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MULTILINEAR BMO ESTIMATES FOR THE COMMUTATORS OF MULTILINEAR FRACTIONAL MAXIMAL AND INTEGRAL OPERATORS ON THE PRODUCT GENERALIZED MORREY SPACES

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Abstract. In this paper, we establish multilinear BMO estimates for commutators of multilinear fractional maximal and integral operators both on product generalized Morrey spaces and product generalized vanishing Morrey spaces, respectively. Similar results are still valid for commutators of multilinear maximal and singular integral operators.

1. Introduction and main results

The classical Morrey spaces $L_{p,\lambda}$ have been introduced by Morrey in [21] to study the local behavior of solutions of second order elliptic partial differential equations(PDEs). In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey-type spaces. It has been obtained that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. Morrey spaces can complement the boundedness properties of operators that Lebesgue spaces can not handle. Morrey spaces which we have been handling are called classical Morrey spaces(see [21]). But, classical Morrey spaces are not totally enough to describe the boundedness properties. To this end, we need to generalize parameters p and q, among others p, but this issue will exceed the scope of the article, so we pass this part. Though we do not consider the direct applications of Morrey spaces to PDEs, Morrey

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spaces can be applied to PDEs. Applications to the second order elliptic partial differential equations can be found in [10] and [26].

We will say that a function $f \in L_{p,\lambda} = L_{p,\lambda}(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n, r>0} \left[r^{-\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy \right]^{1/p} < \infty. \tag{1.1}$$

Here, $1 < p < \infty$ and $0 < \lambda < n$ and the quantity of (1.1) is the (p, λ)-Morrey norm, denoted by $\|f\|_{L_{p,\lambda}}$. In recent years, more and more researches focus on function spaces based on Morrey spaces to fill in some gaps in the theory of Morrey type spaces (see, for example, [11–14, 16, 18, 23, 25, 29]). Moreover, these spaces are useful in harmonic analysis and PDEs. But, this topic exceeds the scope of this paper. Thus, we omit the details here. On the other hand, the study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VL_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [27] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $L_{p,\lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r\to 0} \sup_{\substack{x\in\mathbb{R}^n \\ 0<t<r}} \left[t^{-\lambda} \int\limits_{B(x,t)} |f(y)|^p \, dy \right]^{1/p} = 0,$$

where $1 < p < \infty$ and $0 < \lambda < n$ for brevity, so that

$$VL_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L_{p,\lambda}(\mathbb{R}^n) : \lim_{r\to 0} \sup_{\substack{x\in\mathbb{R}^n \\ 0<t<r}} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0 \right\}.$$

For the properties and applications of vanishing Morrey spaces, see also [1].

After studying Morrey spaces in detail, researchers have passed to the concept of generalized Morrey spaces. Firstly, motivated by the work of [21], Mizuhara [19] introduced generalized Morrey spaces $M_{p,\varphi}$. Then, the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm is defined as follows:

Definition 1.1. ***(generalized Morrey space)*** *Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. If $0 < p < \infty$, then the generalized Morrey space $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ is defined by*

$$\left\{ f \in L_p^{loc}(\mathbb{R}^n) : \|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\}.$$

Obviously, the above definition recover the definition of $L_{p,\lambda}(\mathbb{R}^n)$ if we choose $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$, that is

$$L_{p,\lambda}(\mathbb{R}^n) = M_{p,\varphi}(\mathbb{R}^n) \mid_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Everywhere in the sequel we assume that $\inf_{x\in\mathbb{R}^n, r>0} \varphi(x,r) > 0$ which makes the above spaces non-trivial, since the spaces of bounded functions are contained in these spaces. We point out that $\varphi(x,r)$ is a measurable nonnegative function and no monotonicity type condition is imposed on these spaces.

In [12], [16], [18], [19] and [25], the boundedness of the maximal operator and Calderón-Zygmund operator on the generalized Morrey spaces $M_{p,\varphi}$ has been obtained, respectively.

For brevity, in the sequel we use the notations

$$\mathfrak{M}_{p,\varphi}(f;x,r) := \frac{|B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}}{\varphi(x,r)},$$

and

$$\mathfrak{M}^W_{p,\varphi}(f;x,r) := \frac{|B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))}}{\varphi(x,r)}.$$

In this paper, extending the definition of vanishing Morrey spaces [27], we introduce generalized vanishing Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$ with normalized norm in the following form.

Definition 1.2. ***(generalized vanishing Morrey space)*** *The generalized vanishing Morrey space $VM_{p,\varphi}(\mathbb{R}^n)$ is defined by*

$$\left\{ f \in M_{p,\varphi}(\mathbb{R}^n) : \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f;x,r) = 0 \right\}.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\inf\limits_{x\in\mathbb{R}^n} \varphi(x,r)} = 0, \tag{1.2}$$

and

$$\sup_{0<r<\infty} \frac{1}{\inf\limits_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty, \tag{1.3}$$

which make the spaces $VM_{p,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $VM_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{M}_{p,\varphi}(f;x,r).$$

The spaces $VM_{p,\varphi}(\mathbb{R}^n)$ are also closed subspaces of the Banach spaces $M_{p,\varphi}(\mathbb{R}^n)$, which may be shown by standard means.

Furthermore, we have the following embeddings:

$$VM_{p,\varphi} \subset M_{p,\varphi}, \qquad \|f\|_{M_{p,\varphi}} \leq \|f\|_{VM_{p,\varphi}}.$$

On the other hand, it is well known that, for the purpose of researching non-smoothness partial differential equation, mathematicians pay more attention to the singular integrals. Moreover, the fractional type operators and their weighted boundedness theory play important roles in harmonic analysis and other fields, and the multilinear operators arise in numerous situations involving product-like operations, see [2, 3, 5–8, 14, 17, 20, 24] for instance.

First of all, we recall some basic properties and notations used in this paper.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x=(x_1,...,x_n)$ with norm $|x|=\left(\sum_{i=1}^{n}x_i^2\right)^{1/2}$ and corresponding m-fold product spaces ($m\in\mathbb{N}$) be $(\mathbb{R}^n)^m=\mathbb{R}^n\times\cdots\times\mathbb{R}^n$. Let $B=B(x,r)$ denotes open ball centered at x of radius r for $x\in\mathbb{R}^n$ and $r>0$ and $B^c(x,r)$ its complement. Also $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)|=v_n r^n$, where $v_n=|B(0,1)|$. We also denote by $\overrightarrow{y}=(y_1,\ldots,y_m)$, $d\overrightarrow{y}=dy_1\ldots dy_m$, and by $\overrightarrow{f}$ the m-tuple $(f_1,...,f_m)$, m, n the nonnegative integers with $n\geq 2$, $m\geq 1$.

Let $\overrightarrow{f}\in L_{p_1}^{loc}(\mathbb{R}^n)\times\cdots\times L_{p_m}^{loc}(\mathbb{R}^n)$. Then multi-sublinear fractional maximal operator $M_\alpha^{(m)}$ is defined by

$$M_\alpha^{(m)}\left(\overrightarrow{f}\right)(x)=\sup_{t>0}|B(x,t)|^{\frac{\alpha}{n}}\left[\prod_{i=1}^{m}\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f_i(y_i)|\right]d\overrightarrow{y},\quad 0\leq\alpha<mn.$$

From definition, if $\alpha=0$ then $M_\alpha^{(m)}$ is the multi-sublinear maximal operator $M^{(m)}$ and also; in the case of $m=1$, $M_\alpha^{(m)}$ is the classical fractional maximal operator M_α.

The theory of multilinear Calderón-Zygmund singular integral operators, originated from the works of Coifman and Meyer's [4], plays an important role in harmonic analysis. Its study has been attracting a lot of attention in the last few decades. A systematic analysis of many basic properties of such multilinear singular integral operators can be found in the articles by Coifman-Meyer [4], Grafakos-Torres [7–9], Chen et al. [2], Fu et al. [5].

Let $T^{(m)}$ ($m\in\mathbb{N}$) be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T^{(m)}:S(\mathbb{R}^n)\times\cdots\times S(\mathbb{R}^n)\to S(\mathbb{R}^n).$$

Following [7], recall that the m(multi)-linear Calderón-Zygmund operator $T^{(m)}$ ($m\in\mathbb{N}$) for test vector $\overrightarrow{f}=(f_1,\ldots,f_m)$ is defined by

$$T^{(m)}\left(\overrightarrow{f}\right)(x)=\int\limits_{(\mathbb{R}^n)^m}K(x,y_1,\ldots,y_m)\left\{\prod_{i=1}^{m}f_i(y_i)\right\}dy_1\cdots dy_m,\qquad x\notin\bigcap_{i=1}^{m}supp f_i,$$

where K is an m-Calderón-Zygmund kernel which is a locally integrable function defined off the diagonal $y_0=y_1=\cdots=y_m$ on $(\mathbb{R}^n)^{m+1}$ satisfying the following size estimate:

$$|K(x,y_1,\ldots,y_m)|\leq\frac{C}{|(x-y_1,\ldots,x-y_m)|^{mn}},$$

for some $C>0$ and some smoothness estimates, see [7–9] for details.

The result of Grafakos and Torres [7, 9] shows that the multilinear Calderón-Zygmund operator is bounded on the product of Lebesgue spaces.

Theorem 1.1. *[7, 9] Let $T^{(m)}$ $(m \in \mathbb{N})$ be an m-linear Calderón-Zygmund operator. Then, for any numbers $1 \leq p_1, \ldots, p_m, p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $T^{(m)}$ can be extended to a bounded operator from $L_{p_1} \times \cdots \times L_{p_m}$ into L_p, and bounded from $L_1 \times \cdots \times L_1$ into $L_{\frac{1}{m},\infty}$.*

On the other hand, the multilinear fractional type operators are natural generalization of linear ones. Their earliest version was originated on the work of Grafakos [6] in 1992, in which he studied the multilinear maximal function and multilinear fractional integral defined by

$$M_{\alpha}^{(m)}\left(\overrightarrow{f}\right)(x) = \sup_{t>0} \frac{1}{r^{n-\alpha}} \int\limits_{|y|<r} \left| \prod_{i=1}^{m} f_i\left(x - \theta_i y\right) \right| dy,$$

and

$$I_{\alpha}^{(m)}\left(\overrightarrow{f}\right)(x) = \int\limits_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{i=1}^{m} f_i\left(x - \theta_i y\right) dy,$$

where θ_i $(i = 1, \ldots, m)$ are fixed distinct are nonzero real numbers and $0 < \alpha < n$. We note that, if we simply take $m = 1$ and $\theta_i = 1$, then M_α and I_α are just the operators studied by Muckenhoupt and Wheeden in [22]. In this paper we deal with another kind of multilinear operator which was defined by Kenig and Stein [17] for $\overrightarrow{f} = (f_1, \ldots, f_m)$, which is called multilinear fractional integral operator as follows

$$I_{\alpha}^{(m)}\left(\overrightarrow{f}\right)(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{1}{|(x - y_1, \ldots, x - y_m)|^{mn-\alpha}} \left\{ \prod_{i=1}^{m} f_i\left(y_i\right) \right\} d\overrightarrow{y},$$

whose kernel is

$$|K\left(x, y_1, \ldots, y_m\right)| = |(x - y_1, \ldots, x - y_m)|^{-mn+\alpha}, \quad 0 < \alpha < mn,$$

where $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are measurable and $|(x - y_1, \ldots, x - y_m)| = \sqrt{\sum\limits_{i=1}^{m} |x - y_i|^2}$.

They [17] proved that $I_{\alpha}^{(m)}$ $(m \in \mathbb{N})$ is of strong type $(L_{p_1} \times L_{p_2} \times \cdots \times L_{p_m}, L_q)$ and weak type $(L_{p_1} \times L_{p_2} \times \cdots \times L_{p_m}, L_{q,\infty})$. If we take $m = 1$, $I_{\alpha}^{(m)}$ $(m \in \mathbb{N})$ is the classical fractional integral operator I_α. Moreover, their main result (Theorem 1 in [17]) is the multi-version of well-known Hardy-Littlewood-Sobolev inequality. Later, weighted inequalities for the multilinear fractional integral operators have been established by Moen [20] and Chen-Xue [3], respectively. Yu and Tao [29] have also obtained the boundedness of the operators $I_{\alpha}^{(m)}$, $T^{(m)}$ and $M^{(m)}$ $(m \in \mathbb{N})$ on the product generalized Morrey spaces, respectively.

Now, we will give some properties related to the space of functions of Bounded Mean Oscillation, BMO, which play a great role in the proofs of our main results, introduced by John and Nirenberg [15] in 1961. This space has become extremely important in various areas of analysis including harmonic analysis, PDEs and function theory. BMO-spaces are also of interest since, in the scale of Lebesgue spaces, they may be considered and appropriate substitute for L_∞. Appropriate in the sense that are spaces preserved by a wide class of important operators such as the Hardy-Littlewood maximal function, the Hilbert transform and which can be used as an end point in interpolating L_p spaces.

Let us recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 1.3. *[12, 16] The space $BMO(\mathbb{R}^n)$ of functions of bounded mean oscillation consists of locally summable functions with finite semi-norm*

$$\|b\|_* \equiv \|b\|_{BMO} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}|dy < \infty, \tag{1.4}$$

where $b_{B(x,r)}$ is the mean value of the function b on the ball $B(x,r)$. The fact that precisely the mean value $b_{B(x,r)}$ figures in (1.4) is inessential and one gets an equivalent seminorm if $b_{B(x,r)}$ is replaced by an arbitrary constant c :

$$\|b\|_* \approx \sup_{r>0} \inf_{c\in\mathbb{C}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - c|\, dy.$$

Each bounded function $b \in BMO$. Moreover, BMO contains unbounded functions, in fact log$|x|$ belongs to BMO but is not bounded, so $L_\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.

In 1961 John and Nirenberg [15] established the following deep property of functions from BMO.

Theorem 1.2. *[15] If $b \in BMO(\mathbb{R}^n)$ and $B(x,r)$ is a ball, then*

$$\left|\left\{x \in B(x,r) : |b(x) - b_{B(x,r)}| > \xi\right\}\right| \le |B(x,r)| \exp\left(-\frac{\xi}{C\|b\|_*}\right), \ \xi > 0,$$

where C depends only on the dimension n.

By Theorem 1.2, we can get the following results.

Corollary 1.1. *[12, 16] Let $b \in BMO(\mathbb{R}^n)$. Then, for any $q > 1$,*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{1.5}$$

is valid.

Corollary 1.2. *[12, 16] Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \left(1 + \ln\frac{t}{r}\right) \quad \text{for } 0 < 2r < t, \tag{1.6}$$

and for any $q > 1$, it is easy to see that

$$\|b - (b)_B\|_{L_q(B)} \le C r^{\frac{n}{q}} \|b\|_* \left(1 + \ln\frac{t}{r}\right), \tag{1.7}$$

where C is independent of b, x, r and t.

Now inspired by Definition 1.3, we can give the definition of multilinear BMO ($= \mathfrak{BMO}$). Indeed in this paper we will consider a multilinear version ($=$ multilinear BMO or $\mathfrak{BMO}$) of the BMO.

Definition 1.4. *We say that* $\overrightarrow{b}=(b_1,\dots,b_m)\in\mathfrak{BMO}$ *if*

$$\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}=\sup_{x\in\mathbb{R}^n,r>0}\prod_{i=1}^{m}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}\left|b_i\left(y_i\right)-\left(b_i\right)_{B(x,r)}\right|dy_i<\infty,$$

where

$$\left(b_i\right)_{B(x,r)}=\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}b_i(y_i)dy_i.$$

Remark 1.1. *Notice that* $(BMO)^m$ *is contained in* $\mathfrak{BMO}$ *and we have*

$$\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\leq\prod_{i=1}^{m}\|b_i\|_*,$$

so

$$(BMO)^m\subset\mathfrak{BMO}$$

is valid.

We now make some conventions. Throughout this paper, we use the symbol $A\lesssim B$ to denote that there exists a positive consant C such that $A\leq CB$. If $A\lesssim B$ and $B\lesssim A$, we then write $A\approx B$ and say that A and B are equivalent. For a fixed $p\in[1,\infty)$, p' denotes the dual or conjugate exponent of p, namely, $p'=\frac{p}{p-1}$ and we use the convention $1'=\infty$ and $\infty'=1$.

Remark 1.2. *Let* $0<\alpha<mn$ *and* $1<p_i<\infty$ *with* $\frac{1}{p}=\sum\limits_{i=1}^{m}\frac{1}{p_i}$, $\frac{1}{q_i}=\frac{1}{p_i}-\frac{\alpha}{mn}$, $\frac{1}{q}=\sum\limits_{i=1}^{m}\frac{1}{q_i}=\frac{1}{p}-\frac{\alpha}{n}$ *and* $\overrightarrow{b}=(b_1,\dots,b_m)\in(BMO)^m$ *for* $i=1,\dots,m$. *Then, from Corollary 1.2, it is easy to see that*

$$\prod_{i=1}^{m}\|b_i-(b_i)_B\|_{L_{q_i}(B)}\leq C\prod_{i=1}^{m}|B(x,r)|^{\frac{1}{q_i}}\|b_i\|_*\left(1+\ln\frac{t}{r}\right),\tag{1.8}$$

and

$$\begin{aligned}\prod_{i=1}^{m}\|b_i-(b_i)_B\|_{L_{q_i}(2B)}&\leq\prod_{i=1}^{m}\left(\|b_i-(b_i)_{2B}\|_{L_{q_i}(2B)}+\|(b_i)_B-(b_i)_{2B}\|_{L_{q_i}(2B)}\right)\\&\lesssim\prod_{i=1}^{m}|B(x,r)|^{\frac{1}{q_i}}\|b_i\|_*\left(1+\ln\frac{t}{r}\right).\end{aligned}\tag{1.9}$$

On the other hand, Xu [28] has established the boundedness of the commutators generated by m-linear Calderón-Zygmund singular integrals and $RBMO$ functions with nonhomogeneity on the product of Lebesgue space. Inspired by [2,3,7,9,24,28], commutators $T_{\overrightarrow{b}}^{(m)}$ generated by m-linear Calderón-Zygmund operators $T^{(m)}$ and bounded mean oscillation functions $\overrightarrow{b}=(b_1,\dots,b_m)$ is given by

$$T_{\overrightarrow{b}}^{(m)}\left(\overrightarrow{f}\right)(x)=\int\limits_{(\mathbb{R}^n)^m}K\left(x,y_1,\dots,y_m\right)\left[\prod_{i=1}^{m}\left[b_i\left(x\right)-b_i\left(y_i\right)\right]f_i\left(y_i\right)\right]d\overrightarrow{y},$$

where $K(x, y_1, \dots, y_m)$ is a m-linear Calderón-Zygmund kernel, $b_i \in (BMO)^i(\mathbb{R}^n)$ for $i = 1, \dots, m$. Note that T_b is the special case of $T^{(m)}_{\overrightarrow{b}}$ with taking $m = 1$. Similarly, let $b_i\,(i = 1, \dots, m)$ be a locally integrable functions on $\mathbb{R}^n$, then the commutators generated by m-linear fractional integral operators and $\overrightarrow{b} = (b_1, \dots, b_m)$ is given by

$$I^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{1}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} \left[\prod_{i=1}^{m} [b_i(x) - b_i(y_i)]\, f_i(y_i)\right] d\overrightarrow{y},$$

where $0 < \alpha < mn$, and $f_i\ (i = 1, \dots, m)$ are suitable functions.

The commutators of a class of multi-sublinear maximal operators corresponding to $T^{(m)}_{\overrightarrow{b}}$ and $I^{(m)}_{\alpha,\overrightarrow{b}}$ $(m \in \mathbb{N})$ above are, respectively, defined by

$$M^{(m)}_{\overrightarrow{b}}\left(\overrightarrow{f}\right)(x) = \sup_{t>0}\left[\prod_{i=1}^{m} \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} [|b_i(x) - b_i(y_i)|]\, |f_i(y_i)|\right] d\overrightarrow{y},$$

and

$$M^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)(x) = \sup_{t>0}|B(x,t)|^{\frac{\alpha}{n}}\left[\prod_{i=1}^{m} \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} [|b_i(x) - b_i(y_i)|]\, |f_i(y_i)|\right] d\overrightarrow{y}, \quad 0 \le \alpha < mn.$$

The following result is known.

Lemma 1.1. *[24] (Strong bounds of $I^{(m)}_{\overrightarrow{b},\alpha}$) Let $0 < \alpha_i < n$, $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p} = \sum\limits_{i=1}^{m} \frac{1}{p_i}$, $\alpha = \sum\limits_{i=1}^{m} \alpha_i$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then there is $C > 0$ independent of $\overrightarrow{f}$ and $\overrightarrow{b}$ such that*

$$\left\| I^{(m)}_{\overrightarrow{b},\alpha}\left(\overrightarrow{f}\right)\right\|_{L_q(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \|b_i\|_* \|f_i\|_{L_{p_i}(\mathbb{R}^n)}.$$

Using the idea in the proof of Lemma 3.2 in [13], we can obtain the following Corollary 1.3:

Corollary 1.3. *(Strong bounds of $M^{(m)}_{\alpha,\overrightarrow{b}}$) Under the assumptions of Lemma 1.1, the operator $M^{(m)}_{\alpha,\overrightarrow{b}}$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \cdots L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Moreover, we have*

$$\left\| M^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{L_q(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \|b_i\|_* \|f_i\|_{L_{p_i}(\mathbb{R}^n)}.$$

Proof. Set

$$\widetilde{I}^{(m)}_{\overrightarrow{b},\alpha}(|f|)(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{1}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} \left[\prod_{i=1}^{m} [|b_i(x) - b_i(y_i)|]\, |f_i(y_i)|\right] d\overrightarrow{y} \qquad 0 < \alpha < mn.$$

It is easy to see that Lemma 1.1 holds for $\widetilde{I}^{(m)}_{\overrightarrow{b},\alpha}$. On the other hand, for any $t>0$, we have

$$\widetilde{I}^{(m)}_{\overrightarrow{b},\alpha}(|f|)(x) \geq \int\limits_{(B(x,t))^m} \frac{1}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} \left[\prod_{i=1}^{m}[|b_i(x)-b_i(y_i)|]\,|f_i(y_i)|\right] d\overrightarrow{y}$$
$$\geq \frac{1}{t^{mn-\alpha}} \int\limits_{B(x,t)} \left[\prod_{i=1}^{m}[|b_i(x)-b_i(y_i)|]\,|f_i(y_i)|\right] d\overrightarrow{y}.$$

Taking supremum over $t>0$ in the above inequality, we get

$$M^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)(x) \leq C_{n,\alpha}^{-1}\widetilde{I}^{(m)}_{\overrightarrow{b},\alpha}(|f|)(x) \qquad C_{n,\alpha}=|B(0,1)|^{\frac{mn-\alpha}{n}}. \tag{1.10}$$

□

As a simple corollary of Lemma 1.1 and Corollary 1.3, we can obtain the following result.

Corollary 1.4. *(Strong bounds of $T^{(m)}_{\overrightarrow{b}}$ and $M^{(m)}_{\overrightarrow{b}}$) Let $1<p_1,\dots,p_m<\infty$ and $0<p<\infty$ with $\frac{1}{p}=\sum\limits_{i=1}^{m}\frac{1}{p_i}$. Then there is $C>0$ independent of $\overrightarrow{f}$ and $\overrightarrow{b}$ such that*

$$\left\|T^{(m)}_{\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{L_p(\mathbb{R}^n)} \leq C\prod_{i=1}^{m}\|b_i\|_*\,\|f_i\|_{L_{p_i}(\mathbb{R}^n)},$$

$$\left\|M^{(m)}_{\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{L_p(\mathbb{R}^n)} \leq C\prod_{i=1}^{m}\|b_i\|_*\,\|f_i\|_{L_{p_i}(\mathbb{R}^n)}.$$

The purpose of this paper is to consider the mapping properties on $M_{p_1,\varphi_1}\times\cdots\times M_{p_m,\varphi_m}$ and $VM_{p_1,\varphi_1}\times\cdots\times VM_{p_m,\varphi_m}$ for the commutators of multilinear fractional maximal and integral operators, respectively. Similar results still hold for commutators of multilinear maximal and singular integral operators. Commutators of multilinear fractional maximal and integral operators on product generalized Morrey spaces have not also been studied so far and this paper seems to be the first in this direction. Now, let us state the main results of this paper.

Theorem 1.3. *Let $0<\alpha<mn$ and $1\leq p_i<\frac{mn}{\alpha}$ with $\frac{1}{p}=\sum\limits_{i=1}^{m}\frac{1}{p_i}$, $\frac{1}{q}=\sum\limits_{i=1}^{m}\frac{1}{p_i}+\sum\limits_{i=1}^{m}\frac{1}{q_i}-\frac{\alpha}{n}$ and $\overrightarrow{b}\in(BMO)^m(\mathbb{R}^n)$ for $i=1,\dots,m$. Let functions $\varphi,\varphi_i:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ $(i=1,\dots,m)$ and $(\varphi_1,\dots,\varphi_m,\varphi)$ satisfies the condition*

$$\int\limits_r^{\infty}\left(1+\ln\frac{t}{r}\right)^m \frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\prod\limits_{i=1}^{m}\varphi_i(x,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\frac{1}{q}-\sum\limits_{i=1}^{m}\frac{1}{q_i}\right)+1}}dt \leq C\varphi(x,r), \tag{1.11}$$

where C does not depend on $x\in\mathbb{R}^n$ and $r>0$.

Then, $I^{(m)}_{\alpha,\overrightarrow{b}}$ *and* $M^{(m)}_{\alpha,\overrightarrow{b}}$ $(m\in\mathbb{N})$ *are bounded operators from product space* $M_{p_1,\varphi_1}\times\cdots\times M_{p_m,\varphi_m}$ *to* $M_{q,\varphi}$. *Moreover, we have*

$$\left\|I^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{M_{q,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{M_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{M_{p_i,\varphi_i}}, \tag{1.12}$$

$$\left\|M^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{M_{q,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{M_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{M_{p_i,\varphi_i}}. \tag{1.13}$$

Corollary 1.5. *Let* $1<p_i<\infty$ *and* $0<p<\infty$ *with* $\frac{1}{p}=\sum\limits_{i=1}^{m}\frac{1}{p_i}$ *and* $\overrightarrow{b}\in(BMO)^m(\mathbb{R}^n)$ *for* $i=1,\ldots,m$. *Let functions* $\varphi,\varphi_i:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ $(i=1,\ldots,m)$ *and* $(\varphi_1,\ldots,\varphi_m,\varphi)$ *satisfies the condition*

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)^m\frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\prod\limits_{i=1}^{m}\varphi_i(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt\le C\varphi(x,r),$$

where C *does not depend on* $x\in\mathbb{R}^n$ *and* $r>0$.

Then, $T^{(m)}_{\overrightarrow{b}}$ *and* $M^{(m)}_{\overrightarrow{b}}$ $(m\in\mathbb{N})$ *are bounded operators from product space* $M_{p_1,\varphi_1}\times\cdots\times M_{p_m,\varphi_m}$ *to* $M_{p,\varphi}$. *Moreover, we have*

$$\left\|T^{(m)}_{\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{M_{p,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{M_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{M_{p_i,\varphi_i}},$$

$$\left\|M^{(m)}_{\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{M_{p,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{M_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{M_{p_i,\varphi_i}}.$$

Our another main result is the following.

Theorem 1.4. *Let* $0<\alpha<mn$ *and* $1\le p_i<\frac{mn}{\alpha}$ *with* $\frac{1}{p}=\sum\limits_{i=1}^{m}\frac{1}{p_i}$,
$\frac{1}{q}=\sum\limits_{i=1}^{m}\frac{1}{p_i}+\sum\limits_{i=1}^{m}\frac{1}{q_i}-\frac{\alpha}{n}$ *and* $\overrightarrow{b}\in(BMO)^m(\mathbb{R}^n)$ *for* $i=1,\ldots,m$. *Let functions* $\varphi,\varphi_i:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ $(i=1,\ldots,m)$ *and* $(\varphi_1,\ldots,\varphi_m,\varphi)$ *satisfies conditions (1.2)-(1.3) and*

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)^m\prod_{i=1}^{m}\varphi_i(x,t)\frac{t^{\frac{n}{p}}}{t^{n\left(\frac{1}{q}-\sum\limits_{i=1}^{m}\frac{1}{q_i}\right)+1}}dt\le C_0\varphi(x,r), \tag{1.14}$$

where C_0 *does not depend on* $x\in\mathbb{R}^n$ *and* $r>0$,

$$\lim_{r\to0}\frac{\ln\frac{1}{r}}{\inf\limits_{x\in\mathbb{R}^n}\varphi(x,r)}=0 \tag{1.15}$$

and

$$c_\delta:=\int_\delta^\infty(1+\ln|t|)^m\sup_{x\in\mathbb{R}^n}\prod_{i=1}^{m}\varphi_i(x,t)\frac{t^{\frac{n}{p}}}{t^{n\left(\frac{1}{q}-\sum\limits_{i=1}^{m}\frac{1}{q_i}\right)+1}}dt<\infty \tag{1.16}$$

for every $\delta>0$.

Then, $I^{(m)}_{\alpha,\overrightarrow{b}}$ *and* $M^{(m)}_{\alpha,\overrightarrow{b}}$ $(m\in\mathbb{N})$ *are bounded operators from product space* $VM_{p_1,\varphi_1}\times\cdots\times VM_{p_m,\varphi_m}$ *to* $VM_{q,\varphi}$*. Moreover, we have*

$$\left\|I^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{VM_{q,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{VM_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{VM_{p_i,\varphi_i}}, \tag{1.17}$$

$$\left\|M^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{VM_{q,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{VM_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{VM_{p_i,\varphi_i}}. \tag{1.18}$$

Corollary 1.6. *Let* $1<p_i<\infty$ *and* $0<p<\infty$ *with* $\frac{1}{p}=\sum\limits_{i=1}^{m}\frac{1}{p_i}$ *and* $\overrightarrow{b}\in(BMO)^m(\mathbb{R}^n)$ *for* $i=1,\ldots,m$*. Let functions* $\varphi,\varphi_i:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ $(i=1,\ldots,m)$ *and* $(\varphi_1,\ldots,\varphi_m,\varphi)$ *satisfies conditions (1.2)-(1.3) and*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)^m\prod_{i=1}^{m}\varphi_i(x,t)\frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt\le C_0\varphi(x,r),$$

where C_0 *does not depend on* $x\in\mathbb{R}^n$ *and* $r>0$*,*

$$\lim_{r\to0}\frac{\ln\frac{1}{r}}{\inf\limits_{x\in\mathbb{R}^n}\varphi(x,r)}=0$$

and

$$c_\delta:=\int_\delta^{\infty}(1+\ln|t|)^m\sup_{x\in\mathbb{R}^n}\prod_{i=1}^{m}\varphi_i(x,t)\frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt<\infty$$

for every $\delta>0$*.*

Then, $T^{(m)}_{\overrightarrow{b}}$ *and* $M^{(m)}_{\overrightarrow{b}}$ $(m\in\mathbb{N})$ *are bounded operators from product space* $VM_{p_1,\varphi_1}\times\cdots\times VM_{p_m,\varphi_m}$ *to* $VM_{p,\varphi}$*. Moreover, we have*

$$\left\|T^{(m)}_{\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{VM_{p,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{VM_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{VM_{p_i,\varphi_i}},$$

$$\left\|M^{(m)}_{\overrightarrow{b}}\left(\overrightarrow{f}\right)\right\|_{VM_{p,\varphi}}\lesssim\left\|\overrightarrow{b}\right\|_{\mathfrak{BMO}}\|f_i\|_{VM_{p_i,\varphi_i}}\lesssim\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{VM_{p_i,\varphi_i}}.$$

The article is organized as follows. A key lemma is given and proved in Section 2. Section 3 will be devoted to the proofs of the theorems (Theorems 1.3 and 1.4) stated above.

2. A Key Lemma

In order to prove the main results (Theorems 1.3 and 1.4), we need the following lemma.

Lemma 2.1. *Let* $x_0\in\mathbb{R}^n$*,* $0<\alpha<mn$ *and* $1\le p_i<\frac{mn}{\alpha}$ *with* $\frac{1}{p}=\sum\limits_{i=1}^{m}\frac{1}{p_i}$*,* $\frac{1}{q}=\sum\limits_{i=1}^{m}\frac{1}{p_i}+\sum\limits_{i=1}^{m}\frac{1}{q_i}-\frac{\alpha}{n}$ *and* $\overrightarrow{b}\in(BMO)^m(\mathbb{R}^n)$ *for* $i=1,\ldots,m$*. Then the inequality*

$$\|I^{(m)}_{\alpha,\overrightarrow{b}}\left(\overrightarrow{f}\right)\|_{L_q(B(x_0,r))}\lesssim\prod_{i=1}^{m}\|b_i\|_*r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^m\prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\sum\limits_{i=1}^{m}\frac{1}{q_i}\right)+1}} \tag{2.1}$$

holds for any ball $B(x_0,r)$ *and for all* $\overrightarrow{f}\in L_{p_1}^{loc}(\mathbb{R}^n)\times\cdots\times L_{p_m}^{loc}(\mathbb{R}^n)$.

Proof. In order to simplify the proof, we consider only the situation when $m=2$. Actually, a similar procedure works for all $m\in\mathbb{N}$. Thus, without loss of generality, it is sufficient to show that the conclusion holds for $I_{\alpha,\overrightarrow{b}}^{(2)}\left(\overrightarrow{f}\right)=I_{\alpha,(b_1,b_2)}^{(2)}(f_1,f_2)$.

We just consider the case $p_i>1$ for $i=1,2$. For any $x_0\in\mathbb{R}^n$, set $B=B(x_0,r)$ for the ball centered at x_0 and of radius r and $2B=B(x_0,2r)$. Indeed, we also decompose f_i as $f_i(y_i)=f_i(y_i)\chi_{2B}+f_i(y_i)\chi_{(2B)^c}$ for $i=1,2$. And, we write $f_1=f_1^0+f_1^\infty$ and $f_2=f_2^0+f_2^\infty$, where $f_i^0=f_i\chi_{2B}$, $f_i^\infty=f_i\chi_{(2B)^c}$, for $i=1,2$. Thus, we have

$$\begin{aligned}\left\|I_{\alpha,(b_1,b_2)}^{(2)}(f_1,f_2)\right\|_{L_q(B(x_0,r))}&\le\left\|I_{\alpha,(b_1,b_2)}^{(2)}\left(f_1^0,f_2^0\right)\right\|_{L_q(B(x_0,r))}+\left\|I_{\alpha,(b_1,b_2)}^{(2)}\left(f_1^0,f_2^\infty\right)\right\|_{L_q(B(x_0,r))}\\&+\left\|I_{\alpha,(b_1,b_2)}^{(2)}\left(f_1^\infty,f_2^0\right)\right\|_{L_q(B(x_0,r))}+\left\|I_{\alpha,(b_1,b_2)}^{(2)}\left(f_1^\infty,f_2^\infty\right)\right\|_{L_q(B(x_0,r))}\\&=F_1+F_2+F_3+F_4.\end{aligned}$$

Firstly, we use the boundedness of $I_{\alpha,(b_1,b_2)}^{(2)}$ from $L_{p_1}\times L_{p_2}$ into L_q (see Lemma 1.1) to estimate F_1, and we obtain

$$\begin{aligned}F_1=\left\|I_{\alpha,(b_1,b_2)}^{(2)}\left(f_1^0,f_2^0\right)\right\|_{L_q(B(x_0,r))}&\lesssim\prod_{i=1}^{2}\|b_i\|_*\|f_i\|_{L_{p_i}(2B)}\\&\lesssim r^{\frac{n}{q}}\prod_{i=1}^{2}\|b_i\|_*\|f_i\|_{L_{p_i}(2B)}\int_{2r}^{\infty}\frac{dt}{t^{\frac{n}{q}+1}}\\&\lesssim\prod_{i=1}^{2}\|b_i\|_*r^{\frac{n}{q}}\int_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{\frac{n}{q}+1}}\\&\lesssim\prod_{i=1}^{2}\|b_i\|_*r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.\end{aligned}$$

Secondly, for $F_2=\left\|I_{\alpha,(b_1,b_2)}^{(2)}\left(f_1^0,f_2^\infty\right)\right\|_{L_q(B(x_0,r))}$, we decompose it into four parts as follows:

$$\begin{aligned}F_2&\lesssim\left\|\left[(b_1-\{b_1\}_B)\right]\left[(b_2-\{b_2\}_B)\right]I_\alpha^{(2)}\left(f_1^0,f_2^\infty\right)\right\|_{L_q(B(x_0,r))}\\&+\left\|\left[(b_1-\{b_1\}_B)\right]I_\alpha^{(2)}\left[f_1^0,(b_2-\{b_2\}_B)f_2^\infty\right]\right\|_{L_q(B(x_0,r))}\\&+\left\|\left[(b_2-\{b_2\}_B)\right]I_\alpha^{(2)}\left[(b_1-\{b_1\}_B)f_1^0,f_2^\infty\right]\right\|_{L_q(B(x_0,r))}\\&+\left\|I_\alpha^{(2)}\left[(b_1-\{b_1\}_B)f_1^0,(b_2-\{b_2\}_B)f_2^\infty\right]\right\|_{L_q(B(x_0,r))}\\&\equiv F_{21}+F_{22}+F_{23}+F_{24}.\end{aligned}$$

Let $1 < p_1, p_2 < \frac{2n}{\alpha}$, such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{\overline{q}} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{\overline{r}} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\frac{1}{q} = \frac{1}{\overline{r}} + \frac{1}{\overline{q}}$. Then, using Hölder's inequality and by (1.8) we have

$$
\begin{aligned}
F_{21} &= \left\| \left(b_1 - (b_1)_B\right)\left(b_2 - (b_2)_B\right) I_\alpha^{(2)}\left(f_1^0, f_2^\infty\right) \right\|_{L_q(B(x_0,r))} \\
&\lesssim \left\| \left(b_1 - (b_1)_B\right)\left(b_2 - (b_2)_B\right) \right\|_{L_{\overline{r}}(B(x_0,r))} \left\| I_\alpha^{(2)}\left(f_1^0, f_2^\infty\right) \right\|_{L_{\overline{q}}(B(x_0,r))} \\
&\lesssim \left\| b_1 - (b_1)_B \right\|_{L_{q_1}(B(x_0,r))} \left\| b_2 - (b_2)_B \right\|_{L_{q_2}(B(x_0,r))} \\
&\times r^{\frac{n}{\overline{q}}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{\overline{q}}+1}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_* |B(x_0,r)|^{\frac{1}{q_1}+\frac{1}{q_2}} r^{\frac{n}{\overline{q}}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{\overline{q}}+1}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_* r^{n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)} r^{n\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}\right)} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} t^{n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)} \frac{dt}{t^{\frac{n}{q}+1}} \\
&= \prod_{i=1}^{2} \|b_i\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}},
\end{aligned}
$$

where in the second inequality we have used the following fact:

It is clear that $|(x_0 - y_1,\ x_0 - y_2)|^{2n-\alpha} \geq |x_0 - y_2|^{2n-\alpha}$. By Hölder's inequality, we have

$$
\begin{aligned}
\left| I_\alpha^{(2)}\left(f_1^0, f_2^\infty\right)(x) \right| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f_1^0(y_1)\right| \left|f_2^\infty(y_2)\right|}{|(x-y_1, x-y_2)|^{2n-\alpha}} dy_1 dy_2 \\
&\lesssim \int_{2B} |f_1(y_1)|\, dy_1 \int_{(2B)^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n-\alpha}} dy_2 \\
&\approx \int_{2B} |f_1(y_1)|\, dy_1 \int_{(2B)^c} |f_2(y_2)| \int_{|x_0-y_2|}^{\infty} \frac{dt}{t^{2n-\alpha+1}} dy_2 \\
&\lesssim \|f_1\|_{L_{p_1}(2B)} |2B|^{1-\frac{1}{p_1}} \int_{2r}^{\infty} \|f_2\|_{L_{p_2}(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
&\lesssim \int_{2r}^{\infty} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{\overline{q}}+1}},
\end{aligned}
$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Thus, the inequality

$$
\left\| I_\alpha^{(2)}\left(f_1^0, f_2^\infty\right) \right\|_{L_{\overline{q}}(B(x_0,r))} \lesssim r^{\frac{n}{\overline{q}}} \int_{2r}^{\infty} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{\overline{q}}+1}}
$$

is valid.

On the other hand, for the estimates used in F_{22}, F_{23}, we have to prove the below inequality:

$$\left|I_\alpha^{(2)}\left[f_1^0,(b_2(\cdot)-(b_2)_B)f_2^\infty\right](x)\right| \lesssim \|b_2\|_* \int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+1-\alpha}}. \tag{2.2}$$

To estimate F_{22}, the following inequality

$$\left|I_\alpha^{(2)}\left[f_1^0,(b_2(\cdot)-(b_2)_B)f_2^\infty\right](x)\right| \lesssim \int_{2B}|f_1(y_1)|\,dy_1\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B|\,|f_2(y_2)|}{|x_0-y_2|^{2n-\alpha}}dy_2$$

is satisfied. It's obvious that

$$\int_{2B}|f_1(y_1)|\,dy_1 \lesssim \|f_1\|_{L_{p_1}(2B)}|2B|^{1-\frac{1}{p_1}}, \tag{2.3}$$

and using Hölder's inequality and by (1.6) and (1.7) we have

$$\begin{aligned}
&\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B|\,|f_2(y_2)|}{|x_0-y_2|^{2n-\alpha}}dy_2\\
&\lesssim \int_{(2B)^c}\left|b_2(y_2)-(b_2)_{B(x_0,r)}\right||f_2(y_2)|\left[\int_{|x_0-y_2|}^{\infty}\frac{dt}{t^{2n-\alpha+1}}\right]dy_2\\
&\lesssim \int_{2r}^{\infty}\left\|b_2(y_2)-(b_2)_{B(x_0,t)}\right\|_{L_{q_2}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\left(\frac{1}{p_2}+\frac{1}{q_2}\right)}\frac{dt}{t^{2n-\alpha+1}}\\
&+\int_{2r}^{\infty}\left|(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}\right|\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n-\alpha+1}}\\
&\lesssim \|b_2\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2|B(x_0,t)|^{\frac{1}{q_2}}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\left(\frac{1}{p_2}+\frac{1}{q_2}\right)}\frac{dt}{t^{2n-\alpha+1}}\\
&+\|b_2\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)|B(x_0,t)|\,\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n-\alpha+1}}\\
&\lesssim \|b_2\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\|f_2\|_{L_{p_2}(B(x_0,t))}\frac{dt}{t^{n\left(1+\frac{1}{p_2}\right)+1-\alpha}}.
\end{aligned} \tag{2.4}$$

Hence, by (2.3) and (2.4), it follows that:

$$\begin{aligned}
&\left|I_\alpha^{(2)}\left[f_1^0,(b_2(\cdot)-(b_2)_B)f_2^\infty\right](x)\right|\\
&\lesssim \|b_2\|_*\|f_1\|_{L_{p_1}(2B)}|2B|^{1-\frac{1}{p_1}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\|f_2\|_{L_{p_2}(B(x_0,t))}\frac{dt}{t^{n\left(1+\frac{1}{p_2}\right)+1-\alpha}}\\
&\lesssim \|b_2\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+1-\alpha}}.
\end{aligned}$$

This completes the proof of inequality (2.2). Thus, let $1<\tau<\infty$, such that $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{\tau}$. Then, using Hölder's inequality and from (2.2) and (1.7), we get

$$\begin{aligned}F_{22}&=\left\|\left[\left(b_1-\{b_1\}_B\right)\right]I_\alpha^{(2)}\left[f_1^0,\left(b_2-\{b_2\}_B\right)f_2^\infty\right]\right\|_{L_q(B(x_0,r))}\\&\lesssim\left\|b_1-(b_1)_B\right\|_{L_{q_1}(B)}\left\|I_\alpha^{(2)}\left[f_1^0,\left(b_2-(b_2)_B\right)f_2^\infty\right]\right\|_{L_\tau(B)}\\&\lesssim\prod_{i=1}^2\|b_i\|_*\left|B\left(x_0,r\right)\right|^{\frac{1}{q_1}+\frac{1}{\tau}}\\&\times\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^2\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+1-\alpha}}\\&\lesssim\prod_{i=1}^2\|b_i\|_*r^{\frac{n}{q}}\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^2\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.\end{aligned}$$

Similarly, F_{23} has the same estimate above, here we omit the details, thus the inequality

$$\begin{aligned}F_{23}&=\left\|\left[\left(b_2-\{b_2\}_B\right)\right]I_\alpha^{(2)}\left[\left(b_1-\{b_1\}_B\right)f_1^0,f_2^\infty\right]\right\|_{L_q(B(x_0,r))}\\&\lesssim\prod_{i=1}^2\|b_i\|_*r^{\frac{n}{q}}\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^2\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}\end{aligned}$$

is valid.

Now we turn to estimate F_{24}. Similar to (2.2), we have to prove the following estimate for F_{24}:

$$\left|I_\alpha^{(2)}\left[\left(b_1-(b_1)_B\right)f_1^0,\left(b_2-(b_2)_B\right)f_2^\infty\right](x)\right|\leq\prod_{i=1}^2\|b_i\|_*\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^2\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.\tag{2.5}$$

Firstly, the following inequality

$$\left|I_\alpha^{(2)}\left[\left(b_1-(b_1)_B\right)f_1^0,\left(b_2-(b_2)_B\right)f_2^\infty\right](x)\right|\lesssim\int_{2B}\left|b_1(y_1)-(b_1)_B\right|\left|f_1(y_1)\right|dy_1\int_{(2B)^c}\frac{\left|b_2(y_2)-(b_2)_B\right|\left|f_2(y_2)\right|}{|x_0-y_2|^{2n-\alpha}}dy_2$$

is valid.

It's obvious that from Hölder's inequality and (1.7)

$$\int_{2B}\left|b_1(y_1)-(b_1)_B\right|\left|f_1(y_1)\right|dy_1\lesssim\|b_1\|_*\left|B\left(x_0,r\right)\right|^{1-\frac{1}{p_1}}\|f_1\|_{L_{p_1}(2B)}.\tag{2.6}$$

Then, by (2.4) and (2.6) we have

$$\left|I_\alpha^{(2)}\left[\left(b_1-(b_1)_B\right)f_1^0,\left(b_2-(b_2)_B\right)f_2^\infty\right](x)\right|\leq\prod_{i=1}^2\|b_i\|_*\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^2\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.$$

This completes the proof of inequality (2.5). Therefore, by (2.5) we deduce that

$$F_{24} = \left\| I_\alpha^{(2)} \left[(b_1 - (b_1)_B) f_1^0, (b_2 - (b_2)_B) f_2^\infty \right] \right\|_{L_q(B)}$$
$$\lesssim \prod_{i=1}^{2} \|b_i\|_* r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q} - \left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right)+1}}.$$

Considering estimates F_{21}, F_{22}, F_{23}, F_{24} together, we get the desired conclusion

$$F_2 = \left\| I_{\alpha,(b_1,b_2)}^{(2)} \left(f_1^0, f_2^\infty\right) \right\|_{L_q(B(x_0,r))} \lesssim \prod_{i=1}^{2} \|b_i\|_* r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q} - \left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right)+1}}.$$

Similar to F_2, we can also get the estimates for F_3,

$$F_3 = \left\| I_{\alpha,(b_1,b_2)}^{(2)} \left(f_1^\infty, f_2^0\right) \right\|_{L_q(B(x_0,r))} \lesssim \prod_{i=1}^{2} \|b_i\|_* r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q} - \left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right)+1}}.$$

At last, we consider the last term $F_4 = \left\| I_{\alpha,(b_1,b_2)}^{(2)} \left(f_1^\infty, f_2^\infty\right) \right\|_{L_q(B(x_0,r))}$. We split F_4 in the following way:

$$F_4 \lesssim F_{41} + F_{42} + F_{43} + F_{44},$$

where

$$F_{41} = \left\| (b_1 - (b_1)_B) (b_2 - (b_2)_B) I_\alpha^{(2)} \left(f_1^\infty, f_2^\infty\right) \right\|_{L_q(B)},$$
$$F_{42} = \left\| (b_1 - (b_1)_B) I_\alpha^{(2)} \left[f_1^\infty, (b_2 - (b_2)_B) f_2^\infty\right] \right\|_{L_q(B)},$$
$$F_{43} = \left\| (b_2 - (b_2)_B) I_\alpha^{(2)} \left[(b_1 - (b_1)_B) f_1^\infty, f_2^\infty\right] \right\|_{L_q(B)},$$
$$F_{44} = \left\| I_\alpha^{(2)} \left[(b_1 - (b_1)_B) f_1^\infty, (b_2 - (b_2)_B) f_2^\infty\right] \right\|_{L_q(B)}.$$

Now, let us estimate F_{41}, F_{42}, F_{43}, F_{44} respectively.

For the term F_{41}, let $1 < \tau < \infty$, such that $\frac{1}{q} = \left(\frac{1}{q_1} + \frac{1}{q_2}\right) + \frac{1}{\tau}$, $\frac{1}{\tau} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$. Then, by Hölder's inequality and (1.8), we get

$$F_{41} = \left\| (b_1 - (b_1)_B) (b_2 - (b_2)_B) I_\alpha^{(2)} \left(f_1^\infty, f_2^\infty\right) \right\|_{L_q(B)}$$
$$\lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B)} \|b_2 - (b_2)_B\|_{L_{q_2}(B)} \left\| I_\alpha^{(2)} \left(f_1^\infty, f_2^\infty\right) \right\|_{L_\tau(B)}$$
$$\lesssim \prod_{i=1}^{2} \|b_i\|_* \left|B(x_0, r)\right|^{\frac{1}{q_1} + \frac{1}{q_2}} r^{\frac{n}{\tau}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{\tau}+1}}$$
$$\lesssim \prod_{i=1}^{2} \|b_i\|_* r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q} - \left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right)+1}},$$

where in the second inequality we have used the following fact:

Noting that $|(x_0-y_1,\ x_0-y_2)|^{2n-\alpha} \geq |x_0-y_1|^{n-\frac{\alpha}{2}}\,|x_0-y_2|^{n-\frac{\alpha}{2}}$ and by Hölder's inequality, we get

$$\begin{aligned}
&\left|I_\alpha^{(2)}\left(f_1^\infty, f_2^\infty\right)(x)\right| \\
&\lesssim \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\left|f_1(y_1)\,\chi_{(2B)^c}\right|\left|f_2(y_2)\,\chi_{(B)^c}\right|}{|(x_0-y_1, x_0-y_2)|^{2n-\alpha}}dy_1dy_2 \\
&\lesssim \int\limits_{(2B)^c}\int\limits_{(2B)^c} \frac{|f_1(y_1)|\,|f_2(y_2)|}{|x_0-y_1|^{n-\frac{\alpha}{2}}\,|x_0-y_2|^{n-\frac{\alpha}{2}}}dy_1dy_2 \\
&\lesssim \sum_{j=1}^{\infty}\prod_{i=1}^{2}\int\limits_{2^{j+1}B\setminus 2^jB} \frac{|f_i(y_i)|}{|x_0-y_i|^{n-\frac{\alpha}{2}}}dy_i \\
&\lesssim \sum_{j=1}^{\infty}\prod_{i=1}^{2}\left(2^jr\right)^{-n+\frac{\alpha}{2}}\int\limits_{2^{j+1}B}|f_i(y_i)|\,dy_i \\
&\lesssim \sum_{j=1}^{\infty}\left(2^jr\right)^{-2n+\alpha}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(2^{j+1}B)}\left|2^{j+1}B\right|^{1-\frac{1}{p_i}} \\
&\lesssim \sum_{j=1}^{\infty}\int\limits_{2^{j+1}r}^{2^{j+2}r}\left(2^{j+1}r\right)^{-2n+\alpha-1}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(2^{j+1}B)}\left|2^{j+1}B\right|^{1-\frac{1}{p_i}}dt \\
&\lesssim \sum_{j=1}^{\infty}\int\limits_{2^{j+1}r}^{2^{j+2}r}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\,|B(x_0,t)|^{1-\frac{1}{p_i}}\,\frac{dt}{t^{2n+1-\alpha}} \\
&\lesssim \int\limits_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\,|B(x_0,t)|^{2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}\,\frac{dt}{t^{2n+1-\alpha}} \\
&\lesssim \int\limits_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\,\frac{dt}{t^{\frac{n}{\tau}+1}}.
\end{aligned}$$

Moreover, for $p_1,\ p_2 \in [1,\infty)$ the inequality

$$\left\|I_\alpha^{(2)}\left(f_1^\infty, f_2^\infty\right)\right\|_{L_\tau(B(x_0,r))} \lesssim r^{\frac{n}{\tau}}\int\limits_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\,\frac{dt}{t^{\frac{n}{\tau}+1}} \tag{2.7}$$

is valid.

For the terms F_{42}, F_{43}, similar to the estimates used for (2.2), we have to prove the following inequality:

$$\left|I_\alpha^{(2)}\left[f_1^\infty, (b_2-(b_2)_B)\,f_2^\infty\right](x)\right| \lesssim \|b_2\|_*\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\,\frac{dt}{t^{n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+1-\alpha}}. \tag{2.8}$$

Noting that $|(x_0-y_1,\ x_0-y_2)|^{2n-\alpha} \geq |x_0-y_1|^{n-\frac{\alpha}{2}}\,|x_0-y_2|^{n-\frac{\alpha}{2}}$, we get

$$\begin{aligned}
&\left|I_\alpha^{(2)}\left[f_1^\infty,\left(b_2-(b_2)_B\right)f_2^\infty\right](x)\right| \\
&\lesssim \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{\left|b_2(y_2)-(b_2)_B\right|\left|f_1(y_1)\chi_{(2B)^c}\right|\left|f_2(y_2)\chi_{(2B)^c}\right|}{|(x_0-y_1,x_0-y_2)|^{2n-\alpha}}dy_1dy_2 \\
&\lesssim \int\limits_{(2B)^c}\int\limits_{(2B)^c}\frac{\left|b_2(y_2)-(b_2)_B\right|\left|f_1(y_1)\right|\left|f_2(y_2)\right|}{|x_0-y_1|^{n-\frac{\alpha}{2}}\,|x_0-y_2|^{n-\frac{\alpha}{2}}}dy_1dy_2 \\
&\lesssim \sum_{j=1}^{\infty}\int\limits_{2^{j+1}B\setminus 2^jB}\frac{|f_1(y_1)|}{|x_0-y_1|^{n-\frac{\alpha}{2}}}dy_1\int\limits_{2^{j+1}B\setminus 2^jB}\frac{\left|b_2(y_2)-(b_2)_B\right|\left|f_2(y_2)\right|}{|x_0-y_2|^{n-\frac{\alpha}{2}}}dy_2 \\
&\lesssim \sum_{j=1}^{\infty}\left(2^jr\right)^{-2n+\alpha}\int\limits_{2^{j+1}B}|f_1(y_1)|\,dy_1\int\limits_{2^{j+1}B}\left|b_2(y_2)-(b_2)_B\right|\left|f_2(y_2)\right|dy_2.
\end{aligned}$$

On the other hand, it's obvious that

$$\int\limits_{2^{j+1}B}|f_1(y_1)|\,dy_1 \leq \|f_1\|_{L_{p_1}(2^{j+1}B)}\left|2^{j+1}B\right|^{1-\frac{1}{p_1}}, \tag{2.9}$$

and using Hölder's inequality and by (1.6) and (1.7)

$$\begin{aligned}
&\int\limits_{2^{j+1}B}\left|b_2(y_2)-(b_2)_B\right|\left|f_2(y_2)\right|dy_2 \\
&\leq \left\|b_2-(b_2)_{2^{j+1}B}\right\|_{L_{q_2}(2^{j+1}B)}\|f_2\|_{L_{p_2}(2^{j+1}B)}\left|2^{j+1}B\right|^{1-\left(\frac{1}{p_2}+\frac{1}{q_2}\right)} \\
&+\left|(b_2)_{2^{j+1}B}-(b_2)_B\right|\|f_2\|_{L_{p_2}(2^{j+1}B)}\left|2^{j+1}B\right|^{1-\frac{1}{p_2}} \\
&\lesssim \|b_2\|_*\left|2^{j+1}B\right|^{\frac{1}{q_2}}\left(1+\ln\frac{2^{j+1}r}{r}\right)\|f_2\|_{L_{p_2}(2^{j+1}B)}\left|2^{j+1}B\right|^{1-\left(\frac{1}{p_2}+\frac{1}{q_2}\right)} \\
&+\|b_2\|_*\left(1+\ln\frac{2^{j+1}r}{r}\right)\left|2^{j+1}B\right|\|f_2\|_{L_{p_2}(2^{j+1}B)}\left|2^{j+1}B\right|^{1-\frac{1}{p_2}} \\
&\lesssim \|b_2\|_*\left(1+\ln\frac{2^{j+1}r}{r}\right)^2\left|2^{j+1}B\right|^{1-\frac{1}{p_2}}\|f_2\|_{L_{p_2}(2^{j+1}B)}.
\end{aligned} \tag{2.10}$$

Hence, by (2.9) and (2.10), it follows that:

$$\begin{aligned}
&\left|I_\alpha^{(2)}\left[f_1^\infty,\left(b_2-(b_2)_B\right)f_2^\infty\right](x)\right| \\
&\lesssim \sum_{j=1}^{\infty}\left(2^jr\right)^{-2n+\alpha}\int\limits_{2^{j+1}B}|f_1(y_1)|\,dy_1\int\limits_{2^{j+1}B}\left|b_2(y_2)-(b_2)_B\right|\left|f_2(y_2)\right|dy_2 \\
&\lesssim \|b_2\|_*\sum_{j=1}^{\infty}\left(2^jr\right)^{-2n+\alpha}\left(1+\ln\frac{2^{j+1}r}{r}\right)^2\left|2^{j+1}B\right|^{2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(2^{j+1}B)}
\end{aligned}$$

$$\lesssim \|b_2\|_* \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} \left(2^{j+1}r\right)^{-2n+\alpha-1} \left(1+\ln\frac{2^{j+1}r}{r}\right)^2 \left|2^{j+1}B\right|^{2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(2^{j+1}B)}\, dt$$

$$\lesssim \|b_2\|_* \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} \left(1+\ln\frac{2^{j+1}r}{r}\right)^2 \left|2^{j+1}B\right|^{2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(2^{j+1}B)} \frac{dt}{t^{2n-\alpha+1}}$$

$$\lesssim \|b_2\|_* \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \left|B\left(x_0,t\right)\right|^{2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{2n-\alpha+1}}$$

$$\lesssim \|b_2\|_* \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+1-\alpha}}.$$

This completes the proof of (2.8).

Now we turn to estimate F_{42}. Let $1<\tau<\infty$, such that $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{\tau}$. Then, by Hölder's inequality, (1.7) and (2.8), we obtain

$$F_{42} = \left\|\left(b_1-(b_1)_B\right) I_\alpha^{(2)}\left[f_1^\infty, \left(b_2-(b_2)_B\right) f_2^\infty\right]\right\|_{L_q(B)}$$

$$\lesssim \left\|\left(b_1-(b_1)_B\right)\right\|_{L_{q_1}(B)} \left\|I_\alpha^{(2)}\left[f_1^\infty, \left(b_2-(b_2)_B\right) f_2^\infty\right]\right\|_{L_{\tau(B)}}$$

$$\lesssim \prod_{i=1}^{2}\|b_i\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.$$

Similarly, F_{43} has the same estimate above, here we omit the details, thus the inequality

$$F_{43} = \left\|\left(b_2-(b_2)_B\right) I_\alpha^{(2)}\left[\left(b_1-(b_1)_B\right) f_1^\infty, f_2^\infty\right]\right\|_{L_q(B)}$$

$$\lesssim \prod_{i=1}^{2}\|b_i\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}$$

is valid.

Finally, to estimate F_{44}, similar to the estimate of (2.8), we have

$$\left|I_\alpha^{(2)}\left[\left(b_1-(b_2)_B\right) f_1^\infty, \left(b_2-(b_2)_B\right) f_2^\infty\right](x)\right|$$

$$\lesssim \sum_{j=1}^{\infty} \left(2^j r\right)^{-2n+\alpha} \int_{2^{j+1}B} \left|b_1(y_1)-(b_1)_B\right| \left|f_1(y_1)\right| dy_1 \int_{2^{j+1}B} \left|b_2(y_2)-(b_2)_B\right| \left|f_2(y_2)\right| dy_2$$

$$\lesssim \prod_{i=1}^{2}\|b_i\|_* \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.$$

Thus, we have

$$F_{44} = \left\| I_{\alpha}^{(2)} \left[\left(b_1 - (b_1)_B \right) f_1^{\infty}, \left(b_2 - (b_2)_B \right) f_2^{\infty} \right] \right\|_{L_q(B)}$$
$$\lesssim \prod_{i=1}^{2} \|b_i\|_* r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.$$

By the estimates of F_{4j} above, where $j = 1{,}2{,}3{,}4$, we know that

$$F_4 = \left\| I_{\alpha,(b_1,b_2)}^{(2)} \left(f_1^{\infty}, f_2^{\infty} \right) \right\|_{L_q(B(x_0,r))} \lesssim \prod_{i=1}^{2} \|b_i\|_* r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right)+1}}.$$

Consequently, combining all the estimates for F_1, F_2, F_3, F_4, we complete the proof of Lemma 2.1. □

3. Proofs of the main results

Now we are ready to return to the proofs of Theorems 1.3 and 1.4.

3.1. **Proof of Theorem 1.3.**

Proof. To prove Theorem 1.3, we will use the following relationship between essential supremum and essential infimum

$$\left(\operatorname*{essinf}_{x\in E} f(x) \right)^{-1} = \operatorname*{esssup}_{x\in E} \frac{1}{f(x)}, \tag{3.1}$$

where f is any real-valued nonnegative function and measurable on E (see [30], page 143). Indeed, we consider (1.12) firstly.

Since $\overrightarrow{f} \in M_{p_1,\varphi_1} \times \cdots \times M_{p_m,\varphi_m}$, by (3.1) and the non-decreasing, with respect to t, of the norm $\prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))}$, we get

$$\frac{\prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))}}{\operatorname*{essinf}_{0<t<\tau<\infty} \prod_{i=1}^{m} \varphi_i(x,\tau)\tau^{\frac{n}{p}}} \leq \operatorname*{esssup}_{0<t<\tau<\infty} \frac{\prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))}}{\prod_{i=1}^{m} \varphi_i(x,\tau)\tau^{\frac{n}{p}}}$$
$$\leq \operatorname*{esssup}_{0<\tau<\infty, x\in\mathbb{R}^n} \frac{\prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))}}{\prod_{i=1}^{m} \varphi_i(x,\tau)\tau^{\frac{n}{p}}} \leq \prod_{i=1}^{m} \|f_i\|_{M_{p_i,\varphi_i}}. \tag{3.2}$$

For $1 < p_1, \ldots, p_m < \infty$, since $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies (1.11) and by (3.2), we have

$$\begin{aligned}
&\int_r^{\infty} \left(1+\ln\frac{t}{r}\right)^m \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\sum_{i=1}^m \frac{1}{q_i}\right)+1}} \\
&\le \int_r^{\infty} \left(1+\ln\frac{t}{r}\right)^m \frac{\prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))}}{\operatorname*{essinf}_{t<\tau<\infty} \prod_{i=1}^m \varphi_i(x,\tau)\tau^{\frac{n}{p}}} \frac{\operatorname*{essinf}_{t<\tau<\infty} \prod_{i=1}^m \varphi_i(x,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\frac{1}{q}-\sum_{i=1}^m \frac{1}{q_i}\right)+1}} dt \\
&\le C \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_i}} \int_r^{\infty} \left(1+\ln\frac{t}{r}\right)^m \frac{\operatorname*{essinf}_{t<\tau<\infty} \prod_{i=1}^m \varphi_i(x,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\frac{1}{q}-\sum_{i=1}^m \frac{1}{q_i}\right)+1}} dt \\
&\le C \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_i}} \varphi(x,r).
\end{aligned} \tag{3.3}$$

Then by (2.1) and (3.3), we get

$$\begin{aligned}
\left\| I_{\alpha,\vec{b}}^{(m)}\left(\vec{f}\right)\right\|_{M_{q,\varphi}} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{q}} \left\| I_{\alpha,\vec{b}}^{(m)}\left(\vec{f}\right)\right\|_{L_q(B(x,r))} \\
&\lesssim \prod_{i=1}^m \|b_i\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi(x_0,r)^{-1} \int_r^{\infty} \left(1+\ln\frac{t}{r}\right)^m \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{q}-\sum_{i=1}^m \frac{1}{q_i}\right)+1}} \\
&\lesssim \prod_{i=1}^m \|b_i\|_* \|f_i\|_{M_{p_i,\varphi_i}}.
\end{aligned}$$

Thus we obtain (1.12).

The conclusion of (1.13) is a direct consequence of (1.10) and (1.12). Indeed, from the process proving (1.12), it is easy to see that the conclusions of (1.12) also hold for $\widetilde{I}_{\vec{b},\alpha}^{(m)}$. Combining this with (1.10), we can immediately obtain (1.13), which completes the proof. □

3.2. **Proof of Theorem 1.4.**

Proof. Since the inequalities (1.17) and (1.18) hold by Theorem 1.3, we only have to prove the implication

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{r^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,r))}}{\prod_{i=1}^m \varphi_i(x,r)} = 0 \text{ implies } \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{r^{-\frac{n}{q}} \left\| I_{\alpha,\vec{b}}^{(m)}\left(\vec{f}\right)\right\|_{L_q(B(x,r))}}{\varphi(x,r)} = 0. \tag{3.4}$$

To show that

$$\sup_{x\in\mathbb{R}^n} \frac{r^{-\frac{n}{q}} \left\| I_{\alpha,\vec{b}}^{(m)}\left(\vec{f}\right)\right\|_{L_q(B(x,r))}}{\varphi(x,r)} < \varepsilon \text{ for small } r,$$

we use the estimate (2.1):

$$\sup_{x\in\mathbb{R}^n}\frac{r^{-\frac{n}{q}}\left\|I^{(m)}_{\alpha,\vec{b}}\left(\vec{f}\right)\right\|_{L_q(B(x,r))}}{\varphi(x,r)} \lesssim \sup_{x\in\mathbb{R}^n}\frac{\prod\limits_{i=1}^{m}\|b_i\|_*}{\varphi(x,r)}\int\limits_r^\infty\left(1+\ln\frac{t}{r}\right)^m\prod_{i=1}^m\|f_i\|_{L_{p_i}(B(x,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\sum\limits_{i=1}^m\frac{1}{q_i}\right)+1}}.$$

We take $r<\delta_0$, where δ_0 is small enough and split the integration:

$$\frac{r^{-\frac{n}{q}}\left\|I^{(m)}_{\alpha,\vec{b}}\left(\vec{f}\right)\right\|_{L_q(B(x,r))}}{\varphi(x,r)} \le C\left[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)\right], \tag{3.5}$$

where $\delta_0>0$ (we may take $\delta_0<1$), and

$$I_{\delta_0}(x,r):=\frac{\prod\limits_{i=1}^{m}\|b_i\|_*}{\varphi(x,r)}\int\limits_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)^m\prod_{i=1}^m\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\sum\limits_{i=1}^m\frac{1}{q_i}\right)+1}},$$

and

$$J_{\delta_0}(x,r):=\frac{\prod\limits_{i=1}^{m}\|b_i\|_*}{\varphi(x,r)}\int\limits_{\delta_0}^{\infty}\left(1+\ln\frac{t}{r}\right)^m\prod_{i=1}^m\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{q}-\sum\limits_{i=1}^m\frac{1}{q_i}\right)+1}},$$

and $r<\delta_0$. Now we can choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\frac{t^{-\frac{n}{p}}\prod\limits_{i=1}^m\|f_i\|_{L_{p_i}(B(x,t))}}{\prod\limits_{i=1}^m\varphi_i(x,t)}<\frac{\varepsilon}{2CC_0},\qquad t\le\delta_0,$$

where C and C_0 are constants from (1.14) and (3.5), which is possible since $\vec{f}\in VM_{p_1,\varphi_1}\times\cdots\times VM_{p_m,\varphi_m}$. This allows to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$\prod_{i=1}^m\|b_i\|_*\sup_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\qquad 0<r<\delta_0$$

by (1.14).

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, by the choice of r sufficiently small because of the conditions (1.15) we obtain

$$J_{\delta_0}(x,r)\le\frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ln\frac{1}{r}}{\varphi(x,r)}\prod_{i=1}^m\|b_i\|_*\,\|f_i\|_{VM_{p_i,\varphi_i}},$$

where c_{δ_0} is the constant from (1.16) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (1.15) we can choose r small enough such that

$$\sup_{x\in\mathbb{R}^n}J_{\delta_0}(x,r)<\frac{\varepsilon}{2},$$

which completes the proof of (3.4).

For $M^{(m)}_{\alpha,\vec{b}}$, we can also use the same method to obtain the desired result, but we omit the details. Therefore, the proof of Theorem 1.4 is completed. □

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