مجلة ابن الهيثم للعلوم الصرفة والتطبيقية المجلد 23 (2) 2010

النظريات المباشرة والعكسية لمتعددات حدود جاكسون للدوال الدورية المقيدة القابلة للقياس في الفضاءات المحلية

صاحب كحيط جاسم ، نادية جاسم محمد
قسم الرياضيات ،كلية العلوم ، الجامعة المستنصرية
قسم الرياضيات ، كلية التربية – ابن الهيثم ، جامعة بغداد

الخلاصة

تم في هذا البحث ايجاد تقدير افضل لاقتراب الدوال المقيدة القابلة للقياس في الفضاءات $L_{\delta,p}$ $(1 \le p < \infty)$ من التقديرات التي وجدها بوبوف.

كذلك تم ايجاد العلاقة بين نموذج القياس $\tau_1(f,\frac{1}{n})_p$ والفرق بين الدالة f ومتعددة جاكسون اي وجدنا ان

$$\tau_1(f,\frac{1}{n})_p \approx \|f - J_n(f)\|_p$$

Direct and Inverse Inequalities for Jackson Polynomials of 2π-Periodic Bounded Measurable Functions in Locally Clobal Norms

S.K.Jassim, N.J.Mohamed
Department of Mathematics,College of Science, University of Al-Mustansirya
Department of Mathematics-Ibn-Al-Haitham, College of Education, University of Baghdad

Abstract

Convergence properties of Jackson polynomials have been considered by Zugmund [1,ch.X] in (1959) and J.Szbados [2], ($p=\infty$) while in (1983) V.A.Popov and J.Szabados [3] ($1 \le p \le \infty$) have proved a direct inequality for Jackson polynomials in L_p-space of 2π-periodic bounded Riemann integrable functions ($f \in R$) in terms of some modulus of continuity.

In 1991 S.K.Jassim proved direct and inverse inequality for Jackson polynomials in locally global norms ($L_{\delta,p}$) of 2π-periodic bounded measurable functions ($f \in L_\infty$) in terms of suitable Peetre K-functional [4].

Now the aim of our paper is to proved direct and inverse inequalities for Jackson polynomials of ($f \in L_\infty$) in ($L_{\delta,p}$) in terms of the average modulus of continuity.

Introduction

We denote the set of 2π-periodic bounded measurable functions with usual sup-norm by L_∞ ∍

1. $L_\infty(X) = \left\{ f : \| f \|_{L_\infty(X)} = \sup\{ | f(x) | \forall x \in X\} < \infty \right\}, \| f \|_{L_\infty(X)} = \| f \|_\infty$

and the L_p-norm ($1 \le p < \infty$) of $f \in L_p$ by $\| f \|_{L_p}$ ∍

2. $L_p(X) = \left\{ f : \| f \|_{L_p(X)} = (\int_X | f(x) |^p dx)^{\frac{1}{p}} < \infty \right\}; \| f \|_{L_p(X)} = \| f \|_p$.

and the direct norm is defined by:

3. $\ell_n^p(X) = \left\{ f : \| f \|_{\ell_n^p(x)} = (\frac{1}{n+1} \sum_{k=0}^{n} | f(x_{k,n}) |^p)^{\frac{1}{p}} < \infty \right\},$

where $(x_{k,n}) = \frac{2k\pi}{n+1}, (k = 0,1,2,...,n), \| f \|_{\ell_n^p(x)} = \| f \|_{\ell_n^p}$.

Now let us consider the Dirich let kernel of degree n

4. $D_n(u) = \frac{1}{2} + \sum_{v=1}^{n} \cos(vu)$ ∍ $u \in R$, n=0,1,… then it is easy to get the following:-

$$D_n(u)=\frac{\sin(n+\frac{1}{2})u}{2\sin(\frac{u}{2})} \text{ and } \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)=1.$$

5. let $K_n(u)=\frac{1}{n+1}[D_0(u)+D_1(u)+\ldots+D_n(u)]$ be the Fejer kernel of degree not grater than n, also it is easy to get the following:-

$$K_n(u)=\frac{1}{2(n+1)}\frac{\sin^2(n+1)\frac{u}{2}}{\sin^2(\frac{u}{2})}, K_n(u)=\frac{1}{2}+\frac{1}{n+1}\sum_{k=1}^{n}(n-k+1)\cos(ku) \text{ and } \int_{-\pi}^{\pi} K_n(u)\,du=1.$$

6. $J_n(f,x)=\frac{2}{n+2}\sum_{k=0}^{n} f(x_{k,n})K_n(x-x_{k,n})$ is the so called Jackson polynomial of function $f\in L_\infty$.

Now we will use the so called average modulus of continuity to solve the problem in locally global norm ($L_{\delta,p}$).

The locally modulus of continuity for ($f\in L_\infty$) is defined by

7. $\omega(f,x,h)=\sup\left\{f(x')-f(x^n),x',x^n\in\left[x-\frac{h}{2},x+\frac{h}{2}\right]\right\}$, h is a constant number while k^{th} average modulus of smoothness for $f\in L_p$ is defined by:

8. $\tau_K(f,\delta)_{L_p}=\|\omega_K(f,\bullet,\delta)\|_{L_p} \ni 1\le p\le\infty, \delta>0$ and $k\in\square$.

where the k^{th} modulus of smoothness for $f\in L_p$, $k\in\square$ is defined by:

9. $\omega_k(f,x,\delta)=\sup\left\{\left|\Delta_h^k f(t)\right|: t,t+kh\in\left[x-\frac{k\delta}{2},x+\frac{k\delta}{2}\right]\cap X\right\}$.

Now we set

$$\Delta_h^k f(x)=\begin{cases}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}f(x+mh) & \text{if } x \text{ or } x+h\in X\\ 0 & \text{otherwise.}\end{cases}$$

Then we introduce the definition of k^{th} local modulus of L_p-continuity for $f\in L_p$, ($1\le p\le\infty$) and $k\in\square$.

10. $\omega_k(f,x,\delta(x))_p=\frac{1}{2\delta(x)}\left[\int_{-\delta(x)}^{\delta(x)}\left|\Delta_h^k f(x)\right|^p dv\right]^{\frac{1}{p}}$

where $\delta(x)$ is an arbitrary positive function of x, but here we shall consider only the case $\delta(x)$ is constant.

The k^{th} average modulus of smoothness was first introduced by B.Sendov in 1983 and proved to be very useful in some approximation theoretical problems where the ordinary modulus of continuity $\omega(f,\delta)$ is defined by the following:-

11. $\omega(f,\delta)=\sup\{|f(x')-f(x'')|, |x'-x''|\le\delta, x', x''\in X\}$.

while the k^{th} ordinary modulus of continuity is defined by the following:-

12. $\omega_k(f,\delta)=\sup\left\{\left|\Delta_h^k f(x)\right|: |h|\le\delta, x, x+kh\in X\right\}$.

Now let us consider the definition of ordinary L_p-modulus of continuity ($\omega(f,\delta)_p$).

13. $\omega(f,\delta)_{L_p} = \sup_{|t|\le\delta} \| f(\cdot + t) - f(\cdot)\|_{L_p}$ for δ constant and

14. $\omega_k(f,\delta) = \sup\left\{ \|\Delta_h^k f(\bullet)\|_p : |h| \le \delta \right\}$ also for δ constant, be the k^{th} ordinary L_p-modulus of continuity of $f\in L_p$.

Now for $f\in L_p$ instead of usual sup-norm, let us consider the family of semi norm (locally global norm) for $\delta > 0$, $(1\le p\le\infty)$.

15. $\|f\|_{\delta,p} = \left(\int_X |f_\delta(x)|^p\, dx \right)^{1/p}$,where $f_\delta(X)=\sup\{|f(t)| : t\in U(\delta,x)\}$ and $U(\delta,x)=\{y\in X : |x-y|\le\delta\}$.

The main property of these semi norm is that they do not necessarily vanish if f = 0 Lebesgue almost every where.

Let us denoted by $L_{\delta,p}$ the set of functions from L_∞ which equipped with semi norm $\|\bullet\|_{\delta,p}$.

By T_n we denote the set of all trigonometric polynomials in R of degree not greater than n.

The best approximation to a given continuous function with trigonometric polynomials from T_n on the interval X is given by:

16. $E_n^T(f : X) = \inf\left\{ \|f(\cdot) - T(\cdot)\|_\infty : T \in T_n \right\}$.

While the best approximation of a function $f\in L_p(X)$ with trigonometric polynomials from T_n in the metric of the space L_p is given by:

17. $E_n^T(f)_p = \inf\left\{ \|f(\cdot) - T(\cdot)\|_p : T \in T_n \right\}$.

We also define the best approximation of a function $f\in L_\infty(X)$ with trigonometric polynomials from T_n in the metric of the spaces L_p or $L_{\delta,p}$ are respectively given by:-

18. $E_n^T(f)_{\delta,p} = \inf\left\{ \|f(\cdot) - T(\cdot)\|_{\delta,p(X)} : T \in T_n \right\}$.

The best one sided approximation of a function $f\in L_\infty(X)$ with trigonometric polynomials from T_n in the metric of the space L_p or $L_{\delta,p}$ are respectively given by

19. $\tilde{E}_n^T(f)_p = \inf\left\{ \|T^+(\cdot) - T^-(\cdot)\|_{p(X)} : T^\mp \in T_n, T_n^-(x) \le f(x) \le T_n^+(x), x \in X \right\}$.

20. $\tilde{E}_n^T(f)_{\delta,p} = \inf\left\{ \|T^+(\cdot) - T^-(\cdot)\|_{\delta,p(X)} : T^\mp \in T_n, T_n^-(x) \le f(x) \le T_n^+(x), x \in X \right\}$.

Now let f be a function defined on a domain D then the Stecklov transformation is given by

21. $f_n(x) = \frac{n}{2} \int_{-1/n}^{1/n} f(x+t)dt$.

Let f and g be two functions then we say that f(x) = O{g(x)} if $|f(x)| < A\cdot g(x)$, x goes to some given limit, A is constant and $g(x) \ne 0$.

In particular, O(1) means bounded function.

By f(x) = o{g(x)} we mean that $(f(x) / g(x)) \to 0$ as x tends to a given limit. In particular, o(1) means a function which tends to zero.

Let us introduce some definitions about Peeter K-functionals, let $W_p^1 = \{g : g' \in L_p\}, \tilde{W}_p^1 = \{g : \tilde{g}' \in L_p\}$ have semi norms $\|g'\|_p$ and $\|\tilde{g}'\|_p$ respectively, where $\tilde{g}$ is the conjugate function of a function g.

We shall consider the opppropriate Peetre K-functionals as follows:
$(t > 0), (1 \le p \le \infty)$.

22. $K(f, t, L_p, w_p^1) = \inf\left\{\|f - g\|_p + t\|g'\|_p : g \in w_p^1\right\}$,

23. $K(f; t, L_{t,p}, w_p^1) = \inf\left\{\|f - g\|_{t,p} + t\|g'\|_p : g \in w_p^1\right\}$,

24. $K(f, t, L_p, w_p^1, \tilde{w}_p^1) = \inf\left\{\|f - g\|_p + t\|g'\|_p + t\|\tilde{g}\|_p : g \in w_p^1 \cap \tilde{w}_p^1\right\}$,

25. $K(f, t, L_{t,p}, w_p^1, \tilde{w}_p^1) = \inf\left\{\|f - g\|_{t,p} + t\|g'\|_p + t\|\tilde{g}'\|_p : g \in w_p^1 \cap \tilde{w}_p^1\right\}$.

Next Bernstein inequalities are given since we use them further on

26. $\|T_n'\|_p \le n\|T_n\|_p, T_n \in T_n, 1 \le p \le \infty.$

27. $\|\tilde{T}_n'\|_p \le n\|T_n\|_p, T_n \in T_n, 1 \le p \le \infty.$

1- Assertions

1.1 Lemma: [3]

Let $f \in L_p$, then $(1 \le p \le \infty)$

28. $\|J_n(f, \bullet)\|_p = O(1)\left\{\frac{1}{n}\sum_{k=0}^{n}|f(t_k)|^p\right\}^{\frac{1}{p}} = O\|f\|_{\ell_n^p}$.

1.2 Lemma: [5]

Let $T \in T_n$, then $(1 \le p \le \infty)$

29. $\|T\|_{\delta, p(\Pi)} \le C(1 + n\delta)^{\frac{1}{p}}\|T\|_{p(\Pi)}$.

1.3 Lemma: [3]

For $f, f' \in L_p$ we have $(1 \le p \le \infty)$

30. $\omega(f, h)_p = O(h)\|f'\|_{L_p}$, h is constant.

1.4 Lemma: [6]

Let $f \in L_\infty$, then we have $(1 \le p \le \infty)$

31. $\omega_1(f, \delta)_p \le \tau_1(f, \delta)_p \le \omega_1(f, \delta)_\infty$.

1.5 Lemma: [7]

Let $f \in L_\infty$, then we have $(1 \le p \le \infty)$

32. $\|f\|_{\ell_m^p} \le C\|f\|_{\frac{1}{m}, p}$, (m = 1, 2, 3, …).

1.6 Lemma: [8]

For $f \in L_\infty$, we have

33. $\|f\|_{p(\Omega)} \le \|f\|_{\delta, p(\Omega)} \le \|f\|_{\delta, \infty(\Omega)} \le \|f\|_{\infty(\Omega)}$.

1.7 Lemma: [6]

Let $f \in L_\infty$ and $(1 \le p \le \infty)$ we have

34. $\tau_k(f,\lambda\delta)_p \le (2(\lambda+1))^{k+1}\tau_k(f;\delta)_{L_p}, \lambda > 0$.

1.8 Lemma: [6]

For g, g' $\in L_p$, we have $(1\le p\le\infty)$

35. $\tau_1(g,\delta)_p \le, \delta\|g'\|_p$.

1.9 Lemma: [5]

For $f \in L_\infty$ and $(1\le p\le\infty)$, we have

36. $\tilde{E}_n(f)_p \le E_n(f)_{\frac{1}{n},p} \le c\tilde{E}_n(f)_p$.

1.10 Theorem: [6]

For $f \in L_{\infty(\Pi)}$, we have

37. $\tilde{E}_n(f)_p \le c_k\tau_k(f,\frac{1}{n})_p$, $1\le p\le\infty$.

38. $\tau_k(f,\frac{1}{n})_p \le c_k n^{-k}\sum_{v=0}^{n}(v+1)^{k-1}\tilde{E}_v(f)_p$.

1.11 Theorem: [3]

For 2π-periodic bounded Reimann integrable functions we have the following:-

39. $\|f - J_n(f,x)\|_p = O(1)\left[\tau_1(f,\frac{1}{n})_p + W_1(\tilde{f},\frac{1}{n})_p\right], p = 1,\infty$

40. $\|f - J_n(f,x)\|_p = O(1)\tau_1(f,\frac{1}{n})_p, 1 < p < \infty.$

where $\tilde{f}(x)$ is the conjugate function to f(x).

1.12 Theorem: [4]

For $f \in L_\infty$, then

41. $K(f,\frac{1}{n},L_{\frac{1}{n},p}, W_p^1,\tilde{W}_p^1) \le c\left[\tau_1(f,\frac{1}{n})_p + W_1(\tilde{f},\frac{1}{n})\right], p = 1,\infty.$

42. $\tau_1(f,\frac{1}{n})_p \le K(f;\frac{1}{n};L_{\frac{1}{n},p}, W_p^1,\tilde{W}_p^1) \le c\tau_1(f,\frac{1}{n})_p 1 < p < \infty.$

1.13 Theorem: [9]

For $f \in L_p$, we have

43. $W_K(f,\frac{1}{n})_p \le \frac{C(k)}{n^k}\sum_{v=0}^{n}(1+v)^{k-1}E_v(f)_p$.

2- Main Results

The aim of his paper is to find a better estimation for the rate of convergence in $L_{\delta,p}$ space of $(f \in L_\infty)$ by Jackson polynomials in terms of some modulus of functions.

We shall prove direct and inverse theorems of $(f \in L_\infty)$ by Jackson polynomials in locally global norms in terms of ordinary L_p-modulus of continuity and average modulus of continuity.

2.1 Theorem: (Direct Theorem)

Let $f \in L_\infty$, then

$$\|f - J_n(f)\|_{\frac{1}{n},p} \le \begin{cases} c\tau_1(f,\frac{1}{n})_p + c\omega_1(\tilde{f},\frac{1}{n})_p, & p = 1,\infty \\ c\tau_1(f,\frac{1}{n})_p, & 1 < p < \infty. \end{cases}$$

2.2 Theorem: (Inverse Theorem)

Let $f \in L_\infty$, then

$$\tau_1(f,\frac{1}{n})_p \le \frac{C}{n}\sum_{s=0}^{n} \begin{cases} \|f - J_n(f)\|_{(1/s+1),p} + \|\tilde{f} - J_s(\tilde{f})\|_{(1/s+1),p}, & p = 1,\infty \\ \|f - J_s(f)\|_{(1/s+1),p}, & 1 < p < \infty. \end{cases}$$

2.3 Lemma:

Let $T^{\pm}_n \in T_n$ such that $\tilde{E}^T_n(f)_p = \|T^+_n - T^-_n\|_p$, $f \in L_\infty$ and $T^-_n(x) \le f(x) \le T^+_n(x)$ then

44. $\|\tilde{T}^{+\prime}_n\|_p \le C_n\left[\tau_1(f,\frac{1}{n})_p + \omega_1(\tilde{f},\frac{1}{n})_p\right]$, $1 \le p \le \infty$.

Proof: Let $f_n(x) = n\int_{\frac{1}{2\pi}} f(x+u)\,du$ be the Stecklov transformation $h(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_n(x)D_n(t)\,dt$ be a trigonometric polynomial of degree n, such that $D_n(t)$ is the Dirichlet kernel, then by using Bernstein inequality, we get

$$\|\tilde{T}^{+\prime}_n\|_p \le n\|T^+_n - h\|_p + \|\tilde{h}'\|_p$$

$$\le n\|T^+_n - f\|_p + n\|f - h\|_p + \|\tilde{h}'\|$$

$$\le n\|T^+_n - T^-_n\|_p + n\|f - h\|_p + \|\tilde{h}'\|_p$$

Now let us assume that $\|f - h\|_p = A_1$ and $\|\tilde{h}'\|_p = A_2$ then we get

$$\|\tilde{T}^{+\prime}_n\|_p \le n\tilde{E}^T_n(f)_p + nA_1 + A_2$$

$$A_1 = \|f - h\|_p = \left\|\frac{1}{\pi}\int_{-\pi}^{\pi}[f(\cdot) - f_n(\cdot)]D_n(t)dt\right\|_p$$

$$= \left\|\frac{1}{\pi}\int_{-\pi}^{\pi} n\int_{\frac{-1}{2n}}^{\frac{1}{2n}}[f(\cdot) - f_n(\cdot+u)]duD_n(t)dt\right\|_p$$

$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} n\int_{\frac{-1}{2n}}^{\frac{1}{2n}}\|f(\cdot) - f_n(\cdot+u)\|_p\,duD_n(t)dt$$

This by using definition of average modulus of continuity and by (34), we get

$$A_1 \le \frac{1}{\pi}\int_0^{2\pi} n\int_{\frac{-1}{2n}}^{\frac{1}{2n}} \tau_1(f,\frac{1}{2n})_p \, du D_n(t)dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} \tau_1(f,\frac{1}{2n})_p D_n(t)dt \le C\tau_1(f,\frac{1}{2n})_p$$

$$A_2 = \|\tilde{h}'\|_p = \left\|\frac{1}{\pi}\int_{-\pi}^{\pi} n\int_{\frac{-1}{2n}}^{\frac{1}{2n}} \tilde{f}'(\cdot+u)du D_n(t)dt\right\|_p$$

$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} n\left\|\tilde{f}'(\cdot+\frac{1}{2n}) - \tilde{f}(\cdot-\frac{1}{2n})\right\| D_n(t)dt$$

Then by using definition of ordinary L_p-modulus of continuity we get

$$A_2 \le C_2 n\omega_1(\tilde{f},\frac{1}{n})_p$$

Thus

$$\|\tilde{T}_n^{+}{}'\|_p \le n\tilde{E}_n^T(f)_p + nC_1\tau_1(f,\frac{1}{n})_p + C_2 n\omega(\tilde{f},\frac{1}{n})_p$$

By using (37)

$$\|\tilde{T}_n^{+}{}'\|_p \le C_n\left[\tau_1(f,\frac{1}{n})_p + \omega_1(\tilde{f},\frac{1}{n})_p\right].$$

2.4 Lemma:

Let $f \in L_\infty$ then we have $1 \le p \le \infty$

45. $\|J_n(f)\|_{\frac{1}{n},p} \le C\|f\|_{\frac{1}{n},p}$.

Proof: By using (29), (28) and (32), we get

$$\|J_n(f)\|_{\frac{1}{n},p} \le C_1(1+n\frac{1}{n})^{\frac{1}{p}}\|J_n(f)\|_p$$

$$\le C_2 O(1)\frac{1}{n}\sum_{k=0}^{n}|f(t_k)|$$

$$\le C_3\|f\|_{\ell_n^p}$$

$$\le C_4\|f\|_{\frac{1}{n},p}$$

2.5 Lemma:

Let $f \in L_\infty$ then we have $1 \le p \le \infty$

46. $\tau_k(f,\delta)_p \le C\|f\|_p$.

Proof: By using definition of average modulus of continuity, we get

$$\tau_k(f,\delta)_p = \|\omega_k(f,\delta)\|_p = \left\|\sup\left\{|\Delta_h^k f(t)|; t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap X\right\}\right\|_p$$

$$\le \sum_{i=0}^{k}\binom{k}{i}\left\|\sup|f(\tau+ih)|; t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap X\right\|_p$$

$$= C\|f\|_p$$

2.6 Lemma:

Let $t > 0$, $f \in L_\infty$, we have $1 \le p \le \infty$

47. $K(f;t;L_p,W_p^1,\tilde{W}_p^1) \le K(f;t;L_{t,p},W_p^1,\tilde{W}_p^1)$

Proof: By using definition of Peetre K-functional and (33), we get

$$K(f;t;L_p,W_p^1,\tilde{W}_p^1) = \inf\left\{\|f-g\|_p + t\|g'\|_p + t\|\tilde{g}'\|_p, g \in W_p^1 \cap \tilde{W}_p^1\right\}$$

$$\le \inf\left\{\|f-g\|_{t,p} + t\|g'\|_p + t\|\tilde{g}'\|_p, g \in W_p^1 \cap \tilde{W}_p^1\right\}$$

$$= K(f;t;L_{t,p},W_p^1,\tilde{W}_p^1)$$

2.7 Lemma:

Let f, g be two function define on the same domain then for $1 \le p \le \infty$, we have

48. $\tau_k(f,\frac{1}{n})_p \le \tau_k(f-g,\frac{1}{n})_p + \tau_k(g,\frac{1}{n})_p$.

Proof: The proof follows from the definition of $\tau_k(f,\delta)_p$.

2.8 Lemma:

For $t > 0$, $f \in L_\infty$, we have

49. $K(f;t;L_p,W_p^1) \le K(f;t;L_p,W_p^1,\tilde{W}_p^1)$

Proof: By using definition of Peetre K-functional

$$K(f;t;L_p,W_p^1) = \inf\left\{\|f-g\|_p + t\|g'\|_p ; g \in W_p^1\right\}$$

$$\le \inf\left\{\|f-g\|_p + t\|g'\|_p + t\|\tilde{g}'\|_p ; g \in W_p^1 \cap \tilde{W}_p^1\right\}$$

$$= K(f;t;L_p,W_p^1,\tilde{W}_p^1)$$

2.9 Lemma:

Let $f \in L_\infty$, $1 \le p \le \infty$, then we have

50. $K(f;\frac{1}{n};L_p,W_p^1,\tilde{W}_p^1) \le \frac{C}{n}\sum_{s=0}^{n}\begin{cases} E_S^T(f)_{(1/s+1),p} + E_S^T(\tilde{f})_{(1/s+1),p}, & p = 1,\infty \\ E_S^T(f)_{(1/s+1),p}, & 1 < p < \infty \end{cases}$

Proof: By using (49), (41), (49), (38), (43), (36) and (33), we get

$$K(f;\frac{1}{n};L_p,W_p^1,\tilde{W}_p^1) \le K(f;\frac{1}{n};L_{(1/n),p},W_p^1,\tilde{W}_p^1)$$

$$\le \begin{cases} C_1\left[\tau_1(f,\frac{1}{n})_p + \omega_1(\tilde{f},\frac{1}{n})_p\right], & p = 1,\infty \\ C_2\left[\tau_1(f,\frac{1}{n})_p\right], & 1 < p < \infty. \end{cases}$$

$$\le \begin{cases} C_1\frac{C_1'}{n}\sum_{S=0}^{n}\tilde{E}_S^T(f)_p + C_1\frac{C_1'}{n}\tilde{E}_S^T(\tilde{f})_p, & p = 1,\infty \\ C_2\frac{C_1'}{n}\sum_{S=0}^{n}\tilde{E}_S^T(f)_p, & 1 < p < \infty. \end{cases}$$

$$\le \frac{C}{n}\sum_{s=0}^{n}\begin{cases} E_S^T(f)_{(1/s+1),p} + E_S^T(\tilde{f})_{(1/s+1),p}, & p = 1,\infty \\ E_S^T(f)_{(1/s+1),p}, & 1 < p < \infty \end{cases}$$

Proof of Theorem 2.1 (Direct Theorem)

Let $T_n^{\pm} \in T_n$ such that $\tilde{E}_n^T(f)_p = \|T_n^+ - T_n^-\|_p$ then by using linearity of Jackson polynomials, (45), (29), (39), (40), (30), (44) and (46)

$$\|f - J_n(f)\|_{(1/n),p} \le \|f - T_n^+\|_{(1/n),p} + \|T_n^+ - J_n(T_n^+)\|_{(1/n),p} + \|J_n(T_n^+ - f)\|_{(1/n),p}$$

$$\le \|T_n^- - T_n^+\|_{(1/n),p} + \|T_n^+ - J_n(T_n^+)\|_{(1/n),p} + C_1\|T_n^+ - T_n^-\|_{(1/n),p}$$

$$= C_2\tilde{E}_n^T(f)_p + C_3\|T_n^+ - J_n(T_n^+)\|_p$$

$$\le C_2\tilde{E}_n^T(f)_p + C_3\begin{cases} O(1)\tau_1(T_n^+, \frac{1}{n})_p + \omega_1(\tilde{T}_n^+, \frac{1}{n})_p, & p = 1,\infty \\ O(1)\tau_1(T_n^+, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

$$\le C_2\tilde{E}_n^T(f)_p + \begin{cases} C_4\tau_1(T_n^+, \frac{1}{n})_p + C_4 O(\frac{1}{n})\|\tilde{T}_n^+\|_p, & p = 1,\infty \\ C_4\tau_1(T_n^+, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

$$\le C_2\tilde{E}_n^T(f)_p + \begin{cases} C_4\tau_1(T_n^+ - f, \frac{1}{n})_p + C_4\tau_1(f, \frac{1}{n})_p O(\frac{C_4}{n}) \\ C_5\tau_1(f, \frac{1}{n})_p + \omega_1(\tilde{f}, \frac{1}{n})_p\|\tilde{T}_n^+\|_p, & p = 1,\infty \\ C_4\tau_1(T_n^+ - f, \frac{1}{n})_p + C_4\tau_1(f, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

$$\le C_2\tilde{E}_n^T(f)_p + \begin{cases} C_6\|T_n^+ - f\|_p + C_7\tau_1(f, \frac{1}{n})_p + C_8\omega_1(\tilde{f}, \frac{1}{n})_p, & p = 1,\infty \\ C_6\|T_n^+ - f\|_p + C_4\tau_1(f, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

$$\|f - J_n(f)\|_{(1/n),p} \le C_2\tilde{E}_n^T(f)_p + \begin{cases} C_6\|T_n^+ - T_n^-\|_p + C_7\tau_1(f, \frac{1}{n})_p + C_8\omega_1(\tilde{f}, \frac{1}{n})_p, & p = 1,\infty \\ C_6\|T_n^+ - T_n^-\|_p + C_4\tau_1(f, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

$$= C_2\tilde{E}_n^T(f)_p + \begin{cases} C_6\tilde{E}_n^T(f)_p + C_7\tau_1(f, \frac{1}{n})_p + C_8\omega_1(\tilde{f}, \frac{1}{n})_p, & p = 1,\infty \\ C_6\tilde{E}_n^T(f)_p + C_4\tau_1(f, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

$$\le C_9\tilde{E}_n^T(f)_p + C_{10}\begin{cases} \tau_1(f, \frac{1}{n})_p + C_8\omega_1(\tilde{f}, \frac{1}{n})_p, & p = 1,\infty \\ \tau_1(f, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

$$\le C\begin{cases} \tau_1(f, \frac{1}{n})_p + \omega_1(\tilde{f}, \frac{1}{n})_p, & p = 1,\infty \\ \tau_1(f, \frac{1}{n})_p, & 1 < p < \infty \end{cases}$$

Proof of Theorem 2.2

Let $g\in W_p^1$ then by using the following (48), (46), (35), (47) and (50), we get

$$\tau_1(f,\frac{1}{n})_p \le \tau_1((f-g),\frac{1}{n})_p + \tau_1(f,\frac{1}{n})_p$$

$$\le C_1\left\{\|f-g\|_p + \frac{1}{n}\|g'\|_p + \frac{1}{n}\|\tilde{g}'\|_p\right\}$$

Now if we take the infimum of $gW_p^1 \in \cap \tilde{W}_p^1$, we obtain

$$\tau_1(f,\frac{1}{n})_p \le C_1K(f;\frac{1}{n},L_p,W_p^1,\tilde{W}_p^1)$$

$$\le C_1K(f;\frac{1}{n},L_{(1/n),p},W_p^1,\tilde{W}_p^1)$$

$$\le C_1\frac{C_2}{n}\sum_{S=0}^{n}\begin{cases} E_S^T(f)_{(1/S+1),p} + E_S^T(f)_{(1/S+1),p}, & p=1,\infty \\ E_S^T(f)_{(1/S+1),p}, & 1<p<\infty \end{cases}$$

$$\le \frac{C}{n}\sum_{S=0}^{n}\begin{cases} \|f-J_S(f)\|_{(1/S+1),p} + \|\tilde{f}-J_S(\tilde{f})\|_{(1/S+1),p}, & p=1,\infty \\ \|f-J_S(f)\|_{(1/S+1),p}, & 1<p<\infty. \end{cases}$$

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