International Journal of Analysis and Applications
Volume 16, Number 4 (2018), 462-471
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-462

CONVEXITY OF INTEGRAL OPERATORS INVOLVING DINI FUNCTIONS

SADDAF NOREEN, MUHEY U DIN AND MOHSAN RAZA*

Department of Mathematics, Government College University Faisalabad, Pakistan

* *Corresponding author:* mohsan976@yahoo.com

Abstract. In this article, we are mainly interested to find some covexity properties for certain families of integral operators involving Dini functions in the open unit disc. The main tool in the proofs of our results are some functional inequalities of Dini functions. Some particular cases involving Dini functions are also a part of our investigations.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denote the class of all functions which are univalent in $\mathcal{U}$. Let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of starlike, convex and close-to-convex functions of order α and are defined as:

$$\begin{aligned}
\mathcal{S}^*(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathcal{U}, \alpha \in [0,1) \right\}, \\
\mathcal{C}(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathcal{U}, \alpha \in [0,1) \right\}
\end{aligned}$$

and

$$\mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \quad z \in \mathcal{U}, \alpha \in [0,1), \ g \in \mathcal{S}^* \right\}.$$

Received 2018-01-31; accepted 2018-04-06; published 2018-07-02.

2010 *Mathematics Subject Classification.* 30C45, 33C10, 30C20, 30C75.

Key words and phrases. analytic functions; convex functions; integral operators; Bessel functions; Dini functions.

It is clear that

$$\mathcal{S}^*(0) = \mathcal{S}^*,\ \mathcal{C}(0) = \mathcal{C} \text{ and } \mathcal{K}(0) = \mathcal{K}.$$

Special functions have great importance in pure and applied mathematics. The widely use of these functions have attracted many researchers to work on the different directions. Geometric properties of special functions such as Hypergeometric functions, Bessel functions, Struve functions, Mittage-Lefller functions, Wright functions and some other related functions is an ongoing part of research in geometric function theory. We refer for some geometric properties of these functions [1,2,5,6,11] and references therein.

The Bessel function of the first kind J_v is defined by

$$J_v(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\Gamma(v+n+1)} \left(\frac{z}{2}\right)^{2n+v}, \tag{1.2}$$

where Γ stands for Euler gamma function. It is a particular solution of the second order linear homogeneous differential equation

$$z^2 w''(z) + z w'(z) + (z^2 - v^2) w(z) = 0,$$

where $v \in \mathbb{C}$. For some details see [2,13]. Bessel functions are indispensable in many branches of pure and applied mathematics. Thus, it is important to study their properties in many aspects. Recently Baricz et al [3] studied the close-to-convexity of Dini functions and some monotonicity properties and functional inequalities for the modified Dini function are discussed in [4]. Further some geometric properties of Dini functions are studied in [3,4,7]. Now, we consider the normalized Dini functions $q_v : \mathcal{U} \to \mathbb{C}$ defined as

$$\begin{aligned} q_v(z) &= 2^v \Gamma(v+1) z^{1-\frac{v}{2}} \left((1-v) J_v(\sqrt{z}) + \sqrt{z} J_v'(\sqrt{z})\right) \\ &= z + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1) \Gamma(v+1)}{4^n n! \Gamma(v+n+1)} z^{n+1},\ z \in \mathcal{U}. \end{aligned} \tag{1.3}$$

The Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions is given as $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)...(x+n-1)$.

Recently, Deniz et al. [8], Din et al. [9], Din et al. [10] and Srivastava et al. [12] have obtained sufficient conditions for the univalence of certain families of integral operators defined by Bessel, Dini, Struve and Mittage-Leffler functions respectively. The families of integral operators are defined below:

$$F_{\alpha_1,\ldots,\alpha_n,\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt \right\}^{1/\zeta}, \tag{1.4}$$

$$G_{\xi,n}(z) = \left\{ (n\xi+1) \int_0^z \prod_{i=1}^{n} (f_i(t))^{\xi} dt \right\}^{1/(n\xi+1)}, \tag{1.5}$$

$$H_{\delta_1,\ldots,\delta_n,\mu}(z) = \left\{ \mu \int_0^z t^{\mu-1} \prod_{i=1}^{n} (f_i'(t))^{\delta_i} dt \right\}^{1/\mu} \tag{1.6}$$

and

$$Q_{\lambda}(z)=\left\{\lambda\int_{0}^{z}t^{\lambda-1}\left(e^{f(t)}\right)^{\lambda}dt\right\}^{1/\lambda}. \tag{1.7}$$

In this paper, we are mainly interested in the convexity of the integral operators involving Dini function q_v. These integral operators are defined as

$$F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)=\int_{0}^{z}\prod_{i=1}^{n}\left(\frac{q_{v_i}(t)}{t}\right)^{\alpha_i}dt \tag{1.8}$$

and

$$H_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)=\int_{0}^{z}\prod_{i=1}^{n}\left(q'_{v_i}(t)\right)^{\delta_i}dt. \tag{1.9}$$

Now we prove some functional inequalities which are useful in establishing our main results.

Lemma 1.1. *Let $v\in\mathbb{R}$ and consider the normalized Dini function $q_v:\mathcal{U}\to\mathbb{C}$, defined by*

$$q_v(z)=2^{v}\Gamma(v+1)\,z^{1-\frac{v}{2}}\left((1-v)\,J_v\left(\sqrt{z}\right)+\sqrt{z}J'_v\left(\sqrt{z}\right)\right),$$

where J_v is the Bessel function of first kind. Then the following inequalities hold for all $z\in\mathcal{U}$.

(i) $\left|\frac{zq'_v(z)}{q_v(z)}-1\right|\leq\frac{3v+6}{4v^2+5v-2},\qquad v>\frac{-5+\sqrt{57}}{8},$

(ii) $\left|\frac{zq''_v(z)}{q'_v(z)}\right|\leq\frac{3v+6}{2v^2+v-4},\qquad v>\frac{-1+\sqrt{33}}{4}.$

Proof. (i) By using the well known triangle inequality with the equality

$$\frac{\Gamma(v+1)}{\Gamma(v+n+1)}=\frac{1}{(v+1)(v+2)\cdots(v+n)}=\frac{1}{(v+1)_n},\ n\in\mathbb{N}$$

and the inequality

$$4^{n}n!\,(v+2)_{n-1}\geq\frac{4}{3}n\,(2n+1)\,(v+2)^{n-1},\ n\in\mathbb{N},$$

we obtain

$$\begin{aligned}\left|q'_v(z)-\frac{q_v(z)}{z}\right| &\leq \frac{3}{4(v+1)}\sum_{n=1}^{\infty}\left(\frac{1}{v+2}\right)^{n-1}\\ &= \frac{3(v+2)}{4(v+1)^2}.\end{aligned}$$

Furthermore, if we use the reverse triangle inequality and the inequality

$$4^{n}n!\,(v+2)_{n-1}\geq\frac{1}{3}(2n+1)\,(v+2)^{n-1},\ n\in\mathbb{N},$$

then we get

$$\begin{aligned}\left|\frac{q_v(z)}{z}\right| &= \left|1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)\Gamma(v+1)}{4^n n!\Gamma(v+n+1)}\right| \\ &\geq 1-\frac{3}{4(v+1)}\sum_{n=0}^{\infty}\left(\frac{1}{v+2}\right)^{n-1} \\ &= \frac{4(v+1)^2-3(v+2)}{4(v+1)^2}.\end{aligned}$$

By combining the above inequalities, we get

$$\left|\frac{zq_v'(z)}{q_v(z)}-1\right| \leq \frac{3v+6}{4v^2+5v-2}, \qquad v>\frac{-5+\sqrt{57}}{8}.$$

(ii) By using the well known triangle inequality with equality

$$\frac{\Gamma(v+1)}{\Gamma(v+n+1)}=\frac{1}{(v+1)(v+2)\cdots(v+n)}=\frac{1}{(v+1)_n},\ n\in\mathbb{N}$$

and the inequality

$$4^n n!(v+2)_{n-1}\geq\frac{4}{3}(2n+1)(v+2)^{n-1},\ n\in\mathbb{N},$$

we have

$$\begin{aligned}|zq_v''(z)| &\leq \left|\sum_{n=1}^{\infty}\frac{n(n+1)(2n+1)\Gamma(v+1)}{4^n n!\Gamma(v+n+1)}\right| \\ &\leq \frac{3}{2(v+1)}\sum_{n=1}^{\infty}\left(\frac{1}{v+2}\right)^{n-1} \\ &= \frac{3(v+2)}{2(v+1)^2}.\end{aligned}$$

Furthermore, if we use the reverse triangle inequality and the inequality

$$4^n n!(v+2)_{n-1}\geq\frac{2}{3}\left(2n^2+3n+1\right)(v+2)^{n-1},\ n\in\mathbb{N},$$

then we get

$$\begin{aligned}|q_v'(z)| &\left|\geq 1-\sum_{n=1}^{\infty}\frac{(2n+1)(n+1)\Gamma(v+1)}{4^n n!\Gamma(v+n+1)}\right| \\ &\geq 1-\frac{3}{2(v+1)}\sum_{n=1}^{\infty}\left(\frac{1}{v+2}\right)^{n-1} \\ &= \frac{2v^2+v-4}{4(v+1)^2}\end{aligned}$$

By combining the above inequalities, it can be easily obtained

$$\left|\frac{zq_v''(z)}{q_v'(z)}\right| \leq \frac{3v+6}{2v^2+v-4}, \qquad v>\frac{-1+\sqrt{33}}{4}.$$

□

2. Convexity of Integral Operators Defined by Generalized Dini Functions

The main objective of this paper is to give convexity properties of integral operators involving Dini function. The main results are given as follows.

Theorem 2.1. *Let $v_1, \ldots, v_n > \frac{-5+\sqrt{57}}{8}$, where $n \in \mathbb{N}$. Let $q_{v_i} : \mathcal{U} \to \mathbb{C}$ be defined as*

$$q_{v_i}(z) = 2^{v_i}\Gamma(v+1)\, z^{1-\frac{v_i}{2}} \left((1-v_i) J_{v_i}\left(\sqrt{z}\right) + \sqrt{z} J'_{v_i}\left(\sqrt{z}\right)\right). \tag{2.1}$$

Suppose that $v = \min\{v_1, \ldots, v_n\}$ and $\alpha_1, \ldots, \alpha_n$ be positive real numbers such that these numbers satisfy the following inequality

$$0 \leq 1 - \frac{3v+6}{4v^2+5v-2} \sum_{i=1}^{n} \alpha_i < 1.$$

Then, the function $F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}$ defined by (1.8), is in the class $\mathcal{C}(\beta)$, where

$$\beta = 1 - \frac{3v+6}{4v^2+5v-2} \sum_{i=1}^{n} \alpha_i.$$

Proof. We observe that $q_{v_i}, \forall i = 1, 2, \cdots n$ are such that $q_{v_i}(0) = q'_{v_i}(0) - 1 = 0$. It is also clear that $F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n} \in \mathcal{A}$. That is $F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(0) = F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(0) - 1 = 0$. On the other hand, it is easy to see that

$$F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z) = \prod_{i=1}^{n} \left(\frac{q_{v_i}(z)}{z}\right)^{\alpha_i}. \tag{2.2}$$

Differentiating logarithmically, we get

$$\frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zq'_{v_i}(z)}{q_{v_i}(z)} - 1\right). \tag{2.3}$$

This implies that

$$Re\left\{1 + \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}\right\} = \sum_{i=1}^{n} \alpha_i Re\left(\frac{zq'_{v_i}(z)}{q_{v_i}(z)}\right) + \left(1 - \sum_{i=1}^{n} \alpha_i\right).$$

Now, by using the assertion (i) of Lemma 1.1 for each v_i, where $i = 1, 2, \cdots n$, we obtain

$$\begin{aligned} Re\left\{1 + \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}\right\} &\geq \sum_{i=1}^{n} \alpha_i \left(1 - \frac{3v_i+6}{4v_i^2+5v_i-2}\right) + \left(1 - \sum_{i=1}^{n} \alpha_i\right) \\ &= 1 - \sum_{i=1}^{n} \alpha_i \frac{3v_i+6}{4v_i^2+5v_i-2}. \end{aligned}$$

Consider the function $\phi : \left(\frac{-5+\sqrt{57}}{8}, \infty\right) \to \mathbb{R}$ defined as

$$\phi(x) = \frac{3x+6}{4x^2+5x-2}$$

is decreasing function such that

$$\frac{3v_i+6}{4v_i^2+5v_i-2} \leq \frac{3v+6}{4v^2+5v-2}, \ \forall i = 1, 2, \cdots n.$$

Therefore

$$Re\left\{1+\frac{zF''_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}{F'_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}\right\}>1-\frac{3v+6}{4v^2+5v-2}\sum_{i=1}^{n}\alpha_i.$$

Since $0\leq 1-\frac{3v+6}{4v^2+5v-2}\sum\limits_{i=1}^{n}\alpha_i<1$, therefore $F_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}\in\mathcal{C}(\beta)$, where

$$\beta=1-\frac{3v+6}{4v^2+5v-2}\sum_{i=1}^{n}\alpha_i,$$

which completes the proof. □

By setting $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$ in Theorem 2.1, we obtain the result given below.

Corollary 2.1. *Let $v_1,\dots,v_n>\frac{-5+\sqrt{57}}{8}$, where $n\in\mathbb{N}$. Let $q_{v_i}:\mathcal{U}\to\mathbb{C}$ be defined as*

$$q_{v_i}(z)=2^{v_i}\Gamma(v+1)\,z^{1-\frac{v_i}{2}}\left((1-v_i)\,J_{v_i}\left(\sqrt{z}\right)+\sqrt{z}J'_{v_i}\left(\sqrt{z}\right)\right). \tag{2.4}$$

Suppose that $v=\min\{v_1,\dots,v_n\}$ and α be positive real numbers such that these numbers satisfy the following inequality

$$0\leq 1-\frac{n\alpha(3v+6)}{4v^2+5v-2}<1.$$

Then, the function $F_{v_1,\dots,v_n,\alpha}$ defined by

$$F_{v_1,\dots,v_n,\alpha}(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{q_{v_i}(t)}{t}\right)^{\alpha}dt$$

is in the class $\mathcal{C}(\beta_1)$, where

$$\beta=1-\frac{n\alpha(3v+6)}{4v^2+5v-2}.$$

The next theorem gives convexity properties of the integral operator defined in (1.9). The key tool in the proof is inequality (ii) of Lemma 1.1.

Theorem 2.2. *Let $v_1,\dots,v_n>\frac{-1+\sqrt{33}}{4}$, where $n\in\mathbb{N}$. Let $q_{v_i}:\mathcal{U}\to\mathbb{C}$ be defined as*

$$q_{v_i}(z)=2^{v_i}\Gamma(v+1)\,z^{1-\frac{v_i}{2}}\left((1-v_i)\,J_{v_i}\left(\sqrt{z}\right)+\sqrt{z}J'_{v_i}\left(\sqrt{z}\right)\right). \tag{2.5}$$

Suppose that $v=\min\{v_1,\dots,v_n\}$ and $\delta_1,\dots,\delta_n$ be positive real numbers such that these numbers satisfy the following inequality

$$0\leq 1-\frac{3v+6}{2v^2+v-4}\sum_{i=1}^{n}\delta_i<1.$$

Then, the function $H_{v_1,\dots,v_n,\delta_1,\dots,\delta_n}$ defined by (1.9), is in the class $\mathcal{C}(\gamma)$, where

$$\gamma=1-\frac{3v+6}{2v^2+v-4}\sum_{i=1}^{n}\delta_i.$$

Proof. It can easily be observed that, the operator defined in (1.9) belongs to class $\mathcal{A}$, that is $H_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(0) = H'_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(0) - 1 = 0$. Differentiating (1.9), we have

$$H'_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z) = \prod_{i=1}^{n} \left(q'_{v_i}(z)\right)^{\delta_i}.$$

Differentiating logarithmically, we obtain

$$\frac{zH''_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)}{H'_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)} = \sum_{i=1}^{n} \delta_i \left(\frac{zq''_{v_i}(z)}{q'_{v_i}(z)}\right).$$

This implies that

$$Re\left\{1 + \frac{zH''_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)}{H'_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)}\right\} = 1 + \sum_{i=1}^{n} \delta_i Re\left(\frac{zq''_{v_i}(z)}{q'_{v_i}(z)}\right).$$

Now, by using the assertion (ii) of Lemma 1.1 for each v_i, where $i = 1, 2, \cdots n$, we obtain

$$Re\left\{1 + \frac{zH''_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)}{H'_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)}\right\} > 1 - \sum_{i=1}^{n} \delta_i \left(\frac{3v_i + 6}{2v_i^2 + v_i - 4}\right).$$

Consider the function $\varphi : \left(\frac{-1+\sqrt{33}}{4}, \infty\right) \to \mathbb{R}$ defined as

$$\varphi(x) = \frac{3x + 6}{2x^2 + x - 4}$$

is decreasing function such that

$$\frac{3v_i + 6}{2v_i^2 + v_i - 4} \leq \frac{3v + 6}{2v^2 + v - 4}, \ \forall i = 1, 2, \cdots n.$$

It follows that

$$Re\left\{1 + \frac{zH''_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)}{H'_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n}(z)}\right\} > 1 - \frac{3v + 6}{2v^2 + v - 4} \sum_{i=1}^{n} \delta_i.$$

Since

$$0 \leq 1 - \frac{3v + 6}{2v^2 + v - 4} \sum_{i=1}^{n} \delta_i < 1,$$

therefore $H_{v_1,\ldots,v_n,\delta_1,\ldots,\delta_n} \in \mathcal{C}(\gamma)$, where

$$\gamma = 1 - \frac{3v + 6}{2v^2 + v - 4} \sum_{i=1}^{n} \delta_i,$$

which completes the proof. □

By setting $\delta_1 = \delta_2 = \delta_n = \delta$ in Theorem 2.2, we obtain the result given below.

Corollary 2.2. *Let $v_1, \ldots, v_n > \frac{-1+\sqrt{33}}{4}$, where $n \in \mathbb{N}$. Let $q_{v_i} : \mathcal{U} \to \mathbb{C}$ be defined as*

$$q_{v_i}(z) = 2^{v_i}\Gamma(v + 1) z^{1-\frac{v_i}{2}} \left((1 - v_i) J_{v_i}\left(\sqrt{z}\right) + \sqrt{z} J'_{v_i}\left(\sqrt{z}\right)\right). \tag{2.6}$$

Suppose that $v = \min\{v_1, \ldots, v_n\}$ and δ be positive real numbers such that these numbers satisfy the following inequality

$$0 \leq 1 - \frac{n\delta(3v + 6)}{2v^2 + v - 4} < 1.$$

Then, the function $H_{v_1,\ldots,v_n,\delta}$ *defined as*

$$H_{v_1,\ldots,v_n,\delta}(z)=\int_0^z\prod_{i=1}^{n}\left(q'_{v_i}(t)\right)^{\delta}dt$$

is in the class $\mathcal{C}(\gamma_1)$, *where*

$$\gamma_1=1-\frac{n\delta(3v+6)}{2v^2+v-4}.$$

3. Some particular cases of Dini function

By choosing $v=\frac{1}{2}$ and $v=\frac{3}{2}$ in (1.2), we get the following forms of the normalized Dini function

$$\begin{aligned} q_{\frac{1}{2}}(z) &= \frac{3}{2}\sqrt{z}\left(\sin\sqrt{z}+\sqrt{z}\cos\sqrt{z}\right), \\ q_{\frac{3}{2}}(z) &= \frac{3}{2\sqrt{z}}\left((z-1)\sin\sqrt{z}+\sqrt{z}\cos\sqrt{z}\right). \end{aligned}$$

In particular, the results of the above mentioned theorems are given below.

Corollary 3.1. *Let* $v>\frac{-5+\sqrt{57}}{8}$ *and* $q_v:\mathcal{U}\to\mathbb{C}$ *be defined as*

$$q_v(z)=2^{v}\Gamma(v+1)z^{1-\frac{v}{2}}\left((1-v)J_v\left(\sqrt{z}\right)+\sqrt{z}J'_v\left(\sqrt{z}\right)\right).$$

Suppose that α *be positive real number such that these numbers satisfy the following inequality*

$$0\le 1-\frac{\alpha(3v+6)}{4v^2+5v-2}<1.$$

Then, the function $F_{v,\alpha}$ *defined by*

$$F_{v,\alpha}(z)=\int_0^z\left(\frac{q_v(t)}{t}\right)^{\alpha}dt$$

is in the class $\mathcal{C}(\beta_2)$, *where*

$$\beta_2=1-\frac{\alpha(3v+6)}{4v^2+5v-2}.$$

In particular,

(i) if $\alpha\le 15$, *then the function* $F_{\frac{1}{2},\alpha}:\mathcal{U}\to\mathbb{C}$ *defined by*

$$F_{\frac{1}{2},\alpha}(z)=\int_0^z\left(\frac{\frac{3}{2}\left(\sin\sqrt{t}+\sqrt{t}\cos\sqrt{t}\right)}{\sqrt{t}}\right)^{\alpha}dt$$

is in the class $\mathcal{C}(\beta_3)$, *where* $\beta_3=1-\frac{\alpha}{15}$.

(ii)If $\alpha\le\frac{29}{21}$, *then the function* $F_{\frac{3}{2},\alpha}:\mathcal{U}\to\mathbb{C}$ *defined by*

$$F_{\frac{3}{2},\alpha}(z)=\int_0^z\left(\frac{\frac{3}{2}\left((t-1)\sin\sqrt{t}+\sqrt{t}\cos\sqrt{t}\right)}{t}\right)^{\alpha}dt$$

is in the class $\mathcal{C}(\beta_4)$, where $\beta_4=1-\frac{21\alpha}{29}$.

Corollary 3.2. *Let $\upsilon > \frac{-1+\sqrt{33}}{4}$ and $q_\upsilon : \mathcal{U} \to \mathbb{C}$ be defined as*

$$q_\upsilon(z) = 2^\upsilon \Gamma(\upsilon+1) z^{1-\frac{\upsilon}{2}} \left((1-\upsilon) J_\upsilon\left(\sqrt{z}\right) + \sqrt{z} J'_\upsilon\left(\sqrt{z}\right)\right).$$

Suppose that δ be positive real numbers such that these numbers satisfy the following inequality

$$0 \le 1 - \frac{\delta(3v+6)}{2v^2+v-4} < 1.$$

Then, the function $H_{v,\delta}$ defined as

$$H_{v,\delta}(z) = \int_0^z \left(q'_v(t)\right)^\delta dt$$

is in the class $\mathcal{C}(\gamma_2)$, where

$$\gamma_2 = 1 - \frac{n\delta(3v+6)}{2v^2+v-4}.$$

In particular,

(i) if $\delta \le \frac{4}{21}$, then the function $H_{\frac{3}{2},\alpha} : \mathcal{U} \to \mathbb{C}$ defined by

$$H_{\frac{3}{2},\alpha}(z) = \int_0^z \left(q'_{\frac{3}{2}}(t)\right)^\delta dt$$

is in the class $\mathcal{C}(\gamma_3)$, where $\gamma_3 = 1 - \frac{21}{4}\delta$.

References

[1] Á. Baricz, Bessel transforms and Hardy space of generalized Bessel functions, Mathematica, 48(71) (2006), 127-136.

[2] Á. Baricz, Generalized Bessel Functions of the First Kind, Lecture Notes in Mathematics Vol. 1994, SpringerVerlag, Berlin, 2010.

[3] Á. Baricz, E. Deniz and N. Yagmur, Close-to-convexity of normalized Dini functions, Math. Nech., 289 (2016), 1721-1726.

[4] Á. Baricz, S. Ponnusamy and S. Singh, Modified Dini functions: monotonicity patterns and functional inequalities, Acta Math. Hungrica, 149(2016), 120-142.

[5] Á. Baricz and R. Szász, The radius of convexity of normalized Bessel functions, Anal. Math., 41(3) (2015) 141-151.

[6] D. Bansal and J. K. Prajapat. Certain geometric properties of the Mittag-Leffler functions, Complex Var. Ellipt. Equ., 61(3) (2016), 338-350.

[7] E. Deniz, Ş. Gören and M. Çağlar, Starlikeness and convexity of the generalized Dini functions, AIP Conference Proceedings 1833, 020004 (2017); doi: 10.1063/1.4981652.

[8] E. Deniz, H. Orhan and H. M. Srivastava, Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions, Taiwanese J. Math. 15(2) (2011), 883-917.

[9] M. Din, M. Raza and E. Deniz, Sufficient conditions for univalence of integral operators involving Dini Functions, (Submitted).

[10] M. Din, H. M. Srivastava, M. Raza, Univalence of integral operators involving generalized Struve functions, Hacettepe J. Math. Stat., Doi: 10.15672/HJMS.2017.485.

[11] H. Orhan and N. Yagmur, Geometric properties of generalized Struve functions, In The International Congress in Honour of Professor Hari M. Srivastava, (2012), 23-26.

[12] H. M. Srivastava, B. A. Frasin and Virgil Pescar, Univalence of integral operators involving Mittag-Leffler functions, Appl. Math. Inf. Sci. 11(3) (2017), 635-641.

[13] G. N. Watson. A Treatise on the Theory of Bessel Functions, Second edition, Cambridge University Press, Cambridge, London and New York, 1944.