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THE ARGUMENT AND DEMONSTRATION EXEMPLIFIED IN A MATHEMATICAL DIALOGUE

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ABSTRACT

The teaching-learning process is analyzed in a course for a group of professors who were taught subjects on Calculus, to study the episodes of problem-solving in them, focused on the identification of patterns and argumentation using counterexamples. The explanation and the argument in the classroom can be used together so that the argument (issued as a counterexample) supports the explanation (conjecture). Developing the mathematics class so that the above occurs is a form of interaction and how to encourage students to move from explanation to argumentation (placing a hybrid system). Furthermore, both forms of reasoning can influence dialogue protocols and strategies. In this work, the dialogue model is described as a tool to address the problem that arises when working with students.

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1. INTRODUCTION

Since ancient times, explanation and argumentation are resources used to improve learning. For example, when Socrates and his young disciple Teeteto discussed the meaning of Science, Socrates questioned the conjecture of his disciples in this dialogue, when Teeteto expo exposed by Plato (2003), “What can be learned with Theodore, such as Geometry and the other arts you have mentioned, are so many other sciences and even all the arts, whether that of a shoemaker or any other trade, are nothing but science”.

Socrates, with a counterexample, answered his disciple stating that when you ask about what Science is, it is to make a fool of yourself by giving an answer to the name of a science. This is to answer about the object of science, and not about science itself, which is what the question refers to, with a rebuttal that helps change Teeteto's answer. Here we see how the teacher induces his disciple to "conjecture" to modify his answers, allowing him to develop a better-structured thought.

There are multiple definitions of what is a guess in Mathematics, and most do not make it very clear how it is structured? Will it be possible to build a conjecture with the help of certain elements or how is it built? There is no clear idea or established manual that followed by mathematicians to structure a guess. However, it is intended to identify some forms presented by researchers in this area. An approach presented in the books is the idea of using particular cases to seek regularities and establish a conjecture such as Canadas et al. (2008). Although some authors today highlight the difficulties in separating these two in practice, conjecture and counterexample (García & Morales, 2013; Ibañes, 2001; Marrades & Gutiérrez, 2000; Stenning & Monaghan, 2005), we strive to focus our research in the inductive reasoning process. The authors claim about inductive reasoning, which is a cognitive process that allows to advance knowledge by obtaining more information than the initial data with which the process begins. Human thought then takes a stance that produces affirmations and reaches conclusions based on particular cases and identifying patterns. In other words, in order to structure a conjecture, inductive reasoning must be developed to lead into possible generalization.

In any Calculus course, the teacher intervenes in the planting of ideas (Socratic approach) by asking his students to propose their argument. Students learn to listen sympathetically to the ideas of other peers and to contribute their own. Then, they have to learn to criticize the defects that appear in the development of discussions and accept the corrections that are made to them; establishing a sufficiently broad theoretical scenario for the approach of course members' projects.

In general terms, reasoning involves extracting references from principles and evidence from which individuals draw new conclusions or evaluate conclusions based on what is already known (Johnson-Laird & Byrne, 1993). There are two main types of reasoning, deductive reasoning and inductive reasoning. Deductive reasoning refers to the process of reasoning from a set of general premises to arrive at a valid logical conclusion, while inductive reasoning is the process of reasoning from specific premises or observations to arrive at a general conclusion or rule. general. Thus, deductive reasoning draws conclusive conclusions from given information, while inductive reasoning adds information (Klauer, 2001).

This educational recommendation addresses only inductive Mathematical reasoning for students in calculus courses. Mathematical induction contains information about all instances of a class (for example, the class of all positive integers) and thus can draw conclusions with certainty, whereas students' inductive reasoning generally refers to a certain instance. Therefore, the conclusions it draws are not necessarily applicable to all possible situations (Sternberg & Gardner, 1983). In many cases, however, inductive reasoning is valid and provides an important foundation for understanding Mathematical laws. Both regularity and unity are the basis for the generation of concepts and categories, which play an important role in our daily lives (Klauer & Phye, 1994). Our research focuses on the inductive reasoning required for learning programs in Higher Mathematics (MAT-121) and for most intelligence tests (eg, analogies, classifications, series completion problems, arrays).

Neubert and Binko (1992) relate inductive reasoning in mathematics to the search for patterns and relationships between numbers and figures. This idea goes back to the work of Polya (1967), who defined inductive reasoning as one that allows us to obtain scientific knowledge. Polya (1967) also believes that inductive reasoning in mathematics education is a method of discovering properties of phenomena and logically discovering laws. Inductive reasoning as a method consists of four steps: experience with specific cases, formation of conjectures, testing of conjectures, and verification of new specific cases (Polya, 1967). Based on these steps, Cañadas (2002) developed a system consisting of secondary school students' thinking actions to solve proof and inductive reasoning problems related to the justification of a statement where inductive reasoning appears.

Although in most elementary mathematical problems students are tasked with discovering patterns of relationships or characteristics between the various given elements of the problem, the stimulation of thinking skills is not cleanly pursued. These skills are often seen as byproducts of what is taught in traditional curriculum definitions. For different topics (Sánchez et al., 2021). As a result, most students do not master basic mathematical concepts and have difficulty solving problems (Godino et al., 2007; Godino et al., 2011; Mallart et al., 2018; Maure et al., 2018; Morales-Maure et al., 2022). The latter has been demonstrated in many studies, especially international research assessments such as the OECD-PISA 2018 (Schleicher, 2019) and TIMSS 2019 (Martin et al., 2020).

2. METHOD

The experience described in this work was taken from a Calculus course whose participants were math teachers. This was done with the intention of encouraging problem-solving episodes, as well as the use of examples and counterexamples to encourage argumentation with their students. Taking into account that, as future mathematics teachers, they should receive training with processes similar to those that they are expected to develop in their classes.

First, a diagnostic test on previous knowledge was carried out and then the document "Is argumentation an obstacle? Invitation to a debate by Nicolas Balacheff was read. In this document, the author arguments on the thesis that men live immersed in a context of arguments (Balacheff, 1999). With this in mind, we must say that all the argumentation is part of men's daily world. There is no conversation, discussion, or opinion in which there is no effort of conviction, because not all individuals think the same way. Many of the analyses developed as spontaneous, informal and intuitive. Therefore, the purpose of an argument is, above all, to increase attachment to a point of view submitted to an audience (students, teachers). However, it does not demonstrate the veracity of a conclusion as that belongs to the field of scientific demonstration.

3. RESULT AND DISCUSSION

3.1. The proposed mathematics inductive reasoning framework

In the mathematical sense, a conjecture can be built by looking for patterns or regularities in the classroom, which help to promote an environment that contributes to the development of fundamental processes of mathematical thinking such as the search of patterns, use of multiple representations and communication of mathematical ideas. This concurs with (Benitez, 2006; Castro et al., 2021) who claims that this mathematical sense serves in the learning of students as an axis in structuring their reasoning processes. However, finding numerical, geometric or algebraic patterns should not be considered as easy activities for students. Therefore, it should be gradually encouraged by the teacher, to maximize the possibility of favorable cases for the formulation of a generality.

Following the idea of developing guesswork by students, in this first calculation course, the teacher presented and then explained a problem on the board. In this example, students designed the exponential equation where it is supposed that a single bacterium that starts dividing every hour. After an hour, we have 2 bacteria. After two hours, we have 2^2, which equals 4 bacteria. After three hours, we have 2^3 which equals 8 bacteria, and so on (See Figure 1). The population of bacteria is modeled after t hours and, by means of the developed heuristics, led them to work the exponential equation that represents a growth of the population of bacteria $f(t)=2^t$.

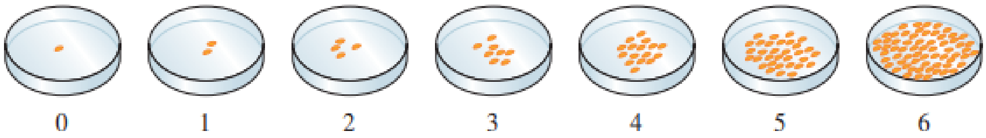

Figure 1. Exponential bacterial growth

The equation proposed by the teacher during the class implies the understanding of the duplication of previous events. However, the lack of mathematical content development in the explanation limited students to argue about the search for solutions.

Students often have difficulty recognizing data, graphs or figures because their identification requires the mastery of special conceptions of the subjects involved. The second activity developed in the same session aims to conjecture the recognition of squares built and plotted in different positions and hidden in other figures and to advance the use of some of the properties that characterize them (summaries). (See Figure 2) This approach is an invitation to episodes compatible with the problem-solving approach, this time the teacher asked: How many rectangles are in the figure shown?

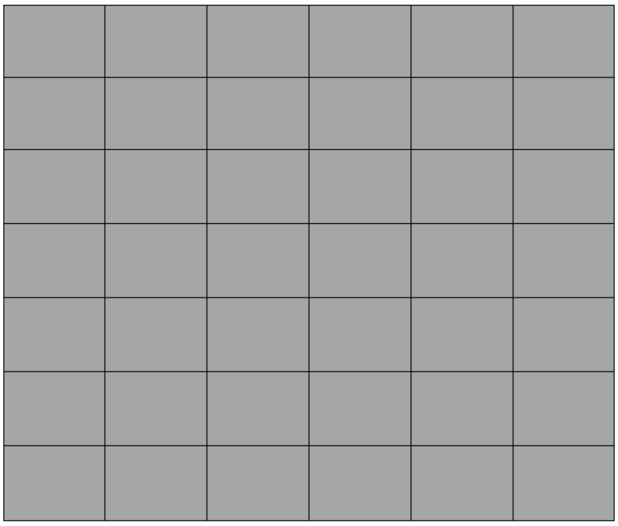

Figure 2. Image made up of several divisions

The objective that the teacher pursues in asking this question is to identify the possible heuristics that students use when addressing this type of problem, and to conjecture some possible solutions, following the steps indicated by Schoenfeld (2016). Students may also be asked questions that aim to identify each of the squares, giving them as data the number of squares that the model hides (see Figure 2). The first step taken was the particular cases to observe the behavior and thus express an equation with the pattern of movement of the numbers.

Table 1. Breakdown of rectangles found in Figure 2

2 x 3					2 x 4					2 x 5				
Area	Amount				Area	Amount				Area	Amount			
	Factor #1		Factor #2			Factor #1		Factor #2			Factor #1		Factor #2	
1x1	3	x	2	=6	1x1	4	x	2	=8	1x1	5	X	2	=10
1x2	2	x	2	=4	1x2	3	x	2	=6	1x2	4	X	2	=8
1x3	1	x	2	=2	1x3	2	x	2	=4	1x3	3	X	2	=6
2x1	3	x	1	=3	1x4	1	x	2	=2	1x4	2	X	2	=4
2x2	2	x	1	=2	2x1	4	x	1	=4	1x5	1	x	2	=2
2x3	1	x	1	=1	2x2	3	x	1	=3	2x1	5	x	1	=5
					2x3	2	x	1	=2	2x2	4	x	1	=4
					2x4	1	x	1	=1	2x3	3	x	1	=3
										2x4	2	x	1	=2
										2x5	1	x	1	=1

It is observed that the #1 factor decreases very differently from the #2 factor (see Table 1). The first guess was to maintain a fixed constant as shown below.

$$\sum_{i=1}^{m} i(n) + \sum_{i=1}^{n} i(n-1) + \ \ldots + \sum_{i=1}^{n} i(1)$$

But, when giving values to it, did not show the pattern of results. So, it was assumed that the guess was wrong and the students realized that it was simply a double summation, which is commonly presented when you have values classified into separate groups. Suppose we have *k* groups of values, and in each group, there are *n* values.

$$\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (m-j)\,(n-i)$$

Here values were given back to *m* and *n*, the proposed guess complied with the initial conditions that decreased both factors *m* and *n* to 1 (the factors). Students induced the development of the double summation as follows:

$$\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (m-j)\,(n-i) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (mn - im - jn + ij)$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (mn) - \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (im) - \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (jn) + \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (ij)$$

$$= mn \sum_{j=0}^{m-1} (1) \sum_{i=0}^{n-1} (1) - m \sum_{j=0}^{m-1} (1) \sum_{i=0}^{n-1} (i) - n \sum_{j=0}^{m-1} (j) \sum_{i=0}^{n-1} (1) + \sum_{j=0}^{m-1} (j) \sum_{i=0}^{n-1} (i)$$

$$= mnmn + mm\frac{n(1-n)}{2} + nn\frac{m(1-m)}{2} + \frac{m(m-1)}{2}\frac{n(n-1)}{2}$$

$$= m^2n^2 + \frac{m^2n - m^2n^2}{2} + \frac{mn^2 - m^2n^2}{2} + \frac{m^2n^2 - mn^2 - m^2n + mn}{4}$$

$$= \frac{4m^2n^2 + 2m^2n - 2m^2n^2 + 2mn^2 - 2m^2n^2 + m^2n^2 - mn^2 - m^2n + mn}{4}$$

$$= \frac{m^2n + mn^2 + m^2n^2 + mn}{4} = \frac{mn(m + n + mn + 1)}{4} = \frac{mn(m+1)(n+1)}{4}$$

Therefore, the summation represents the number of rectangles that are formed to the following formula:

$$\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (m - j)\,(n - i) = \frac{mn(m+1)(n+1)}{4}$$

Continuing with the narration of the episode, the students then wondered how many squares are formed into a rectangular figure of *n*m* area? To present their arguments, the students highlighted the relationships found as results of explorations on behaviors that remain fixed (particular situations) observing facts as Osorio (2002) claims. There they found particular cases to observe what information it produced.

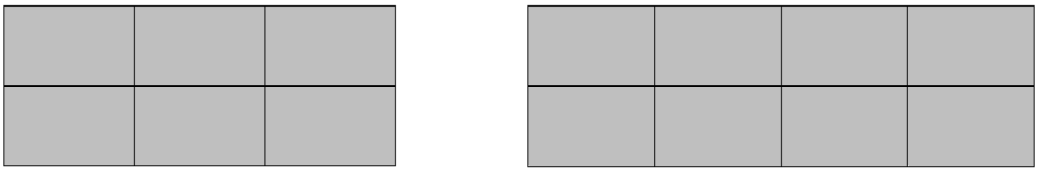

Figure 3. Case 2 by 3 and Case 2 by 4

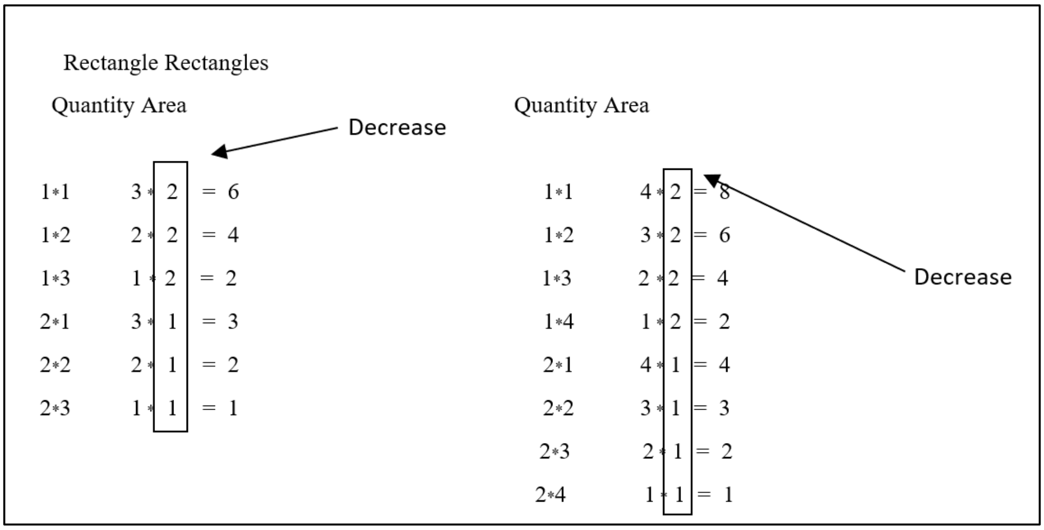

Figure 4. Expanding case development according to the area

Figure 3 offered a guide in which each result students realized there was a decrease in the construction of values (see Figure 4) and suggested a summation where there is a fixed constant proposing as a guess:

$$\sum_{i=1}^{m} i(n) + \sum_{i=1}^{n} i(n-1) + \ \dots + \sum_{i=1}^{n} i(1)$$

But this summary had an impact on two other summaries where *n* also varied. Then, a heuristic was proposed that included two summations and that both values would decrease by changing the initial idea. Thus, you have a double summation,

$$\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(m-j)\,(n-i)$$

Once presented, the idea was taken in a particular case where *m* is 4 and *n* is 2, to check if we got the given products that are 4∗2 + 4∗1 + 3∗2 + 3∗1 + 2∗2 + 2∗1 + 1∗2 + 1∗1. Indeed, the proposed guess complies with the initial conditions of decreased both *m* and *n* numbers to 1 (values). And finally, it develops to this double summation:

$$\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(m-j)\,(n-i) = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(mn-im-jn+ij)$$

$$=\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(mn) - \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(im) - \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(jn) + \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(ij)$$

$$= mn\sum_{j=0}^{m-1}(1)\sum_{i=0}^{n-1}(1) - m\sum_{j=0}^{m-1}(1)\sum_{i=0}^{n-1}(i) - n\sum_{j=0}^{m-1}(j)\sum_{i=0}^{n-1}(1) + \sum_{j=0}^{m-1}(j)\sum_{i=0}^{n-1}(i)$$

$$= mnmn + mm\,\frac{n(1-n)}{2} + nn\frac{m(1-m)}{2} + \frac{m(m-1)}{2}\,\frac{n(n-1)}{2}$$

$$= m^2n^2 + \frac{m^2n - m^2n^2}{2} + \frac{mn^2 - m^2n^2}{2} + \frac{m^2n^2 - mn^2 - m^2n + mn}{4}$$

$$= \frac{4m^2n^2 + 2m^2n - 2m^2n^2 + 2mn^2 - 2m^2n^2 + m^2n^2 - mn^2 - m^2n + mn}{4}$$

$$= \frac{m^2n + mn^2 + m^2n^2 + mn}{4} = \frac{mn(m+n\ + mn + 1)}{4} = \frac{mn(m+1)(n+1)}{4}$$

Then,

$$\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(m-j)\,(n-i) = \frac{mn(m+1)(n+1)}{4}$$

Thus, mathematical development establishes relationships that lead it $\frac{mn(m+1)(n+1)}{4}$ to indicate how many rectangles are formed in a rectangular area *m*∗*n*.

3.2. Discussion

By bringing more elements to the discussion, it can be observed that the typical textbooks available for Calculus content courses are designed specifically for future engineers or mathematicians, who provide solutions to the problems they pose and provide practical examples. This puts course students in the position of being math consumers, rather than doers and creators.

According to Association of Mathematics Teacher Educators [AMTE] (2017), well-prepared beginner teachers "strive to position students as authors of ideas, students who discuss, explain and justify their reasoning using various representations and tools" (p. 16). Thus, it is argued that textbooks in content courses should be focused on student development as the author of mathematical ideas in the course. Consequently, an alternative is not to use a standard textbook in the course but have developed their own series of texts for math students (i.e. engineers and possible mathematicians) (Beam et al., 2019a, 2019b).

On the other hand, it is noted that the curriculum carried out many activities that do not propose the use of manipulable, whether physical as cardboard or virtual tokens, such as those that can be developed with dynamic geometry. The decision to use them is based on whether they can contribute substantially to students' ability to represent information or understand and think about a problem with the help of teaching resources that use various representation systems (literal, symbolic and concrete) as argued by Yáñez et al. (2013).

In teaching various geometric contents through observation and manipulation to help students, there is a guide for the teacher, who presents and defines them. Underlying this mode of teaching, there is also the belief that the learning occurs not only by simple observation but also through external information provided to a person. Contrastingly, from the perspective of the didactics of mathematics, apprenticeships should appear progressively to the extent students are exposed to problem-solving episodes (i.e., various representation systems), in which they need to identify patterns by drawing up guesses and, to a large extent, find ways to justify such conjectures.

In addition, the need for understanding mathematics teachers in constructing mathematical objects and their meanings can be done by formulating conjectures and counter-examples so that later they can make appropriate didactic transpositions in their classes. This process can also be used to help students independently organize their mathematical thinking.

In the traditional classroom, the teacher focuses on following the textbook and curriculum contents to the letter, not leading to episodes like those described in this experience. In the classroom with a traditional approach, mathematical knowledge is presented as something finished that makes it impossible to ask questions that make students reflect when an assertion is true or not. Motivating to validate or invalidate all mathematical ideas that arise in the learning process. Teacher training is something that should not be overlooked and must always be taken into account within the Didactics of Mathematics. Therefore, there is a need to work to influence the practice of teachers and to be able to train them in the use of teaching resources such as the construction of conjectures and argumentation through counterexamples, which will allow them to expand their mathematical discourse.

Mathematical discovery has been addressed as a methodology to be included in educational practices. Such a discovery has a close relationship with maieutic. In the sense, the game that is established between the teacher and the student to find the "truth". A truth in Mathematics requires a constant cautious reassessment of its purposes, which in this case is intended to change naive thoughts and, for others, it is better developed to structure a mathematical thought.

4. CONCLUSION

Based on this experience, we reflected on the different logical theories and epistemologies in the research processes so that the thematic approaches and developments of our research projects are also at the forefront; besides being viable, solid, and unfold with the best possible structure.

In general, urgent changes are needed in traditional educational practices where it can be incorporated into conjecture and counterexample, so that teachers can help their students change their naive thoughts toward structuring appropriate mathematical thinking. Thus, students may have tools to be somehow competitive, critical and analytical in a society that underpins communication, that in many respects appear in mathematical language.

It is necessary to explore into other works on how to structure mathematical thinking at different educational levels, as teachers are required to incorporate such resources into their practice.

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