International Journal of Analysis and Applications
Volume 17, Number 3 (2019), 342-360
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-17-2019-342

# PATA-TYPE FIXED POINT RESULTS IN $b_\nu(s)$-METRIC SPACES

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Abstract. The aim of this is to study fixed point theorems in $b_\nu(s)$-metric spaces under the Pata-type conditions. As consequences, we establish common fixed point results of Pata-type for two maps in $b_\nu(s)$-metric spaces.

## 1. Introduction

The Banach contraction principle introduced by Banach [6] is one of the most important results in mathematical analysis. It is the most widely applied fixed point result in many branches of mathematics and generalized in many different directions. Some generalizations of the notion of a metric space have been proposed by some authors, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, D metric spaces, and cone metric spaces (see [1, 2, 7, 20, 24, 26, 29–33, 36]).

The other direction of investigation is concerned with generalizations of contractive condition (see [3, 10, 11, 18, 19] and others in literature). One of the interesting recent results of this kind was obtained by V. Pata in [23]. Several scholars have already used Pata-type conditions to obtain new fixed point results (see [5, 9, 14–17]).

V. Pata obtained the following interesting refinement of the classical Banach Contraction Principle.

Received 2018-12-01; accepted 2019-01-30; published 2019-05-01.

2010 *Mathematics Subject Classification.* 54H25; 47H10.

*Key words and phrases.* Fixed point; $b_\nu(s)$-metric space; Pata-type contractive mapping; triangular $\alpha$-admissible mapping.

**Theorem 1.1.** *[23] Let $(X, d)$ be a metric space, $f : X \to X$, Let $\Lambda \geq 0, \eta \geq 1$ and $\beta \in [0, \eta]$ be fixed constants and $\psi : [0, 1] \to [0, \infty)$ be an increasing function, vanishing with continuity at 0. If the inequality*

$$d(fx, fy) \leq (1 - \varepsilon)d(x, y) + \Lambda\varepsilon^{\eta}\psi(\varepsilon)[1 + ||x|| + ||y||]^{\beta}$$

*is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$, then $f$ has a unique fixed point $z \in X$. Here, $||x|| = d(x, x_0)$ for a chosen point $x_0 \in X$.*

It was also shown by an example that the previous theorem is real generalization of Banach's result. More results of this kind were subsequently obtained by various authors.

$b$-metric spaces were firstly used by I.A. Bakhtin and S. Czerwik.

**Definition 1.1.** *[4, 8] Let $X$ be a nonempty set, $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0, \infty)$ is called a $b$-metric with parameter $s$ if for all $x, y \in X$ the following holds:*

(1) $d(x, y) = 0$ *if and only if* $x = y$*;*

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *;*

(3) $d(x, y \leq s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$ *($b$-triangular inequality).*

*Then the pair $(X, d)$ is called a $b$-metric space.*

**Remark 1.1.** *In general, $b$-metric might not be continuous functions (see example in [4, 8]).*

**Definition 1.2.** *[7] Let $X$ be a nonempty set. Let $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from $x$ and $y$:*

(1) $d(x, y) = 0$ *if and only if* $x = y$*;*

(2) $d(x, y) = d(y, x)$*;*

(3) $d(x, y \leq d(x, u) + d(u, v) + d(v, y)$ *(rectangular inequality).*

*Then $d$ is called a generalized metric and the pair $(X, d)$ is called generalized metric space (or shortly GMS).*

**Remark 1.2.** *Obviously, each metric space is a generalized metric space, but the converse is not true. Moreover, Sarma et al. [32] and Samet [31] presented examples showing that generalized metric spaces might not be Hausdorff and, again, that generalized metric might be discontinuous. Also, Suzuki showed in [35] that, in general, generalized metric spaces do not have a compatible topology.*

As a combination of $b$-metric and generalized metric spaces, $b$-rectangular metric spaces were introduced and used in [12, 22, 28].

**Definition 1.3.** *[12] Let $X$ be a nonempty set and $s \geq 1$ be a fixed real number.. Let $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from $x$ and $y$:*

(1) $d(x, y) = 0$ *if and only if* $x = y$*;*

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y \leq s[d(x,u) + d(u,v) + d(v,y)]$ *(b-rectangular inequality).*

*Then d is called a b-rectangular metric and the pair* $(X,d)$ *is called b-rectangular metric space with parameter* $s$.

In 2017, Z.D. Mitrovic and S. Radenovic [21] introduced the concept of $b_\mu(s)$-metric space as follows.

**Definition 1.4.** *[21] Let* $X$ *be a nonempty set. Let* $d : X \times X \to [0,\infty)$ *be a mapping and let* $\nu \in N$, $s \geq 1$. *Then* $(X,d)$ *is said to be a* $b_\nu(s)$*-metric space if for all* $x,y \in X$ *and for all distinct points* $u_1, u_2, \cdots, u_\nu \in X$, *each of them different from* $x$ *and* $y$, *the following hold:*

(1) $d(x,y) = 0$ *if and only if* $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y \leq s[d(x,u_1) + d(u_1,u_2) + \cdots + d(u_\nu,y)]$ *(*$b_\nu(s)$*-metric inequality).*

They note that:

$b_1(1)$-metric space is usual metric space,

$b_1(s)$-metric space is b-metric space with coefficient of Bakhtin and Czerwik,

$b_2(1)$-metric space is generalized metric space,

$b_2(s)$-metric space is rectangular b-metric space with coefficient $s$ of George et al.,

$b_\nu(1)$-metric space is $\nu$-generalized metric space of Branciari.

**Definition 1.5.** *[21] Let* $(X,d)$ *be a* $b_\nu(s)$*-metric space,* $\{x_n\}$ *be a sequence in* $X$ *and* $x \in X$. *Then*

(i) *The sequence* $\{x_n\}$ *is said to be convergent in* $(X,d)$ *and converges to* $x$, *if for every* $\varepsilon > 0$ *there exists* $n_0 \in N$ *such that* $d(x_n,x) < \varepsilon$ *for all* $n > n_0$ *and this fact represented by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to \infty$ *as* $n \to \infty$.

(ii) *The sequence* $\{x_n\}$ *is said to be Cauchy sequence in* $(X,d)$ *if for every* $\varepsilon > 0$ *there exists positive integer* $N(\varepsilon)$ *such that* $d(x_m,x_n) < \varepsilon$ *for all* $m,n > N(\varepsilon)$.

(iii) $(X,d)$ *is said to be a complete* $b_\mu(s)$*-metric space if for every Cauchy sequence in* $X$ *converges to some* $x$.

And they proved the following Theorem:

**Theorem 1.2.** *[21] Let* $(X,d)$ *be a complete* $b_\nu(s)$*-metric space and suppose that* $T : X \to X$ *be a self-mapping satisfying:*

$$d(Tx,Ty) \leq \lambda d(x,y)$$

*for all* $x,y \in X$, *where* $\lambda \in [0,1)$. *Then* $T$ *has a unique fixed point.*

**Definition 1.6.** *[34] Let $X$ be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that $T$ is a triangular $\alpha$-admissible mapping if*

(1) *$\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, for $x, y \in X$;*
(2) *$\alpha(x, z) \geq 1$, $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$, for all $x, y, z \in X$.*

**Lemma 1.1.** *[34] Let $T$ is a triangular $\alpha$-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \; for \; all \; m, n \in N \; with \; m < n.$$

The following lemmas will be used for proving our main results.

**Lemma 1.2.** *Let $(X, d)$ be a $b_\nu(s)$-metric space and let $\{x_n\}$ be a sequence in $X$ with distinct elements ( $x_n \neq x_m$ for $n \neq m$). Suppose that $d(x_n, x_{n+p})$ tends to $0$ as $n \to \infty$ for all $p = 1, 2, \cdots, \nu$, and $x_n \to x$ as $n \to \infty$. Then*

$$\frac{1}{s} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y) \leq \limsup_{n\to\infty} d(x_n, y) \leq s d(x, y),$$

*for all $y \in X$ with $y \neq x$.*

**Proof.** Since $\{x_n\}$ be a sequence in $X$ with distinct elements, we can assume that $x_n$ is different from $x$ and $y$ for all $n \in N$. By the $b_\nu(s)$-metric inequality, we have

$$d(x, y) \leq s[d(x, x_{n+\nu-1}) + d(x_{n+\nu-1}, x_{n+\nu-2}) + \cdots + d(x_{n+1}, x_n) + d(x_n, y)],$$

$$d(x_n, y) \leq s[d(x_n, x_{n+\nu-1}) + d(x_{n+\nu-1}, x_{n+\nu-2}) + \cdots + d(x_{n+1}, x) + d(x, y)].$$

Since $d(x_n, x_{n+p})$ tends to $0$ as $n \to \infty$ for all $p = 1, 2, \cdots, \nu$, and $x_n \to x$ as $n \to \infty$, taking $\liminf_{n\to\infty}$ on the both sides of the first inequality and taking $\limsup_{n\to\infty}$ on the both sides of the second inequality, it follows that

$$\frac{1}{s} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y) \leq \limsup_{n\to\infty} d(x_n, y) \leq s d(x, y).$$

**Lemma 1.3.** *Let $(X, d)$ be a $b_\nu(s)$-metric space and let $\{x_n\}$ be a sequence in $X$ with distinct elements ( $x_n \neq x_m$ for $n \neq m$). Suppose that $d(x_n, x_{n+p})$ tends to $0$ as $n \to \infty$ for all $p = 1, 2, \cdots, \nu$ and $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two sequence $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k + \nu$, $m_k \geq k$ and*

$$\epsilon \leq \liminf_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \limsup_{n\to\infty} d(x_{n_k}, x_{m_k}) \leq s\epsilon,$$

$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq \limsup_{n\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq s\epsilon.$$

*Proof.* Since $\{x_n\}$ is not a Cauchy sequence, there exists $\epsilon > 0$ for which we can choose two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_k$ is the smallest index for which

$$n_k > m_k \geq k \text{ and } d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{1.1}$$

This means that

$$d(x_{m_k}, x_{m_k+1}) < \epsilon, d(x_{m_k}, x_{m_k+2}) < \epsilon, \cdots, d(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{1.2}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p = 1, 2, \cdots, \nu$, we can assume that $n_k > m_k + \nu$. Using (1.1), (1.2) and $b_\nu(s)$-metric inequality, we have

$$\begin{aligned}\varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq s[d(x_{m_k}, x_{n_k-\nu}) + d(x_{n_k-\nu}, x_{n_k-\nu+1}) + \cdots \\ &+ d(x_{n_k-2}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})] \\ &\leq s[\epsilon + d(x_{n_k-\nu}, x_{n_k-\nu+1}) + \cdots + d(x_{n_k-2}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})].\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq s\epsilon.$$

Using (1.1) and $b_\nu(s)$-metric inequality, we have

$$\begin{aligned}\epsilon \leq d(x_{m_k}, x_{n_k}) \leq &s[d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) \\ &+ d(x_{n_k-1}, x_{n_k-2}) + \cdots + d(x_{n_k-\nu+1}, x_{n_k})].\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p = 1, 2, \cdots, \nu$, we get

$$\epsilon \leq s \liminf_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}),$$

that is

$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}).$$

Using (1.2) and $b_\nu(s)$-metric inequality, we have

$$\begin{aligned}d(x_{m_k-1}, x_{n_k-1}) &\leq s[d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k-\nu}) + d(x_{n_k-\nu}, x_{n_k-\nu+1}) + \cdots \\ &+ d(x_{n_k-3}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k-1})] \\ &\leq s[d(x_{m_k-1}, x_{m_k}) + \epsilon + d(x_{n_k-\nu}, x_{n_k-\nu+1}) + \cdots \\ &+ d(x_{n_k-3}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k-1})].\end{aligned}$$

By taking the upper limit as $k \to \infty$ in the above inequality, since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we get

$$\limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq s\epsilon.$$

Thus

$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq \limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq s\varepsilon.$$

□

**Lemma 1.4.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative numbers. If*

$$\lim_{n\to\infty} b_n = 0, \lim_{n\to\infty} \max\{a_n, b_n\} = a,$$

*then $\lim_{n\to\infty} a_n = a$.*

## 2. Main Results

Throughout the paper, $F(T)$ denotes the set of fixed points of the mapping $T$. For a given $b_\nu(s)$-metric space $(X, d)$ and a fixed $x_0 \in X$, we will denote $||x|| = d(x, x_0)$ for $x \in X$. We denote by $\Psi$ the family of all functions $\psi : [0, 1] \to [0, \infty)$ which is an increasing function, continuous at 0, with $\psi(0) = 0$.

**Theorem 2.1.** *Let $(X, d)$ be a complete $b_\nu(s)$-metric space with $s \geq 1$, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ a given function. Suppose that following conditions are satisfied:*

(1) *$T$ is a triangular $\alpha$-admissible mapping;*

(2) *there exist $\Lambda \geq 0, \eta \geq 1$, $\beta \in [0, \eta]$ and $\psi \in \Psi$ such that for every $\varepsilon \in [0, 1]$ and for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $d(Tx, Ty) > 0$,*

$$\begin{aligned} sd(Tx, Ty) <&(1-\varepsilon)M(x, y) \\ &+ \Lambda\varepsilon^\eta \psi(\varepsilon)[1 + ||x|| + ||y|| + ||Tx|| + ||Ty||]^\beta \end{aligned} \tag{2.1}$$

*where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\};$$

(3) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(4) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in N$.*

*Then $T$ has a fixed point $u$ and $\{T^n x_0\}$ converges to $u$. Further, if all $x, y \in F(T)$, we have $\alpha(x, y) \geq 1$, then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ satisfies $\alpha(x_0, Tx_0) \geq 1$. We construct the sequence $\{x_n\}$ in $X$ by $x_n = Tx_{n-1} = T^n x_0$ for $n \in N$. If $x_n = x_{n+1}$ for some $n \in N$, then $x_n$ is a fixed point of $T$. Consequently, we suppose that $x_n \neq x_{n+1}$ for all $n \in N$.

Since $T$ is a triangular $\alpha$-admissible mapping, by Lemma 1.1, we have

$$\alpha(x_n, x_m) \geq 1, \text{ for all } n, m \in N \text{ with } n < m. \tag{2.2}$$

*Step I.* We will show that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing. Indeed, putting $\varepsilon = 0$, $x = x_n$, $y = x_{n+1}$ in (2.1), we obtain

$$sd(x_n, x_{n+1}) = sd(Tx_{n-1}, Tx_n) < M(x_{n-1}, x_n), \tag{2.3}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$sd(x_n, x_{n+1}) < \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Hence

$$d(x_n, x_{n+1}) < \frac{1}{s} d(x_{n-1}, x_n) \tag{2.5}$$

for all $n \in N$. Thus the sequence $\{d(x_n, x_{n+1})\}$ is decreasing.

*Step II.* We will prove that $x_n \neq x_m$ for all $n \neq m$. Suppose that $x_n = x_m$ for some $n > m$, so we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$. By (2.5), we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n) < \cdots < d(x_m, x_{m+1}) = d(x_n, x_{n+1})$ a contradiction. Thus $x_n \neq x_m$ for all $n \neq m$.

*Step III.* We will show that for $p = 1, 2, \cdots, \nu$, the sequence $\{d(x_n, x_{n+p})\}$ is bounded. Indeed, since $\alpha(x_n, x_{n+p}) \geq 1$ and $d(x_n, x_{n+p}) > 0$, putting $\varepsilon = 0$, $x = x_n$, $y = x_{n+p}$ in (2.1), we obtain

$$sd(x_n, x_{n+p}) = sd(Tx_{n-1}, Tx_{n+p-1}) < M(x_{n-1}, x_{n+p-1}), \tag{2.6}$$

where

$$
\begin{aligned}
M(x_{n-1}, x_{n+p-1}) &= \max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+p-1}, Tx_{n+p-1}),\\
&\frac{d(x_{n-1}, Tx_{n-1})d(x_{n+p-1}, Tx_{n+p-1})}{1 + d(x_{n-1}, x_{n+p-1})}\}\\
&= \max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p}),\\
&\frac{d(x_{n-1}, x_n)d(x_{n+p-1}, x_{n+p})}{1 + d(x_{n-1}, x_{n+p-1})}\}\\
&\leq \max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n)^2\}.
\end{aligned}
$$

Combining (2.6), we have

$$sd(x_n, x_{n+p}) < \max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n)^2\}.$$

Taking $a_n = d(x_n, x_{n+p})$ and $b_n = d(x_n, x_{n+1})$, since $s \geq 1$, we have

$$sa_n < \max\{a_{n-1}, b_{n-1}, b_{n-1}^2\}.$$

Since $sb_n < b_{n-1} \leq \max\{a_{n-1}, b_{n-1}, b_{n-1}^2\}$ and $sb_n^2 < b_{n-1}^2 \leq \max\{a_{n-1}, b_{n-1}, b_{n-1}^2\}$, we have

$$\max\{a_n, b_n, b_n^2\} < \frac{1}{s}\max\{a_{n-1}, b_{n-1}, b_{n-1}^2\} \tag{2.7}$$

for all $n \in N$. Thus the sequence $\{\max\{a_n, b_n, b_n^2\}\}_{n\in N}$ is decreasing. Thus

$$K = \sup\{d(x_n, x_{n+p}), d(x_n, x_{n+1}), d(x_n, x_{n+1})^2 : n = 1, 2, \cdots ; p = 1, 2, \cdots , \nu\} < \infty. \tag{2.8}$$

*Step IV.* We will prove that the sequence $c_n = d(x_n, x_0)$ is bounded.

Using (2.5), we deduce the following estimate

$$
\begin{aligned}
c_n = d(x_n, x_0) &\leq s[d(x_0, x_{\nu-1}) + d(x_{\nu-1}, x_{\nu-2}) + \cdots\\
&+ d(x_2, x_1) + d(x_1, x_{n+1}) + d(x_{n+1}, x_n)]\\
&\leq s[c_{\nu-1} + (\nu - 1)c_0] + sd(Tx_0, Tx_n).
\end{aligned}
$$

Therefore, we infer from (2.1) that

$$
\begin{aligned}
c_n \leq &(1 - \varepsilon)M(x_0, x_n)\\
&+ \Lambda\varepsilon^{\eta}\psi(\varepsilon)[1 + ||x_n|| + ||x_1|| + ||x_{n+1}||]^{\beta} + s[c_{\nu-1} + (\nu - 1)c_0],
\end{aligned} \tag{2.9}
$$

where

$$\begin{aligned} M(x_0, x_n) &= \max\{d(x_0, x_n), d(x_0, Tx_0), d(x_n, Tx_n), \frac{d(x_0, Tx_0)d(x_n, Tx_n)}{1 + d(x_0, x_n)}\} \\ &= \max\{d(x_0, x_n), d(x_0, x_1), d(x_n, x_{n+1}), \frac{d(x_0, x_1)d(x_n, x_{n+1})}{1 + d(x_0, x_n)}\} \\ &\leq \max\{c_n, c_1, c_1^2\}. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), as $\beta \leq \eta$ we have

$$\begin{aligned} c_n \leq &(1 - \varepsilon)[\max\{c_n, c_1, c_1^2\}] \\ &+ \Lambda\varepsilon^\eta \psi(\varepsilon)[1 + c_n + c_1 + c_{n+1}]^\eta + s[c_{\nu-1} + (\nu - 1)c_0]. \end{aligned} \tag{2.11}$$

Suppose that the sequence $c_n = d(x_n, x_0)$ is not bounded. Then there is a subsequence $\{c_{n_i}\}$ satisfying that $c_{n_i} \geq \max\{1, c_1, c_1^2, 1 + \nu K\}$ for all $i \in N$ and $c_{n_i} \to \infty$. Using (2.8), we have

$$\begin{aligned} c_{n+1} = d(x_{n+1}, x_0) &\leq s[d(x_0, x_n) + d(x_n, x_{n+1}) + \cdots \\ &\quad + d(x_{n+\nu-2}, x_{n+\nu-1}) + d(x_{n+\nu-1}, x_{n+1})] \\ &\leq sd(x_0, x_n) + s\nu K = sc_n + s\nu K. \end{aligned}$$

Thus, for all $i \in N$, (2.11) implies that

$$\begin{aligned} c_{n_i} &\leq (1 - \varepsilon)c_{n_i} + \Lambda\varepsilon^\eta \psi(\varepsilon)[1 + (1 + s)c_{n_i} + s\nu K]^\eta + s[c_{\nu-1} + (\nu - 1)c_0] \\ &\leq (1 - \varepsilon)c_{n_i} + \Lambda\varepsilon^\eta \psi(\varepsilon)(3sc_{n_i})^\eta + s[c_{\nu-1} + (\nu - 1)c_0]. \end{aligned}$$

Thus we have

$$c_{n_i} \leq (1 - \varepsilon)c_{n_i} + a\varepsilon^\eta \psi(\varepsilon)c_{n_i}^\eta + b$$

for some $a, b > 0$. Hence

$$\varepsilon c_{n_i} \leq a\varepsilon^\eta \psi(\varepsilon)c_{n_i}^\eta + b.$$

Now, as in [23], the choice $\varepsilon = \varepsilon_i = (1 + b)/c_{n_i}$ leads to the contradiction

$$1 \leq a(1 + b)^\eta \psi(\varepsilon_i) \to 0.$$

Hence the sequence $c_n = d(x_n, x_0)$ is bounded.

*Step V.* For $p = 1, 2, \cdots, \nu$, $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$.

For all $\varepsilon \in (0,1]$ and for $x = x_n, y = x_{n+p}$ we have

$$\begin{aligned} sd(x_n, x_{n+p}) &= sd(Tx_{n-1}, Tx_{n+p-1}) \\ &< (1-\varepsilon)M(x_{n-1}, x_{n+p-1}) \\ &+ \Lambda\varepsilon^\eta\psi(\varepsilon)[1 + ||x_n|| + ||x_{n+p-1}|| + ||x_{n+p}||]^\beta \\ &\leq (1-\varepsilon)M(x_{n-1}, x_{n+p-1}) \\ &+ B\varepsilon^\eta\psi(\varepsilon),\ B > 0 \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+p-1}) &= \max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+p-1}, Tx_{n+p-1}), \\ &\frac{d(x_{n-1}, Tx_{n-1})d(x_{n+p-1}, Tx_{n+p-1})}{1 + d(x_{n-1}, x_{n+p-1})}\} \\ &= \max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p}), \\ &\frac{d(x_{n-1}, x_n)d(x_{n+p-1}, x_{n+p})}{1 + d(x_{n-1}, x_{n+p-1})}\} \\ &\leq \max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n)^2\}. \end{aligned} \tag{2.13}$$

For $p = 1$, using (2.13) and (2.5), the inequality (2.12) implies that

$$\begin{aligned} sd(x_n, x_{n+1}) &= sd(Tx_{n-1}, Tx_n) \\ &\leq (1-\varepsilon)M(x_{n-1}, x_n) + B\varepsilon^\eta\psi(\varepsilon) \\ &\leq (1-\varepsilon)d(x_{n-1}, x_n) + B\varepsilon^\eta\psi(\varepsilon). \end{aligned} \tag{2.14}$$

By (2.5), the sequence $\{d(x_n, x_{n+1})\}$ is converges. If $\lim_{n\to\infty} d(x_n, x_{n+1}) = d^* > 0$, it follows from (2.14) that $d^* \leq B\psi(\varepsilon)$, that is $d^* = 0$. A contradiction. Thus $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

In the following, we assume that $d(x_n, x_{n+1}) < 1$ for all $n \in N$. Thus

$$d(x_n, x_{n+1})^2 < d(x_n, x_{n+1}) \tag{2.15}$$

for all $n \in N$.

Fixed $p \geq 2$. (2.12), (2.13) and (2.15) imply that

$$sd(x_n, x_{n+p}) \leq (1-\varepsilon)\max\{d(x_{n-1}, x_{n+p-1}), d(x_{n-1}, x_n)\} + B\varepsilon^\eta\psi(\varepsilon),$$

that is, using the notations in step III,

$$a_n \leq sa_n \leq (1-\varepsilon)\max\{a_{n-1}, b_{n-1}\} + B\varepsilon^\eta\psi(\varepsilon). \tag{2.16}$$

From step III, we see that $\max\{a_n, b_n\}$ is decreasing. Since $\lim_{n\to\infty} b_n = 0$, by Lemma 1.4, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \max\{a_n, b_n\} = t.$$

If $t > 0$, taking the limit as $n \to \infty$ on both sides of (2.16), we have

$$t \leq B\psi(\varepsilon)$$

for all $\varepsilon \in (0, 1]$, that is $t = 0$. A contradiction.

*Step VI.* We will show that $\{x_n\}$ is Cauchy sequence in $X$. Suppose, to the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. By Step V and Lemma 1.3, there exist $\delta > 0$ and two sequence $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k + \nu$, $m_k \geq k$ and

$$\begin{aligned}
&\delta \leq \liminf_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \limsup_{n\to\infty} d(x_{n_k}, x_{m_k}) \leq s\delta,\\
&\frac{\delta}{s} \leq \liminf_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq \limsup_{n\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq s\delta.
\end{aligned} \tag{2.17}$$

Now putting $x = x_{m_k-1}, y = x_{n_k-1}$ in (2.1), since $d(x_{n_k}, x_{m_k}) > 0$ and $\alpha(x_{n_k}, x_{m_k}) \geq 1$, we obtain

$$\begin{aligned}
s\delta &\leq sd(x_{m_k}, x_{n_k}) = sd(Tx_{m_k-1}, Tx_{n_k-1})\\
&< (1-\varepsilon)M(x_{m_k-1}, x_{n_k-1}) + B\varepsilon\psi(\varepsilon),
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
M(x_{m_k-1}, x_{n_k-1}) &= \max\{d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, Tx_{m_k-1}), d(x_{n_k-1}, Tx_{n_k-1}),\\
&\quad \frac{d(x_{m_k-1}, Tx_{m_k-1})d(x_{n_k-1}, Tx_{n_k-1})}{1 + d(x_{m_k-1}, x_{n_k-1})}\}\\
&= \max\{d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}),\\
&\quad \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k-1}, x_{n_k})}{1 + d(x_{m_k-1}, x_{n_k-1})}\}.
\end{aligned} \tag{2.19}$$

Using Step V and (2.19), we have

$$\limsup_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) \leq s\delta.$$

Taking the limit of supermum as $k \to \infty$ in (2.18),

$$s\delta \leq B\psi(\varepsilon),$$

that is, $\delta = 0$, a contradiction.

Hence $\{x_n\}$ is Cauchy sequence in $X$. Since $(X, d)$ is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0.$$

*Step VII.* We show that $u$ is a fixed point of $T$. Using Theorem 2.1 (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, u) \geq 1$ for all $k \in N$. Suppose that $u \neq Tu$, so $d(u, Tu) > 0$. Since $\{x_n\}$ is a sequence with distinct elements, we can assume that $x_n \neq Tu$ for all $n \in N$. Putting $x = x_{n_k}, y = u$ in (2.1), we get

$$\begin{aligned} & d(u, Tu) \leq s[d(u, x_{n_k+\nu}) + d(x_{n_k+\nu-1}, x_{n_k+\nu-2}) + \cdots + d(x_{n_k+2}, x_{n_k+1}) + d(x_{n_k+1}, Tu)] \\ & < s[d(u, x_{n_k+\nu}) + d(x_{n_k+\nu-1}, x_{n_k+\nu-2}) + \cdots + d(x_{n_k+2}, x_{n_k+1})] + \\ & (1-\varepsilon)M(x_{n_k}, u) + B\varepsilon\psi(\varepsilon), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M(x_{n_k}, u) &= \max\{d(x_{n_k}, u), d(x_{n_k}, Tx_{n_k}), d(u, Tu), \frac{d(x_{n_k}, Tx_{n_k})d(u, Tu)}{1 + d(x_{n_k}, u)}\} \\ &= \max\{d(x_{n_k}, u), d(x_{n_k}, x_{n_k+}), d(u, Tu), \frac{d(x_{n_k}, x_{n_k+1})d(u, Tu)}{1 + d(x_{n_k}, u)}\}. \end{aligned} \tag{2.21}$$

Using Step V and (2.21), we have

$$\lim_{k\to\infty} M(x_{n_k}, u) = d(u, Tu).$$

Taking the limit as $k \to \infty$ in (2.20), using step V, we have

$$d(u, Tu) \leq (1-\varepsilon)d(u, Tu) + B\varepsilon\psi(\varepsilon),$$

from which we have

$$d(u, Tu) \leq B\psi(\varepsilon),$$

that is $d(u, Tu) = 0$, a contradiction. Thus $u = Tu$.

*Step VIII.* Finally, we prove that the fixed point of $T$ is unique. Suppose that $u, v$ are two fixed points of $T$ such that $u \neq v$. Then by the hypothesis, $\alpha(u, v) \geq 1$. Hence, from (2.1) with $\varepsilon = 0$, $x = u$ and $y = v$ we have

$$sd(u, v) = sd(Tu, Tv) < M(u, v)$$

where

$$M(u, v) = \max\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tu)d(v, Tv)}{1 + d(u, v)}\} = d(u, v).$$

Thus $d(u, v) < d(u, v)$, a contradiction. □

**Note.** In Theorem 2.1, if $s > 1$, the inequality (2.1) can be replaced by

$$\begin{aligned} sd(Tx, Ty) \leq & (1-\varepsilon)M(x, y)) \\ & + \Lambda\varepsilon^\eta\psi(\varepsilon)[1 + ||x|| + ||y|| + ||Tx|| + ||Ty||]^\beta. \end{aligned}$$

Moreover, if $\nu \geq 2$, we can give another method to prove that $\{x_n\}$ is a Cauchy sequence. In fact, from (2.7) and (2.8), we have

$$\max\{a_n, b_n, b_n^2\} \leq \frac{K}{s^n}$$

for all $n \in N$. Using $b_\mu(s)$-inequality, for all $n, p \in N$, we have

$$\begin{aligned}
d(x_n, x_{n+p\nu}) \leq\ & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+\nu-1}, x_{n+\nu})] \\
& + s^2[d(x_{n+\nu}, x_{n+\nu+1}) + d(x_{n+\nu+1}, x_{n+\nu+2}) + \cdots + d(x_{n+2\nu-1}, x_{n+2\nu})] \\
& \cdots \\
& + s^p[d(x_{n+(p-1)\nu}, x_{n+(p-1)\nu+1}) + d(x_{n+(p-1)\nu+1}, x_{n+(p-1)\nu+2}) + \cdots \\
& + d(x_{n+p\nu-1}, x_{n+p\nu})] \\
& \leq s[\frac{K}{s^n} + \frac{K}{s^{n+1}} + \cdots + \frac{K}{s^{n+\nu-1}}] \\
& + s^2[\frac{K}{s^{n+\nu}} + \frac{K}{s^{n+\nu+1}} + \cdots + \frac{K}{s^{n+2\nu-1}}] \\
& + \cdots \\
& + s^p[\frac{K}{s^{n+(p-1)\nu}} + \frac{K}{s^{n+(p-1)\nu+1}} + \cdots + \frac{K}{s^{n+p\nu-1}}] \\
& \leq s\nu \frac{\frac{K}{s^n}}{1 - \frac{1}{s^{\nu-1}}},
\end{aligned}$$

$$\begin{aligned}
d(x_n, x_{n+p\nu+1}) \leq\ & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+\nu-1}, x_{n+\nu})] \\
& + s^2[d(x_{n+\nu}, x_{n+\nu+1}) + d(x_{n+\nu+1}, x_{n+\nu+2}) + \cdots + d(x_{n+2\nu-1}, x_{n+2\nu})] \\
& \cdots \\
& + s^p[d(x_{n+(p-1)\nu}, x_{n+(p-1)\nu+1}) + d(x_{n+(p-1)\nu+1}, x_{n+(p-1)\nu+2}) + \cdots \\
& + d(x_{n+p\nu-1}, x_{n+p\nu+1})] \\
& \leq s\nu \frac{\frac{K}{s^n}}{1 - \frac{1}{s^{\nu-1}}}, \\
& \cdots
\end{aligned}$$

$$\begin{aligned}
d(x_n, x_{n+p\nu+\nu-1}) \leq\ & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+\nu-1}, x_{n+\nu})] \\
& + s^2[d(x_{n+\nu}, x_{n+\nu+1}) + d(x_{n+\nu+1}, x_{n+\nu+2}) + \cdots + d(x_{n+2\nu-1}, x_{n+2\nu})] \\
& \cdots
\end{aligned}$$

$$+ s^p[d(x_{n+(p-1)\nu}, x_{n+(p-1)\nu+1}) + d(x_{n+(p-1)\nu+1}, x_{n+(p-1)\nu+2}) + \cdots$$
$$+ d(x_{n+p\nu-1}, x_{n+p\nu+\nu-1})]$$
$$\leq s\nu \frac{\frac{K}{s^n}}{1 - \frac{1}{s^{\nu-1}}},$$

this implies

$$d(x_n, x_{n+m}) \leq s\nu \frac{\frac{K}{s^n}}{1 - \frac{1}{s^{\nu-1}}}.$$

Thus $\{x_n\}$ is a Cauchy sequence.

**Corollary 2.1.** Let $(X, \preceq, d)$ be a partially ordered and complete $b_\nu(s)$-metric space with $s \geq 1$. Suppose that following conditions are satisfied:

(1) $T$ is a increasing mapping with respect $\preceq$, that is $Tx \preceq Ty$ if $x \preceq y$;

(2) there exist $\Lambda \geq 0, L \geq 0, \eta \geq 1$, $\beta \in [0, \eta]$ and $\psi \in \Psi$ such that for every $\varepsilon \in [0, 1]$ and for all $x, y \in X$ with $x \preceq y$ and $d(Tx, Ty) > 0$,

$$sd(Tx, Ty) \leq (1 - \varepsilon)M(x, y) + \Lambda\varepsilon^\eta \psi(\varepsilon)[1 + ||x|| + ||y|| + ||Tx|| + ||Ty||]^\beta$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\};$$

(3) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;.

(4) $x_n \preceq x$ for all $n \in N$ whenever $\{x_n\}$ is nondecreasing sequence in $X$ such that $x_n \to x \in X$.

Then $T$ has a fixed point $u$ and $\{T^n x_0\}$ converges to $u$. Further, if all $x, y \in F(T)$, $x$ and $y$ are comparable, then $T$ has a unique fixed point in $X$.

*Proof.* Define $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, by Theorem 2.1, $T$ has a fixed point. $\square$

## 3. Common fixed point results

In this section, we prove some common fixed point results for two self-mappings. Following [27], we introduce the notion of $f-\alpha$-admissible mapping.

**Definition 3.1.** *Let $X$ be a non-empty set. And let $T, f: X \longrightarrow X$ and $\alpha: X \times X \longrightarrow [0, \infty)$. The mapping $T$ is $f-\alpha$-admissible if, for all $x, y \in X$ such that $\alpha(fx, fy) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$.*

Clearly, if $f$ is the identity mapping, then $T$ is $\alpha$-admissible.

**Theorem 3.1.** *Let $(X, d)$ be a complete $b_\nu(s)$-metric space with $s \geq 1$ and $\alpha : X \times X \to [0, \infty)$ be a given function. Let $T, f : X \to X$ be two mappings. Suppose that following conditions are satisfied::*

(1) *$T$ is an $f-\alpha$-admissible mapping;*
(2) *there exist $\Lambda \geq 0, \eta \geq 1$, $\beta \in [0, \eta]$ and $\psi \in \Psi$ such that for every $\varepsilon \in [0, 1]$ and for all $x, y \in X$ with $\alpha(fx, fy) \geq 1$ and $d(Tx, Ty) > 0$,*

$$\begin{aligned} sd(Tx, Ty) \leq &(1-\varepsilon)M(fx, fy) \\ &+ \Lambda\varepsilon^\eta\psi(\varepsilon)[1 + ||fx|| + ||fy|| + ||Tx|| + ||Ty||]^\beta \end{aligned} \tag{4.1}$$

*where*

$$M(fx, fy) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Tx)d(fy, Ty)}{1 + d(fx, fy)}\};$$

(3) *there exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$;*
(4) *if $\{y_n\}$ is a sequence in $X$ such that $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in N$ and $y_n \to y$ as $n \to \infty$, then there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\alpha(y_{n(k)}, y) \geq 1$ for all $k \in N$.*

*Then $T$ and $f$ have a point of coincidence in $X$. Moreover, if $T$ and $f$ are weakly compatible, then $T$ and $f$ have a common fixed point. Further, if all points of coincidence of $f$ and $T$, we have $\alpha(fx, fy) \geq 1$, then $T$ and $f$ have a unique point of coincidence in $X$..*

Before we prove this theorem, we introduce the following lemma.

**Lemma 3.1.** *[13] Let $X$ be a non-empty set and let $f : X \to X$ be a self-mapping. Then there exists a subset $E$ of $X$ such that $fE = fX$ and $f|_E$ is injective.*

*Proof.* By Lemma 3.1, there exists $E \subseteq X$ such that $fE = fX$ and $f : E \longrightarrow X$ is one-to-one. Now, define a map $h : f(E) \longrightarrow f(E)$ by $h(f(x)) = Tx$. Since $f$ is one-to-one on $E$, $h$ is well defined. Note that for all $fx, fy \in f(E)$ with $\alpha(fx, fy) \geq 1, d(h(fx), h(fy)) > 0$, then (4.1) can be rewrite as

$$\begin{aligned} sd(h(fx), h(fy)) \leq &(1-\varepsilon)M(fx, fy) \\ &+ \Lambda\varepsilon^{\eta}\psi(\varepsilon)[1 + ||fx|| + ||fy|| + ||h(fx)|| + ||h(fy)||]^{\beta}, \end{aligned}$$

$$M(fx, fy) = \max\{d(fx, fy), d(fx, h(fx)), d(fy, h(fy)), \frac{d(fx, h(fx))d(fy, h(fy))}{1 + d(fx, fy)}\};$$

Thus for all $x', y' \in f(E)$ with $\alpha(x'y') \geq 1$ and $d(hx', hy') > 0$, we have

$$\begin{aligned} sd(hx', hy') \leq &(1-\varepsilon)M(x', y') \\ &+ \Lambda\varepsilon^{\eta}\psi(\varepsilon)[1 + ||x'|| + ||y'|| + ||hx'|| + ||hy'||]^{\beta} \end{aligned}$$

where

$$M(x', y') = \max\{d(x', y'), d(x', hx'), d(y', hy'), \frac{d(x', hx')d(y', hy')}{1 + d(x', y')}\}.$$

Since $f(E) = f(X)$ is complete, by using Theorem 2.1, there exists $x_0 \in E$ such that $h(fx_0) = fx_0$. Hence $T$ and $f$ have a point of coincidence in $X$. It is clear that $T$ and $f$ have a common fixed point whenever $T$ and $f$ are weakly compatible. □

In 2017, M. Rangamma and P. M. Reddy [25] established a unique common fixed point theorem for $T$-contraction of two self mappings on generalized cone $b$-metric spaces with solid cone. In the following theorem, a unique common fixed point theorem for $T$-contraction of two self mappings on $b_\nu(s)$-metric spaces is established.

**Theorem 3.2.** *Let $(X, d)$ be a complete $b_\nu(s)$-metric space and $\alpha : X \times X \to [0, \infty)$ be a given function. Let $T, f : X \to X$ be a mappings. Suppose that $T$ is one to one and $T(X)$ is a complete subspace of $X$, and the following conditions are satisfied:*

1. *if $\alpha(Tx, Ty) \geq 1$ then $\alpha(Tfx, Tfy) \geq 1$, and $\alpha(x, y) \geq 1, \alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$, $x, y, z \in X$;*
2. *there exist $\Lambda \geq 0, \eta \geq 1$, $\beta \in [0, \eta]$ and $\psi \in \Psi$ such that for every $\varepsilon \in [0, 1]$ and for all $x, y \in X$ with $\alpha(Tx, Ty) \geq 1$ and $d(Tfx, Tfy) > 0$,*

$$\begin{aligned} sd(Tfx, Tfy) \leq &(1-\varepsilon)M(Tx, Ty) \\ &+ \Lambda\varepsilon^{\eta}\psi(\varepsilon)[1 + ||Tx|| + ||Ty|| + ||Tfx|| + ||Tfy||]^{\beta} \end{aligned} \tag{4.3}$$

*where*

$$M(Tx, Ty) = \max\{d(Tx, Ty), d(Tx, Tfx), d(Ty, Tfy), \frac{d(Tx, Tfx)d(Ty, Tfy)}{1 + d(Tx, Ty)}\};$$

(3) *There exists* $x_0 \in X$ *such that* $\alpha(Tx_0, Tfx_0) \geq 1$;

(4) *if* $\{x_n\}$ *is a sequence in* $X$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in N$ *and* $x_n \to x$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* $k \in N$.

*Then* $f$ *has a fixed point in* $X$. *Further, if all* $x, y \in F(f)$, *we have* $\alpha(Tx, Ty) \geq 1$ *then* $f$ *has a unique fixed point in* $X$. *Moreover, if* $f$ *and* $T$ *are commuting at the fixed point of* $f$, *then* $f$ *and* $T$ *have a unique common fixed point in* $X$.

*Proof.* Since $T$ is one to one, the conditions (i) and (ii) can be restated as
(i′) if $\alpha(Tx, Ty) \geq 1$ then $\alpha(TfT^{-1}Tx, TfT^{-1}Ty) \geq 1$, and $\alpha(x, y) \geq 1, \alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$, $x, y, z \in X$.
(ii′) if $\alpha(Tx, Ty) \geq 1$ and $d(TfT^{-1}Tx, TfT^{-1}Ty) > 0$ implies

$$sd(TfT^{-1}Tx, TfT^{-1}Ty) \leq (1-\varepsilon)M(Tx, Ty)$$
$$+ \Lambda\varepsilon^{\eta}\psi(\varepsilon)[1 + ||Tx|| + ||Ty|| + ||TfT^{-1}Tx|| + ||TfT^{-1}Ty||]^{\beta}$$

where

$$M(Tx, Ty) = \max\{d(Tx, Ty), d(Tx, TfT^{-1}Tx), d(Ty, TfT^{-1}Ty), \frac{d(Tx, TfT^{-1}Tx)d(Ty, TfT^{-1}Ty)}{1 + d(Tx, Ty)}\}.$$

Let $f' = TfT^{-1}$. Then we have
(i″) $f'$ is a triangular $\alpha$-admissible mapping in $TX$.
(ii″) for all $x', y' \in TX$, if $\alpha(x', y') \geq 1$ and $d(f'x, f'y) > 0$ implies

$$sd(f'x', f'y') \leq (1-\varepsilon)M(x', y')$$
$$+ \Lambda\varepsilon^{\eta}\psi(\varepsilon)[1 + ||x'|| + ||y'|| + ||f'x'|| + ||f'y'||]^{\beta}$$

where

$$M(x', y') = \max\{d(x', y'), d(x', f'x'), d(y', f'y'), \frac{d(x', f'x')d(y', f'y)}{1 + d(x', y')}\}.$$

Then, by Theorem 2.1, there exist $x' = Tx \in TX$ such that $f'Tx = Tx$, that is $Tfx = Tx$. Since $T$ is one to one, we get $fx = x$. If $x \in F(f)$, then $Tfx = Tx$ and $TfT^{-1}Tx = Tx$, which mains that $Tx$ is a fixed point of $f'$. Thus if for all $x, y \in F(f)$, $\alpha(Tx, Ty) \geq 1$, then, by Theorem 2.1, $f'$ has a unique fixed point. It follows that $f$ has a unique fixed point. Moreover, if $f$ and $T$ are commuting at the unique fixed point

$x$ of $f$, then $Tx = Tfx = fTx$, i.e., $Tx$ is also a fixed point of $f$. Since $f$ has unique fixed point, we have $Tx = x$, i.e., $x$ is also the fixed point of $T$. So $f$ and $T$ have a unique common fixed point in $X$. $\square$

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