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FIXED POINTS OF NON-SMOOTH FUNCTIONS ON FINITE DIMENSIONAL ORDERED BANACH SPACES VIA CLARKE GENERALIZED JACOBIAN

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ABSTRACT. Considering Lipschitz functions which are not necessarily Fréchet differentiable, we obtain a non-smooth version of Lakshmikantham's theorem in finite dimensional ordered Banach spaces . We also present an application of the obtained result in dynamical Coulomb friction problem.

1. INTRODUCTION

Ordered Banach spaces are very significant class of vector spaces which are studied widely in theory and applications of mathematics. This class of vector spaces is considered in nonlinear integral equations [2], nonlinear boundary value problems [4], optimal control theory [8], operator equations [20] and etc. On the other hand, an important theory in mathematical analysis is fixed point theory. This theory and its applications in orederd Banach spaces have been considered by many researchers. (see [2,3,5,6,11,14,16,17] and the references therein.)

Recently, Lakshmikantham et al. [14] have proved some fixed point theorems in ordered Banach space X for a Fréchet differentiable mapping $T : X \to X$. They showed the applications of their results in ODE initial value problems and semilinear parabolic initial boundary value problems. Vijesh and Kumar [19] and Mouhadjer and Benahmed [16] obtained some generalizations of Lakshmikantham's fixed point theorems.

To obtain fixed points in a class of nonlinear operators in ordered Banach spaces, Mouhadjer and Benahmed introduced a monotone Newton-like method in [15], by using Lakshmikantham's fixed point theorems.

In this paper, we study these fixed point theorems for Lipschitz mappings on finite Bancah spaces which are not necessary Fréchet differentiable. Our main tool is Clarke generalized Jacobian which was firstly introduced by Clarke in [9] for a mapping between two finite dimensional vector spaces. Clarke generalized Jacobian has heavy calculus rules and this paper, to the best of our knowledge, is the first work which deal to non-smooth fixed point theorem using generalized gradient. Since every finite dimensional vector space is isomorphic and homeomorphic to $\mathbb{R}^n$ for some n, we only focus on mappings $F : \mathbb{R}^n \to \mathbb{R}^n$ which are not necessary differentiable and we prove some fixed point theorems for Lipschitzian ones. Finally, to show one of the applications of the obtained results, we consider a non-smooth conic complementary problem and then we investigate the relation between obtained fixed points and the solutions of the problem. This conic complementary problem is arised in the linear discrete Coulomb friction problem which studied by Acary et al. [1].

The paper is organized as follows. In Section 2 preliminaries are given. In Section 3 we introduce some new definitions related to Clarke generalized Jacobian and then we investigate main results. Section 4 is devoted to an application of the obtained results.

2. Preliminaries

Let S be a set. Denote the convex hull of S (the set of all finite convex combinations of members of S) by $co(S)$. Let $C \subseteq \mathbb{R}^n$ be nonempty. Then C is a closed pointed convex cone, when

i) C is a closed convex set,

ii) For every $x \in C$ and every scalar $\lambda \geq 0$, $\lambda x \in C$,

iii) $C \cap -C = \{0\}$. We have $C + C \subseteq C$ for every convex cone C. Using a closed pointed convex cone $C \subseteq \mathbb{R}^n$, we can define the following order relation on $\mathbb{R}^n$:

$$x \leq y \Longleftrightarrow y - x \in C.$$

This relation is reflexive, antisymmetric and transitive. If $x \leq y$, then

i) $x + z \leq y + z$, for each $z \in \mathbb{R}^n$,

ii) $\alpha x \leq \alpha y$, for all scalar $\alpha \geq 0$.

For $\bar{x}, \bar{y} \in \mathbb{R}^n$ such that $\bar{x} \leq \bar{y}$, the order interval $[\bar{x}, \bar{y}]$ is defined as

$$[\bar{x}, \bar{y}] := \{z \in \mathbb{R}^n : \bar{x} \leq z \leq \bar{y}\}.$$

A cone, especially used for many applications, is standard cone in finite dimensional Euclidean spaces:

$$\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_i \geq 0, i = 1 \ldots, n\}.$$

Definition 2.1. *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex cone.*

i) C is called normal if there exists a positive constant δ such that for every $x, y \in \mathbb{R}^n$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq \delta \|y\|,$$

ii) C is called regular if every increasing (decreasing) sequence $\{x_n\}_{n \in \mathbb{N}}$ which is bounded from above (below) converges.

Since every finite-dimensional normed space is reflexive, then for closed pointed convex cone $C \subseteq \mathbb{R}^n$, normality and regularity are equivalent [12].

Definition 2.2. *The mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is strictly differentiable at x, if there is an $m \times n$-matrix M such that*

$$\lim_{y \to x, u \to 0} \frac{F(y + u) - F(y) - M(u)}{\|u\|} = 0.$$

In this case, M is called strict derivative of F at x.

Definition 2.3. *The mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is Fréchet differentiable at x, if there is an $m \times n$-matrix M such that*

$$\lim_{u \to 0} \frac{F(x + u) - F(x) - M(u)}{\|u\|} = 0.$$

In this case, M is called Fréchet derivative of F at x.

Note that, strictly differentiable functions are Fréchet differentiable while the converse is not true in general [13].

Let the vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz at a given point x, that is, there exists a neighborhood U of x and a positive k such that

$$\|F(x_1) - F(x_2)\| \leq k\|x_1 - x_2\|, \ \forall x_1, x_2 \in U.$$

By Rademacher's theorem [9], F is differentiable almost everywhere (in the sense of Lebesgue measure) on U.

Definition 2.4. *[9, 13] Let the vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz at a given point x. The Clarke generalized Jacobian of F at x, denoted by $\partial F(x)$ is defined as*

$$\partial F(x) := co \left\{ \lim_{i \to \infty} \nabla F(x_i) : x_i \in \Omega, x_i \to x \right\},$$

where Ω is the set of points in U at which F is differentiable at them.

Note that $\partial F(x)$ is a nonempty convex and compact subset of $L(\mathbb{R}^n, \mathbb{R}^m)$, the space of real $m \times n$-matrices.

3. Main Results

In [14], Lakshmikantham et al. considered a mapping $T : E \to E$ where E is an ordered Banach space and they worked on an order interval $[v_0, w_0]$ which satisfies some assumptions. Fréchet differentiability of T on $[v_0, w_0]$ is a vital assumption in [14]. One of the Lakshmikantham's abstract fixed point theorems is as follows.

Theorem 3.1. *[14, Theorem 2.1] Let E be an ordered Banach space with regular order cone E_+. Suppose $T : E \to E$ satisfies the following hypotheses:*

(i) There exist $v_0, w_0 \in E$ such that $v_0 \leq Tv_0$, $Tw_0 \leq w_0$ and $v_0 \leq w_0$.

(ii) The Fréchet derivative $T'(u)$ exists for every $u \in [v_0, w_0]$, and the mapping $u \mapsto T'(u)v$ is increasing on $[v_0, w_0]$ for all $v \in E_+$.

(iii) $[I - T'(u)]^{-1}$ exists and is a bounded and positive operator for all $u \in [v_0, w_0]$.

Then, for $n \in \mathbb{N}$, relations

$$v_{n+1} = Tv_n + T'(v_n)(v_{n+1} - v_n), \ w_{n+1} = Tw_n + T'(v_n)(w_{n+1} - w_n),$$

define an increasing sequence $\{v_n\}_{n=0}^{\infty}$ and a decreasing sequence $\{w_n\}_{n=0}^{\infty}$ which both converge to fixed points of T. These fixed points are equal if

(iv) $Tu_1 - Tu_0 < u_1 - u_0$ whenever $v_0 \leq u_0 < u_1 \leq w_0$.

Assumptions (ii) and (iii) are deeply dependent on Fréchet differentiability of T on $[v_0, w_0]$, so even for a simple function $f(x) = |x|$ on $\mathbb{R}$ we cannot use this theorem (since f is not differentiable at $x = 0$).

In this section, we assume that, the mappings are Lipschitz and we impose strictly differentiability only in $\bar{v}$, the left bound of the order interval $[\bar{v}, \bar{w}]$, and we do not need to assume differentiability in other members of the interval.

Definition 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz on $[\bar{v}, \bar{w}]$ and let C be a regular convex cone . We say the set-valued $u \mapsto \partial F(u)$ is semi-increasing on $[\bar{v}, \bar{w}]$ if for all $v, w \in [\bar{v}, \bar{w}]$, with $w - v \in C$,*

$$\partial F(w) \circ z \subseteq \partial F(v) \circ z + C, \qquad \forall z \in C, \tag{3.1}$$

where $\circ$ denotes the matrix multiplication.

Remark 3.1. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is strictly differentiable on $[\bar{v}, \bar{w}]$, then for every $x \in [\bar{v}, \bar{w}]$, $\partial F(\cdot)$ is a singleton and Lipschitz on $[\bar{v}, \bar{w}]$ [9]. We denote this derivative by $F'(x)$. If F is semi-increasing on $[\bar{v}, \bar{w}]$ and $C = \mathbb{R}_+^n$. Then, for all $v, w \in [\bar{v}, \bar{w}]$, with $w - v \in \mathbb{R}_+^n$ we have,*

$$F'(w) \circ z \subseteq F'(v) \circ z + \mathbb{R}_+^n, \qquad \forall z \in \mathbb{R}_+^n.$$

Thus, for some $r \in \mathbb{R}^n_+$, $F'(w) \circ z = F'(v) \circ z + r$. This shows that for $v \leq w$, $F'(v) \circ z \leq F'(w) \circ z$. Therefore, $u \mapsto T'(u)z$ is increasing on $[\bar{v}, \bar{w}]$ for all $z \in \mathbb{R}^n_+$. Thus, semi-increasing map notion is a generalization of increasing map notion.

Example 3.1. *Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $F(x,y) = (|x|, |y|)$ on $[(0,0)^T, (1,1)^T]$ and $C = \mathbb{R}^2_+$. Where T denotes Matrix Transposition. Then,*

$$\partial F(0,0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in [-1,1] \right\}$$

$$\partial F(x,0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} : \beta \in [-1,1] \right\} \qquad 0 < x \leq 1$$

$$\partial F(0,y) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : \alpha \in [-1,1] \right\} \qquad 0 < y \leq 1$$

$$\partial F(x,y) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \qquad\qquad 0 < x, y \leq 1.$$

If $v = (0,0)^T$, $v \leq w$ and $z = (z_1, z_2)^T$ is arbitrary, then Figure 1 shows that $\partial F(w) \circ z \subseteq \partial F(0,0) \circ z + \mathbb{R}^2_+$. It is easy to check that in all other cases, (3.1) holds too. Thus, F is a semi-increasing map on $[(0,0)^T, (1,1)^T]$.
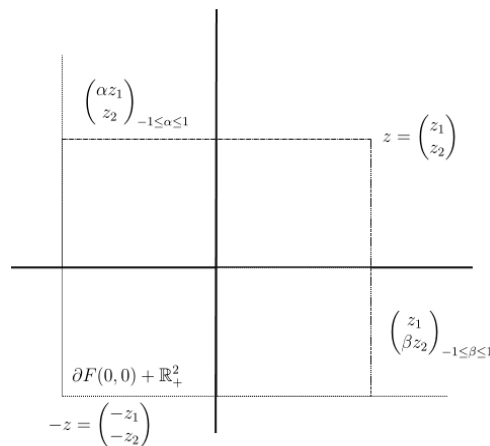

FIGURE 1. semi-increasing map in Example 3.1

Lemma 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz on $[\bar{v}, \bar{w}]$. If the set-valued $u \mapsto \partial F(u)$ is semi-increasing on $[\bar{v}, \bar{w}]$ and $v, w \in [\bar{v}, \bar{w}]$, with $w - v \in C$, then*

$$\partial F(w) \circ z \subseteq \partial F(v) \circ z - C, \qquad \forall z \in -C.$$

Proof. Let $z \in -C$ and $\zeta z \in \partial F(w) \circ z$ be arbitrary. Since $-z \in C$ and $\zeta(-z) \in \partial F(w) \circ (-z)$ we have $\zeta(-z) \in \partial F(v) \circ (-z) + C$. Thus, $\zeta(-z) = \eta(-z) + c$, for some $\eta \in \partial F(v)$ and $c \in C$. On the other hand, $\partial F(u) \subseteq L(\mathbb{R}^n)$ for every $u \in \mathbb{R}^n$, thus, $\zeta(-z) = -\zeta z$ and $\eta(-z) = -\eta z$ and we have $\zeta z = \eta z - c$. Hence, $\zeta z \in \partial F(v) \circ z - C$, which proves the lemma. $\qquad \square$

The following lemma plays an important role in the reminder of this paper.

Lemma 3.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz and semi-increasing on an order interval $[\bar{v}, \bar{w}]$. Then for $\bar{v} \leq v \leq w \leq \bar{w}$ we have,*

$$F(v) - F(w) \in \partial F(v) \circ (v - w) - C.$$

Proof. According to the mean value theorem [9, Proposition 2.6.5],

$$F(v) - F(w) \in co(\partial F(co\{v, w\})) \circ (v - w).$$

The right-hand side above denotes the convex hull of all points of the form $\eta(v - w)$, where $\eta \in \partial F(u)$ for some point $u \in co\{v, w\}$. Assume that $u = v + t(w - v)$ where $0 \leq t \leq 1$. There exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $\lambda_i \geq 0$ for $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^{m} \lambda_i = 1$ and we have,

$$F(v) - F(w) = \left(\sum_{i=1}^{m} \lambda_i \zeta_i \right) (v - w), \tag{3.2}$$

where $\zeta_i \in \partial F(u)$. On the other hand, since $v - w \in -C$ and $u - v \in C$, by Lemma 3.1 we have,

$$\partial F(u) \circ (v - w) \subseteq \partial F(v) \circ (v - w) - C. \tag{3.3}$$

Now, considering the convexity of $\partial F(u)$, also (3.2) and (3.3), we have

$$F(v) - F(w) \in \partial F(v) \circ (v - w) - C.$$

$\qquad \square$

In the following, we prove our main results.

Theorem 3.2. *Let $\mathbb{R}^n$ be ordered with regular cone C. Assume that the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following hypotheses.*

(i) There exist $\bar{v}, \bar{w} \in \mathbb{R}^n$ with $\bar{w} - \bar{v} \in C$, such that F is Lipschitz function on order interval $[\bar{v}, \bar{w}]$, $\bar{v} \leq F(\bar{v})$ and $F(\bar{w}) \leq \bar{w}$.

(ii) The set-valued $u \mapsto \partial F(u)$ is semi-increasing on $[\bar{v}, \bar{w}]$.

(iii) F is strictly differentiable at $\bar{v}$ with $\partial F(\bar{v}) = \{\zeta_{\bar{v}}\}$ such that $[I - \zeta_{\bar{v}}]^{-1}$ exists and is a bounded positive operator, that is, $[I - \zeta_{\bar{v}}]^{-1}(C) \subseteq C$.

Then for $n \in \mathbb{N}$, relations

$$v_{n+1} = F(v_n) + \zeta_{\bar{v}}(v_{n+1} - v_n), \quad w_{n+1} = F(w_n) + \zeta_{\bar{v}}(w_{n+1} - w_n), \tag{3.4}$$

with $v_0 := \bar{v}$, *define an increasing sequence $\{v_n\}_{n \in \mathbb{N}}$ and a decreasing sequence $\{w_n\}_{n \in \mathbb{N}}$ which both converge to fixed points of F. These fixed points are equal if*

(iv) $Fu_1 - Fu_0 < u_1 - u_0$ whenever $\bar{v} \leq u_0 < u_1 \leq w_0$.

Proof. We recall that since F is strictly differentiable at $\bar{v}$, $\partial F(\bar{v})$ is a singleton set (see [13, P. 15]). Consider $v_1 = F(\bar{v}) + \zeta_{\bar{v}}(v_1 - \bar{v})$. Thus, $[I - \zeta_{\bar{v}}]v_1 = F(\bar{v}) - \zeta_{\bar{v}}\bar{v}$. This implies $v_1 = [I - \zeta_{\bar{v}}]^{-1}(F(\bar{v}) - \zeta_{\bar{v}}\bar{v})$. Since $[I - \zeta_{\bar{v}}]^{-1}$ is bounded, v_1 is well-defined. We show that $\bar{v} \leq v_1 \leq \bar{w}$. The assumption (1) implies:

$$\bar{v} - v_1 \leq F(\bar{v}) - [F(\bar{v}) + \zeta_{\bar{v}}(v_1 - \bar{v})] = \zeta_{\bar{v}}(\bar{v} - v_1).$$

Therefore, $[I - \zeta_{\bar{v}}](\bar{v} - v_1) \leq 0$. Since, $[I - \zeta_{\bar{v}}]^{-1}$ is a positive operator, $(\bar{v} - v_1) \leq [I - \zeta_{\bar{v}}]^{-1}(0) = 0$ and we have $\bar{v} \leq v_1$. Now, by (i)

$$v_1 - \bar{w} \in F(\bar{v}) + \zeta_{\bar{v}}(v_1 - \bar{v}) - F(\bar{w}) - C.$$

and using Lemma 3.2 we have,

$$v_1 - \bar{w} \in \partial F(\bar{v}) \circ (\bar{v} - \bar{w}) + \zeta_{\bar{v}}(v_1 - \bar{v}) - C.$$

Therefore, for some $c \in C$ we have,

$$v_1 - \bar{w} = \zeta_{\bar{v}}(\bar{v} - \bar{w}) + \zeta_{\bar{v}}(v_1 - \bar{v}) - c. \tag{3.5}$$

Then (3.5) implies:

$$v_1 - \bar{w} \leq \zeta_{\bar{v}}(v_1 - \bar{w}).$$

Thus $[I - \zeta_{\bar{v}}](v_1 - \bar{w}) \leq 0$ and $(v_1 - \bar{w}) \leq [I - \zeta_{\bar{v}}]^{-1}(0) = 0$ which implies $v_1 \leq \bar{w}$. In a similar way we can show that there exists a point w_1 such that $w_1 = F(\bar{w}) + \zeta_{\bar{v}}(w_1 - \bar{w})$ and $\bar{v} \leq w_1 \leq \bar{w}$. We claim that $v_1 \leq w_1$. To prove this claim, note that:

$$v_1 - w_1 = F(\bar{v}) + \zeta_{\bar{v}}(v_1 - \bar{v}) - [F(\bar{w}) + \zeta_{\bar{v}}(w_1 - \bar{w})].$$

Thus, by Lemma 3.2 we have,

$$v_1 - w_1 \in \partial F(\bar{v}) \circ (\bar{v} - \bar{w}) + \zeta_{\bar{v}}(v_1 - \bar{v}) - \zeta_{\bar{v}}(w_1 - \bar{w}) - C,$$

similarly, $v_1 - w_1 \leq \zeta_{\bar{v}}(v_1 - w_1)$ and we have $v_1 \leq w_1$.

Up to now, we showed that $\bar{v} \leq v_1 \leq w_1 \leq \bar{w}$. We claim for every $j \in \mathbb{N}$, $\bar{v} \leq v_j \leq v_{j+1} \leq w_{j+1} \leq w_j \leq \bar{w}$. Since $\bar{v} \leq v_1$ and $w_1 \leq \bar{w}$, we only need to prove $v_j \leq v_{j+1} \leq w_j$ and $v_{j+1} \leq w_{j+1} \leq w_j$. We just prove $v_j \leq v_{j+1}$ and other inequalities are obtained similarly. By Lemma 3.2

$$v_j - v_{j+1} = F(v_{j-1}) + \zeta_{\bar{v}}(v_j - v_{j-1}) - [F(v_j) + \zeta_{\bar{v}}(v_{j+1} - v_j)]$$

$$\in \partial F(v_{j-1}) \circ (v_{j-1} - v_j) + \zeta_{\bar{v}}(v_j - v_{j-1}) - \zeta_{\bar{v}}(v_{j+1} - v_j) - C$$

Since $v_{j-1} - \bar{v} \in C$ and $v_{j-1} - v_j \in -C$, by Lemma 3.1 we have,

$$v_j - v_{j+1} \in \partial F(\bar{v}) \circ (v_{j-1} - v_j) + \zeta_{\bar{v}}(v_j - v_{j-1}) - \zeta_{\bar{v}}(v_{j+1} - v_j) - C.$$

Therefore,

$$v_j - v_{j+1} \leq \zeta_{\bar{v}}(v_j - v_{j+1}),$$

so we have $v_j - v_{j+1} \leq [I - \zeta_{\bar{v}}]^{-1}(0) = 0$ which proves our claim. We found an increasing sequence $\{v_n\}_{n \in \mathbb{N}}$ and a decreasing sequence $\{w_n\}_{n \in \mathbb{N}}$ such that

$$\bar{v} \leq v_1 \leq \ldots \leq v_n \leq w_n \leq \ldots \leq w_1 \leq \bar{w}.$$

Since C is a regular cone, $\{v_n\}_{n \in \mathbb{N}}$ and $\{w_n\}_{n \in \mathbb{N}}$ are convergent. Suppose $v_n \to v$ and $w_n \to w$. We will prove $v_{n+1} \to F(v)$ which shows v is a fixed point of F. We have

$$v_{n+1} - F(v) = F(v_n) + \zeta_{\bar{v}}(v_{n+1} - v_n) - F(v)$$

Since F is continuous and $v_n \to v$, $v_{n+1} - v_n \to 0$ and $F(v_n) - F(v) \to 0$, thus $v_{n+1} \to F(v)$, so $F(v) = v$. Similarly one can show that $F(w) = w$.

If (iv) holds and $v < w$, then $w - v = Fw - Fv < w - v$ which is a contradiction. And the uniqueness of the fixed point is proved. $\qquad \square$

Note that under some conditions, the sequences in Theorem 3.2, converge quadratically as the following theorem shows.

Theorem 3.3. *Let $\mathbb{R}^n$ be ordered with regular cone C. Let the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the hypotheses of Theorem 3.2 and*

$$\|\zeta_v - \zeta_{\bar{v}}\| \leq L\|u - v\|, \ \forall \zeta_v \in \partial F(v),$$

whenever $\bar{v} \leq v \leq u \leq \bar{w}$. Then, the sequences $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ converge quadratically to the same fixed point of F.

Proof. Note that, by Theorem 3.2, both the sequences $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ converge to the same fixed point u of F. We prove that these sequences converge quadratically. For sequence $\{v_n\}_{n=0}^{\infty}$ we have,

$$v_{n+1} - u = F(v_n) - F(u) + \zeta_{\bar{v}}(v_{n+1} - v_n)$$

$$\subseteq \partial F(v_n) \circ (v_n - u) + \zeta_{\bar{v}}(v_{n+1} - v_n) - C.$$

Thus, for some $\eta_{v_n} \in \partial F(v_n)$ and $c \in C$ we have,

$$v_{n+1} - u = \zeta_{v_n} \circ (v_n - u) + \zeta_{\bar{v}}(v_{n+1} - v_n) - c$$

$$= \zeta_{v_n} \circ (v_n - u) + \zeta_{\bar{v}}(v_{n+1} - u) - \zeta_{\bar{v}}(v_n - u) - c.$$

Hence,

$$u - v_{n+1} = \zeta_{v_n} \circ (u - v_n) - \zeta_{\bar{v}}(v_{n+1} - u) + \zeta_{\bar{v}}(v_n - u) + c.$$

This implies

$$[I - \zeta_{\bar{v}}](u - v_{n+1}) = (\zeta_{v_n} - \zeta_{\bar{v}}) \circ (u - v_n) + c.$$

Since, $[I - \zeta_{\bar{v}}]^{-1}$ is a positive operator, we have

$$0 \leq u - v_{n+1} \leq [I - \zeta_{\bar{v}}]^{-1}(\zeta_{v_n} - \zeta_{\bar{v}}) \circ (u - v_n) + \bar{c},$$

where $\bar{c} = [I - \zeta_{\bar{v}}]^{-1}(c)$. On the other hand, the cone C is regular and therefore normal, thus, there exists a positive constant δ such that

$$\|u - v_{n+1}\| \leq \delta\|[I - \zeta_{\bar{v}}]^{-1}\|\|(\zeta_{v_n} - \zeta_{\bar{v}})\|\|u - v_n\| + \bar{c}$$

$$\leq \delta L\|[I - \zeta_{\bar{v}}]^{-1}\|\|u - v_n\|^2 + \bar{c}.$$

For sequence $\{w_n\}_{n=0}^{\infty}$ the claim can be proved by a similar manner. □

4. Application in the Dynamical Coulomb Friction Problem

The dynamical Coulomb friction problem in finite dimension with discretized time is associated to the problem of simulating the dynamics of mechanical systems which involve unilateral contact between their parts or with external objects. Acary et al. [1] presented a new formulation of this problem. They capture and treat directly the friction model as a parametric quadratic optimization problem with second-order cone constraints coupled with a fixed point equation. In this section, we just use the resulted fixed point theorems obtained in this paper, to investigate problem. Firstly, we recall some preliminaries.

Let $x \in \mathbb{R}^d$. The normal and tangential components of x with respect to a unit vector $e \in \mathbb{R}^d$ are defined respectively as

$$x_N := x^T e \in \mathbb{R}, \qquad x_T := x - x_N e \in \mathbb{R}^d.$$

For the unit vector $e \in \mathbb{R}^d$ and parameter $\mu \in (0, +\infty)$, the second-order cone $K_{e,\mu}$ is defined by

$$K_{e,\mu} := \{x \in \mathbb{R}^d : \|x_T\| \leq \mu x_N\}.$$

If $\mu = 0$ or $\mu = +\infty$, we define

$$K_{e,0} := \{x \in \mathbb{R}^d : x_T = 0, x_N \geq 0\}, \ K_{e,\infty} := \{x \in \mathbb{R}^d : x_N \geq 0\}.$$

The dual cone of the second-order cone $K_{e,\mu}$ with $\mu \in (0, \infty)$ is defined as

$$K_{e,\mu}^* := \{s \in \mathbb{R}^d : x^T s \geq 0, \ \forall x \in K_{e,\mu}\} = K_{e,\frac{1}{\mu}}.$$

Also with the convention $1/0 = \infty$ and $1/\infty = 0$, $(K_{e,0})^* = K_{e,\infty}$ and $(K_{e,\infty})^* = K_{e,0}$. Consider $e^1, \ldots, e^n \in \mathbb{R}^d$ and $\mu^1, \ldots, \mu^n \in [0, +\infty]$. The associated product cone and its dual cone are respectively

$$L \quad = K_{e^1, \mu^1} \times \cdots \times K_{e^n, \mu^n} \subseteq \mathbb{R}^{nd}$$

$$, \quad L^* \quad = K^*_{e^1, \mu^1} \times \cdots \times K^*_{e^n, \mu^n} = K_{e^1, \frac{1}{\mu^1}} \times \cdots \times K_{e^n, \frac{1}{\mu^n}}.$$

Set

$$I := \left\{ i \in \{1, \cdots, n\} : \mu^i \neq 0 \right\}, \quad n_I := Card\ I.$$

Consider the problem:

$$Mv + f = H^T r$$

$$\tilde{u} = Hv + w + Es$$

$$L^* \ni \tilde{u} \perp r \in L$$

$$s^i = \|\tilde{u}^i_T\|\ for\ i \in I \tag{4.1}$$

with respect to the variable $(v, r, \tilde{u}, s) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \mathbb{R}^{nd} \times \mathbb{R}^{n_I}$ where $\tilde{u} := (\tilde{u}^1, \ldots, \tilde{u}^n) \in \mathbb{R}^{nd}$, $r := (r^1, \ldots, r^n) \in \mathbb{R}^{nd}$. Moreover, the data of the problem are

$$e^i \in \mathbb{R}^d, \quad \mu^i \in [0, +\infty], \quad M \in \mathbb{R}^{m \times m}, \quad f \in \mathbb{R}^m, \quad H \in \mathbb{R}^{nd \times m}, \quad w \in \mathbb{R}^{nd}, \quad E \in \mathbb{R}^{nd \times n_I},$$

where matrix M is definite positive and matrix E is constructed by concatenating n_I columns $E_i \in \mathbb{R}^{nd}$, where E_i is itself the concatenation of n vectors of $\mathbb{R}^d$, all zeros except for the i-th which is $\mu^i e^i$.

Define

$$\overline{C}(s) := \{v \in \mathbb{R}^m : Hv + w + Es \in L^*\},$$

$$v(s) := \arg \min_{v \in \overline{C}(s)} \left\{ \frac{1}{2} v^T Mv + f^T v \right\} \in \mathbb{R}^m,$$

$$\tilde{u}(s) := Hv(s) + w + Es \in \mathbb{R}^{nd},$$

and

$$F(s) := \left(\|\tilde{u}^i_T(s)\| \right)_{i \in I} \in \mathbb{R}^{n_I}$$

The following theorem shows the relation between fixed points of F and the solutions of (4.1).

Theorem 4.1. [1, Theorem 3.3] Let $(v^*, r^*, \tilde{u}^*, s^*)$ solves the problem (4.1), then $v^* = v(s^*)$ and $F(s^*) = s^*$.

In dimension two , the inverse is also hold.

Theorem 4.2. [1, Theorem 3.7] Let the dimension $d = 2$. Then $(v^*, r^*, \tilde{u}^*, s^*)$ solves the problem (4.1) if and only if $v^* = v(s^*)$ and $F(s^*) = s^*$.

Using these theorems, we can compute a fixed point of F and then check whether the computed fixed point is a solution of the problem (4.1). Note that F is a non-smooth function and to compute its fixed points we need to use the non-smooth version fixed point theorem presented in this paper. In the following Examples we show the efficiency of the resulted fixed point theorem.

Example 4.1. *Let* $d = 2$, $n = 3$, $e^1 = e^2 = e^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mu^1 = 1$, $\mu^2 = 0$, $\mu^3 = 2$ *and* $m = 2$. *Then*

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Also, $I = \{1, 3\}$ *and* $n_I = 2$. *For* $x = \left(x_1, x_2 \right)^T$ *and* e^i $(i = 1, 2, 3)$, $x_N = x_2$ *and* $x_T = \left(x_1, 0 \right)^T$. *Moreover,*

$$K_{e^1, \mu^1} = \left\{ x \in \mathbb{R}^2 : |x_1| \leq x_2 \right\}, \qquad\qquad K^*_{e^1, \mu^1} = K_{e^1, \mu^1},$$

$$K_{e^2, \mu^2} = \left\{ x \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0 \right\}, \qquad K^*_{e^2, \mu^2} = \left\{ x \in \mathbb{R}^2 : x_2 \geq 0 \right\}$$

$$K_{e^3, \mu^3} = \left\{ x \in \mathbb{R}^2 : |x_1| \leq 2x_2 \right\} \qquad\qquad K^*_{e^3, \mu^3} = \left\{ x \in \mathbb{R}^2 : |x_1| \leq \frac{1}{2}x_2 \right\}.$$
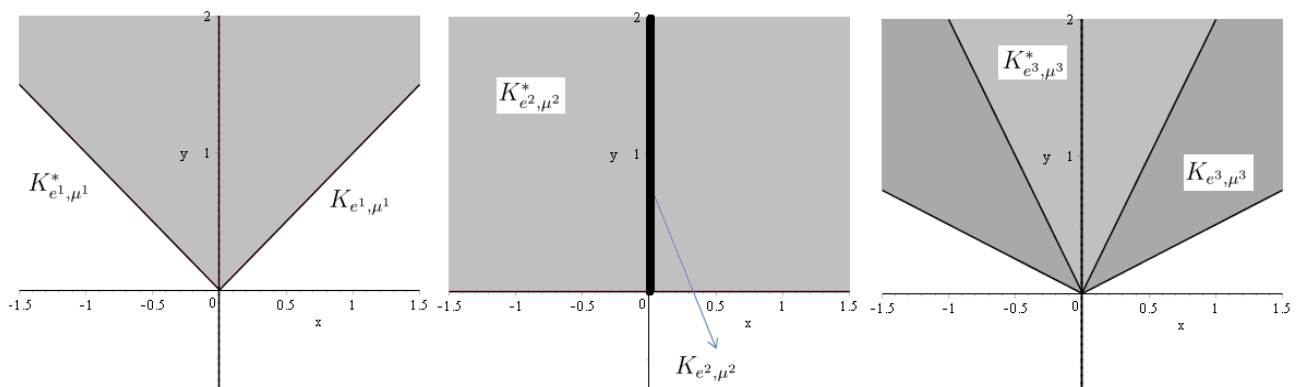
These cones and their dual cones are showed in Figure 2.

FIGURE 2. Cones and their dual cones in Example 4.1

Note that:

$$L = \left\{ (\alpha, \beta, \gamma)^T : \alpha \in K_{e^1, \mu^1}, \beta \in K_{e^2, \mu^2}, \gamma \in K_{e^3, \mu^3} \right\},$$

$$L^* = \left\{ (\alpha^*, \beta^*, \gamma^*) : \alpha^* \in K^*_{e^1, \mu^1}, \beta^* \in K^*_{e^2, \mu^2}, \gamma^* \in K^*_{e^3, \mu^3} \right\}.$$

Assume,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, s = \begin{bmatrix} s^1 \\ s^3 \end{bmatrix}$$

$$r = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)^T \in L, \ \tilde{u} = (\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, \gamma_1^*, \gamma_2^*)^T \in L^*.$$

Let $v \in \overline{C}(s)$. Thus,

$$\begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \beta_1^* \\ \beta_2^* \\ \gamma_1^* \\ \gamma_2^* \end{bmatrix} \in L^* \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} s^1 \\ s^3 \end{bmatrix} \in L^*.$$

Therefore,

$$\begin{bmatrix} v_1 + 1 \\ v_2 + s^1 \\ v_1 \\ v_1 + 1 \\ v_2 \\ v_1 + 2s^3 \end{bmatrix} \in L^* \Rightarrow \begin{bmatrix} v_1 + 1 \\ v_2 + s^1 \end{bmatrix} \in K_{e^1, \mu^1}^*, \begin{bmatrix} v_1 \\ v_1 + 1 \end{bmatrix} \in K_{e^2, \mu^2}^*, \begin{bmatrix} v_2 \\ v_1 + 2s^3 \end{bmatrix} \in K_{e^3, \mu^3}^*.$$

Hence,

$$|v_1 + 1| \leq v_2 + s^1, v_1 + 1 \geq 0, |v_2| \leq \frac{1}{2}(v_1 + 2s^3).$$

So, we have

$$\overline{C}(s) = \left\{ (v_1, v_2)^T \in \mathbb{R}^2 : v_1 + 1 \leq v_2 + s^1, |v_2| \leq \frac{1}{2}(v_1 + 2s^3) \right\}.$$

Note that $\overline{C}(s)$ is a closed convex set and $J(v) := \frac{1}{2}v^T M v + f^T v$ is a convex function. Therefore, if $\overline{C}(s) \neq \emptyset$, then $J(v)$ has a unique solution. In this example, if $s^1 \geq -2s^3 + 1$, then $\overline{C}(s) \neq \emptyset$. Now we must consider the following convex optimization problem:

$$\min \ \frac{1}{2}v^T M v + f^T v = \frac{1}{2}(v_1^2 + v_2^2) + v_1$$

$$\text{subject to } g_1(v) := v_1 - v_2 - s^1 + 1 \leq 0$$

$$g_2(v) := |v_2| - \frac{1}{2}v_1 - s^3 \leq 0$$

According to [10, Page 150], if $v^* = (v_1^*, v_2^*)^T$, then there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1^* + 1 \\ v_2^* \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\frac{1}{2} \\ \alpha \end{pmatrix}, \ \lambda_i g_i(v^*) = 0, i = 1, 2 \tag{4.2}$$

where

$$\begin{cases} \alpha = 1 & \text{if } v_2^* > 0, \\ -1 \leq \alpha \leq 1 & \text{if } v_2^* = 0, \\ \alpha = -1 & \text{if } v_2^* < 0, \end{cases}$$

then v^* is the optimal solution. The system (4.2) *has a solution if $s^3 \leq 1/2$. With these condition, $v_1^* = 0$ and $v_2^* = -2s^3$. Thus, $v(s) = (0, -2s^3)$. We also have ,*

$$\tilde{u}(s) = Hv(s) + w + Es = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2s^3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} s^1 \\ s^3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2s^3 + s^1 \\ 0 \\ 1 \\ -2s^3 \\ 2s^3 \end{bmatrix}$$

and

$$F(s) = (1, -2s^3), \ \text{where } s^3 \leq \frac{1}{2}.$$

Note that from Example 3.1, F is a semi-increasing map with $C = \mathbb{R}_+^2$. Set $\bar{v} = (-1, -1)^T$, $\bar{w} = (2, 0)^T$. We have $\partial F(\bar{v}) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$. With these informations, conditions (i), (ii), (iii) of Theorem 3.2 hold. Set $v_0 = (-1, -1)^T$ and

$$v_{n+1} = (1, 2\|v_n^2\|)^T + \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} (v_{n+1} - v_n).$$

We have $v_n = (1, 0)^T$ for every $n \in \mathbb{N}$. Thus $(1, 0)^T$ is a fixed point of F. It is easy to check that the combination of these resulted quantities with $r = (0, 0, 0, 0, 0, 1)^T$, is a solution of system (4.1).

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