R$_____(16)
Subtracting (16) from (15), we get:
$[\tau(r),\tau(s)]\,\tau(x)(d-g)(y)=0$, for all $x,y,r,s \in R$ _____(17)
Since R has a commutator left nonzero divisor, we get:
$\tau(x)(d-g)(y)=0$, for all $x,y \in R$_____(18)
That is:
$\tau(x)d(y)=\tau(x)g(y)$, for all $x,y \in R$_____(19)
Hence d and g are (σ,τ)-derivations.

References

1. Herstien, I.N., (1969), TOPICS IN RING THEORY, The University of Chicago Press, Chicago.
2. Ashraf, M., Ali, S. and Haetinger, C., (2006), " On Derivations in Rings and their Applications", The Aligarh Bull of Math., 25(2), 79-107.
3. Hamdi, A.D., (2007), "(σ,τ)-Derivations on prime Rings", MSc. Thesis, Baghdad University.
4. Yass, S., (2010), "Strongly Derivation Pairs on Prime and Semiprime Rings", MSc. Thesis, Baghdad University.
5. Cortes, W. and Haetinger, C., (2005),"On Jordan Generalized Higher Derivations in Rings", Turkish J. of Math., 29(1),1-10.

الأشتقاقات المزدوجة القوية-(σ,τ) على الحلقات

اكرام احمد سعيد
الجامعة التكنولوجية

استلم البحث في : 30 تشرين الثاني 2010
قبل البحث في : 27 شباط 2011

الخلاصة

لتكن R حلقة تجميعية. في هذا البحث قدمنا تعريف الأشتقاق المزدوج القوي-(σ,τ) واشتقاق جوردان المزدوج القوي-(σ,τ) في الحلقة R، ودراسة العلاقة بينهم. كذلك، ندرس الحلقات الأولية، الحلقات شبه الأولية، والحلقات التي لها مبدل قاسم غير صفري أيسر مع الأشتقاق المزدوج القوي-(σ,τ) للحصول على الأشتقاق -(σ,τ). أي ان $\sigma,\tau: R \rightarrow R$ هما دالتين على الحلقة R.

الكلمات المفتاحية :

حلقة اولية، حلقة شبه اولية، مشتقة -(σ,τ)، الأشتقاق المزدوج القوي -(σ,τ)، اشتقاق جوردان المزدوج القوي -(σ,τ).