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BIFURCATION AND STABILITY ANALYSIS OF A DISCRETE TIME SIR EPIDEMIC MODEL WITH VACCINATION

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Abstract. In this paper, we study the qualitative behavior of a discrete-time epidemic model with vaccination. Analysis of the model shows forth that the Disease Free Equilibrium (DFE) point is asymptotically stable if the basic reproduction number R_0 is less than one, while the Endemic Equilibrium (EE) point is asymptotically stable if the basic reproduction number R_0 is greater than one. The results are reinforced with numerical simulations and enhanced with graphical representations like time trajectories, phase portraits and bifurcation diagrams for different sets of parameter values.

1. Introduction

Mathematical models defining biological events has an important place in the study of population dynamics. Most of the biological occurrences in nature are illustrated by discrete time, which point to, that there are particular time instants at which the basic events in the system can occur, and it is not essential that at these discrete time instants only a exclusive event happens. The most realistic approach to non-overlapping generations like fish or insect populations, is created with discrete time system ([6], [9], [15], [16], [17], [18]).

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One of the other famous examples identified by these systems are epidemic models [10]. Modeling an outbreak that is progressing through the population allows us to examine the consequences of ways of preventing or controlling the disease. Although the work of Kermack and McKendrick have the basic foundation for epidemics, the first attempt in explaining, predicting or modeling of epidemics dates back to over a century is made by Hamer (1906), Ross (1911). These early models operate on the principle where individuals can be classified by their epidemiological status which are susceptible to the infection, infected and recovered (immune) ([12], [13], [14]).

Discrete time models are more suitable than continuous time models to examine infectious diseases due to many reasons. Statistical data on diseases are collected at a specific time. In this case, the appropriate model defining the disease will be the discrete time model [19]. On the other hand, the studies on discrete time models obtained from the continuous time model by using nonstandard discretization technique are more suitable to avoid mathematical complexity with regularity of solutions ([20], [21]). Furthermore, although the continuous-time logistic equation has only equilibrium dynamics, the well known discrete logistic equation which is discrete counterpart of it exhibits period doubling bifurcation to chaos ([22], [23]).

2. SIR Epidemic Model with Vaccination

The mathematical modelling of infectious diseases has a significant role in the studies of dynamical system. Because studies on the dynamics of these models help us to control diseases like swine flu, bird flue and AIDS. SIR models are suitable to define the transmission of infectious diseases with lifelong immunity such as chicken pox, measles, smallpox, mumps and SARS. The SIR model is one of the simplest and most fundamental of all epidemiological models and in these models with a single epidemic, births and deaths are ignored, and so, only two transitions are possible: infection (moving individuals from the susceptible to the infected class) and recovery (moving individuals from the infected to the recovered class). The assumptions in this model are that the per capita rate that a given susceptible individual becomes infected is proportional to the prevalence of infection in the population and that infected individuals recovers at a constant rate. The fundamental parameter that governs the behavior of the epidemic is the basic reproductive ratio, R_0 which is defined as the average number of secondary cases produced by a single infectious individual in a totally susceptible population.

Vaccination can be included in a epidemic model by assuming a proportion of susceptible individuals are vaccinated during each time interval ([1], [3], [4], [5]). Vaccination operates by reducing the pool of susceptible individuals, and when this is reduced sufficiently, an infectious disease cannot spread within the population. Most importantly, it is not necessary to vaccinate everyone to prevent an epidemic, immunizing someone not only protects that person but confers some protection to the population in general [11].

3. The Discrete Time System

The author of [2] has presented the dynamics of the SIR epidemic model which is as follows:

$$\begin{aligned} S_{t+1} =& S_t - \frac{a}{N} I_t S_t + \beta(R_t + I_t) \\ I_{t+1} =& \frac{a}{N} I_t S_t + (1 - \beta - \gamma) I_t \\ R_{t+1} =& (1 - \beta) R_t + \gamma I_t + p S_t \end{aligned} \tag{3.1}$$

where $a > 0$, $0 < \beta < 1$ and $0 < \gamma < 1$.

In this paper, we focus on the dynamics of a SIR epidemic model by including vaccination to the model as presented in [2]. The general SIR epidemic model is of the following form [1]:

$$\begin{aligned} S_{t+1} =& (1 - p) S_t - \frac{a}{N} I_t S_t + \beta(R_t + I_t) \\ I_{t+1} =& \frac{a}{N} I_t S_t + (1 - \beta - \gamma) I_t \\ R_{t+1} =& (1 - \beta) R_t + \gamma I_t + p S_t \end{aligned}$$

such that the initial conditions S_0, I_0 and R_0 which are positive real numbers with $(S_0 + I_0 + R_0 = N)$. Here $0 < p + a < 1$ and $0 < \beta + \gamma < 1$. Also, β is the probability of birth, γ is the probability of recovery, p is the proportion of vaccinated, a is the contact rate and N is the total population size. Moreover, we have the following equivalent two dimensional system using the relation $S_t + I_t + R_t = N$.

$$\begin{aligned} S_{t+1} =& (1 - p) S_t - \frac{a}{N} I_t S_t + \beta(N - S_t) \\ I_{t+1} =& \frac{a}{N} I_t S_t + (1 - \beta - \gamma) I_t \end{aligned} \tag{3.2}$$

where p, a, β and γ have positive values.

4. Stability of Equilibrium points and Numerical Simulations

In this section, we consider the discrete-time system (3.2). Foremost, we discuss the existence of equilibrium points for (3.2), and then study the stability of the equilibrium points by using the characteristic polynomial or the eigenvalues of the Jacobian matrix evaluated at each of the fixed points.

Lemma 4.1. *[7] Let $Q(x) = x^2 + Bx + C$. Suppose that $Q(1) > 0$, x_1 and x_2 are two roots of $Q(x) = 0$. Then*

(i) *$|x_1| < 1$ and $|x_2| < 1$ if and only if $Q(-1) > 0$ and $C < 1$;*

(ii) *$|x_1| < 1$ and $|x_2| > 1$ (or $|x_1| > 1$ and $|x_2| < 1$) if and only if $Q(-1) < 0$;*

(iii) *$|x_1| > 1$ and $|x_2| > 1$ if and only if $Q(-1) > 0$ and $C > 1$;*

(iv) *$x_1 = -1$ and $|x_2| \neq 1$ if and only if $Q(-1) = 0$ and $B \neq 0, 2$;*

(v) *x_1 and x_2 are complex and $|x_1| = |x_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$.*

Lemma 4.2. *[8]The characteristic polynomial*

$$Q(x) = x^2 + Bx + C$$

where B=-(Trace of the Jacobian matrix) and C= Determinant of the Jacobian matrix has all its roots inside the unit open disk ($|x| < 1$) if and only if (i) $Q(1) > 0$ and $Q(-1) > 0$. (ii) $D_1^+ = 1 + C > 0$ and $D_1^- = 1 - C > 0$

Now, we will investigate the equilibrium points and then analyze the stability of these equilibrium points. For analyzing the local stability of equilibrium points (S^*, I^*), we give the following theorems.

Theorem 4.1. *The model (3.2) has two equilibrium points, $P_0 = \left(\frac{\beta N}{\beta+p}, 0\right)$ and $P_1 = \left(\frac{(\beta+\gamma)N}{a}, \frac{N(a\beta-(\beta+\gamma)(p+\beta))}{a(\beta+\gamma)}\right)$.*

Proof. When we examine the following equilibrium points (S^*, I^*) of the model (3.2), we easily obtain the equilibrium points of the model (3.2) by using $S_t = S_{t+1} = S^*$ and $I_t = I_{t+1} = I^*$:

$$\begin{aligned} S^* &= (1-p)S^* - \frac{a}{N}I^*S^* + \beta(N - S^*) \\ I^* &= \frac{a}{N}I^*S^* + (1-\beta-\gamma)I^* \end{aligned}$$

□

Theorem 4.2. *Suppose that $p + \beta < 1$. The disease-free equilibrium (DFE) point $P_0 = \left(\frac{\beta N}{\beta+p}, 0\right)$ of the system (3.2) is locally asymptotically stable (LAS) if*

$$\frac{a\beta}{(\beta+p)(\beta+\gamma)} < 1. \tag{4.1}$$

Proof. By considering (3.2), we can get the Jacobian matrix evaluated P_0 as

$$J_{P_0} = \begin{pmatrix} 1-p-\beta & \frac{-a\beta}{(\beta+p)} \\ 0 & \frac{a\beta}{(\beta+p)} + (1-\beta-\gamma) \end{pmatrix}.$$

The eigenvalues of this matrix are

$$x_1 = 1 - p - \beta, \quad x_2 = \frac{a\beta}{(p+\beta)} + (1-\gamma-\beta).$$

If $\beta + p < 1$, then it is easy to see that $x_1 = 1 - p - \beta < 1$, and also since $\beta + \gamma < 1$, $x_2 > 0$ is always true. Consequently, if $|x_2| = \frac{a\beta}{(\beta+p)} + (1-\beta-\gamma) < 1$, then we get $\frac{a\beta}{(\beta+p)(\beta+\gamma)} < 1$. □

Corollary 4.1. *The basic reproductive ratio R_0 is referred as $\frac{a\beta}{(\beta+p)(\beta+\gamma)}$. This ratio is a threshold parameter. If $R_0 < 1$, then there exists that the DFE point is LAS.*

We consider the initial conditions $(S(0), I(0)) = (70, 30)$ for numerical study.

Example 4.1. *(a) For the DFE point, we assume the parameter values as* $N = 100$, $p = 0.0005$, $a = 0.1$, $\beta = 0.02$, $\gamma = 0.2$. *The eigenvalues are* $|x_1| = 0.9795 < 1$, $|x_2| = 0.8776 < 1$ *and* $R_0 = 0.4435 < 1$ *then the DFE point* $P_0 = (97.561, 0)$ *of the model (3.2) is LAS (see Figure-1). (b)We take the parameter values as* $N = 100, p = 0.05, a = 1.7, \beta = 1.7, \gamma = 0.1$. *The eigenvalues are* $|x_1| = 0.7500 < 1$, $|x_2| = 0.8514 < 1$ *and* $R_0 = 0.9175 < 1$ *then the DFE point* $P_0 = (97.1429, 0)$ *of the model (3.2) is LAS (see Figure-2). Note that trace* $J_{P_0} > 0$.

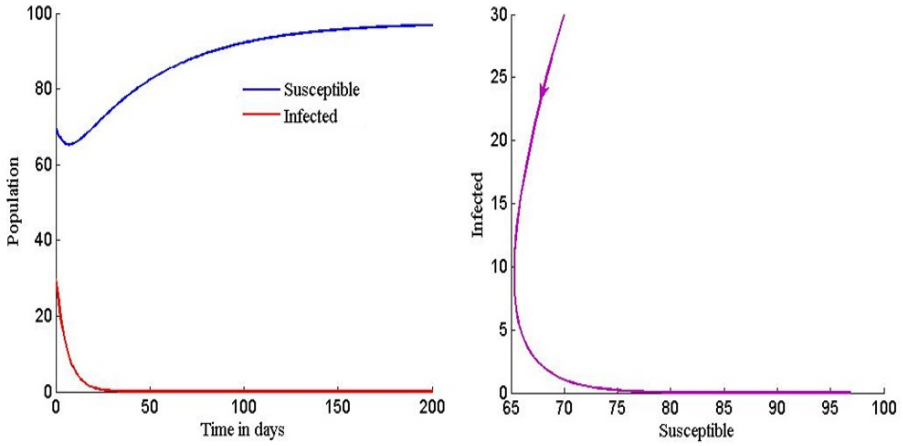

FIGURE 1. Time plots and phase portrait of DFE point P_0 with stability $R_0 < 1$

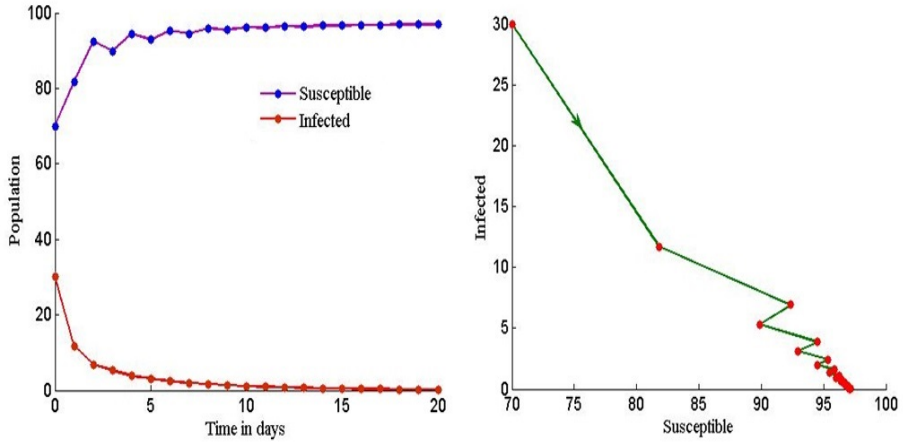

FIGURE 2. Time plots and phase portrait of DFE point P_0 with stability $R_0 < 1$

Theorem 4.3. *If* $1 < R_0 < \frac{2}{(p+\beta)}$, *then the endemic equilibrium (EE) point* $P_1 = \left(\frac{(\beta+\gamma)N}{a}, \frac{N(a\beta-(\beta+\gamma)(p+\beta))}{a(\beta+\gamma)}\right)$ *of the system (3.2) is LAS.*

Proof. By considering (3.2), we can write the Jacobian matrix evaluated at P_1 as

$$J_{P_1} = \begin{pmatrix} 1 - \frac{\beta a}{\beta+\gamma} & -\beta - \gamma \\ \frac{\beta a}{\beta+\gamma} - (p+\beta) & 1 \end{pmatrix}, \tag{4.2}$$

If it is organized as relating to R_0, we find

$$J_{P_1} = \begin{pmatrix} 1 - (\beta+p)R_0 & \frac{-a\beta}{(\beta+p)R_0} \\ (p+\beta)(R_0-1) & 1 \end{pmatrix},$$

The characteristic polynomial of the Jacobian matrix at J_{P_1} is as follows:

$$Q(x) = x^2 - (2 - (\beta+p)R_0)x + 1 - (\beta+p)R_0 + a\beta\left(1 - \frac{1}{R_0}\right). \tag{4.3}$$

For the stability of the EE point of (3.2), we get

$$0 < a\beta\left(1 - \frac{1}{R_0}\right) < R_0(p+\beta),$$

or equivalently

$$1 < R_0 < \frac{\beta a}{(\beta+\gamma)^2(p+\beta)} + 1. \tag{4.4}$$

from Lemma 4.2. Note that

$$\frac{a\beta}{\beta+\gamma} < 2 \tag{4.5}$$

is always provided. Equivalently, we have

$$R_0 < \frac{2}{(p+\beta)}. \tag{4.6}$$

If (4.4) and (4.6) are compared, then we get

$$\frac{2}{(p+\beta)} < \frac{\beta a}{(\beta+\gamma)^2(p+\beta)} + 1. \tag{4.7}$$

Thus the proof is completed. □

Corollary 4.2. *If* $1 < R_0 < \frac{2}{(p+\beta)}$, *then* $Q(1) > 0$, $Q(-1) > 0$ *and* $C < 1$ *is always confirmed such that* $0 < p + a < 1$ *and* $0 < \beta + \gamma < 1$.

Proof. From (4.2), the characteristic polynomial is as follows:

$$Q(x) = x^2 + \left(\frac{a\beta}{\beta+\gamma} - 2\right)x + 1 - \frac{a\beta}{\beta+\gamma} + a\beta - (\beta+\gamma)(\beta+p). \tag{4.8}$$

Obviously, $Q(1) > 0$ is always true, since $Q(1) > 0$, $R_0 > 1$. Also, we obtain

$$Q(-1) = 4 + a\beta - (\beta+\gamma)(\beta+p) - \frac{2a\beta}{\beta+\gamma} \tag{4.9}$$

$$D_1^- = 1 - C = \frac{a\beta}{\beta+\gamma} - a\beta + (\beta+\gamma)(\beta+p) \tag{4.10}$$

and

$$D_1^+ = 1 + C = 2 - \frac{a\beta}{\beta+\gamma} + a\beta - (\beta+\gamma)(\beta+p). \tag{4.11}$$

Here, we take

$$C = Q(0) = 1 - \frac{a\beta}{\beta+\gamma} + a\beta - (\beta+\gamma)(\beta+p) \tag{4.12}$$

such that $R_0 > 1$. Note that whenever $Q(-1) > 0$, $D_1^+ > 0$ is always true. By considering (4.10), we can write,

$$a\beta - a\beta(\beta+\gamma) + (\beta+\gamma)^2(\beta+p) > 0$$

such that $\beta+\gamma < 1$. It clear that $a\beta - a\beta(\beta+\gamma) > 0$. So $1 - C > 0$ is always provided. Similarly, we can write by considering (4.9).

$$(\beta+\gamma)[4 + a\beta - (\beta+\gamma)(\beta+p)] - 2a\beta > 0$$

such that $a\beta - (\beta+\gamma)(\beta+p) > 0$. Then, we get $Q(-1) > 0$. From (4.5) and by the positive state of the EE point of (3.2), the result is clear. □

Example 4.2. *For the EE point, we take the parameter values as (a)* $N = 100$, $p = 0.0005$, $a = 0.6$, $\beta = 0.025$, $\gamma = 0.3$. *Applying the conditions, we get* $Q(1) = 0.0068 > 0$, $Q(-1) = 3.9144 > 0$, $C = 0.9606 < 1$ *and* $R_0 = 1.81 > 1$ *and thus the EE point* $P_1 = (54.1667, 3.4423)$ *of the model (3.2) is LAS (see Figure-3). (b) We take the parameter values as* $N = 100$, $p = 0.05$, $a = 4.2$, $\beta = 1.6$, $\gamma = 0.1$ *and applying the conditions we see that* $Q(1) = 3.9150 > 0$, $Q(-1) = 0.0092 > 0$, $C = 0.9621 < 1$ *and* $R_0 = 2.3957 > 1$ *and so the EE point* $P_1 = (40.4762, 54.8319)$ *of the model (3.2) is LAS (see Figure-4).*

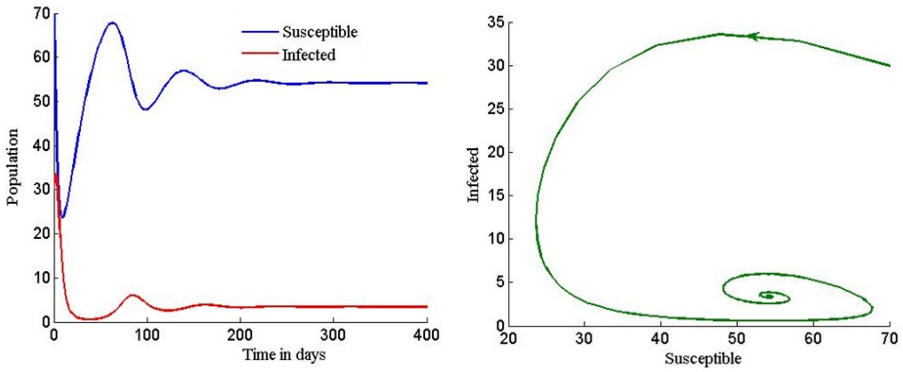

FIGURE 3. Time plots and phase portrait of EE point P_1 with stability $R_0 > 1$

5. BIFURCATION

In this section, we give the bifurcation diagrams of the susceptible and infected populations of the model (3.2). The bifurcation diagrams are considered for four cases:

Case (i): Fixing parameters $N = 100$, $\beta = 0.8$, $p = 0.0005$, $\gamma = 0.1$ and varying a.

The bifurcation diagrams of model (3.2) are plotted with contact rate $a \in (3.0, 4.15)$ as the bifurcation

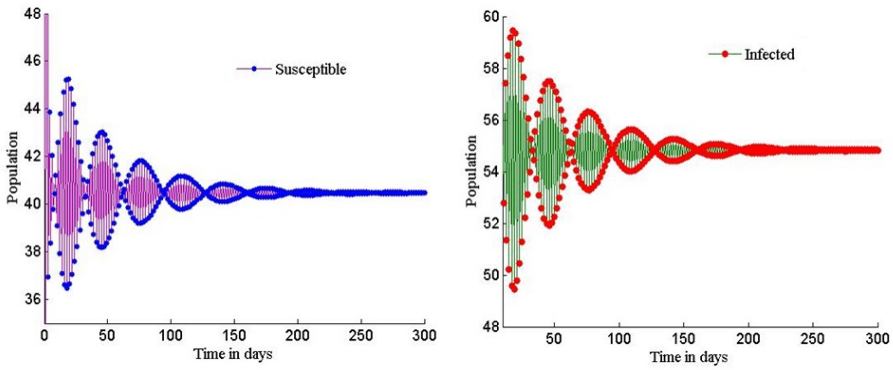

FIGURE 4. Time plots of EE point P_1 with stability $R_0 > 1$

parameter and the system undergoes periodic doubling or flip bifurcation. When $a \in (3.0, 3.36)$ there appears stability. In the range $a \in (3.36, 3.8)$ periodic-2 orbits, for $a \in (3.8, 3.9)$ periodic-4 orbits and $a \in (3.9, 3.95)$ periodic-8 orbits occur, leading to chaos for $a \in (3.95, 4.15)$. Local amplifications corresponding to figure (5) for $a \in [3.75, 4.15]$ can be seen in figure(6).

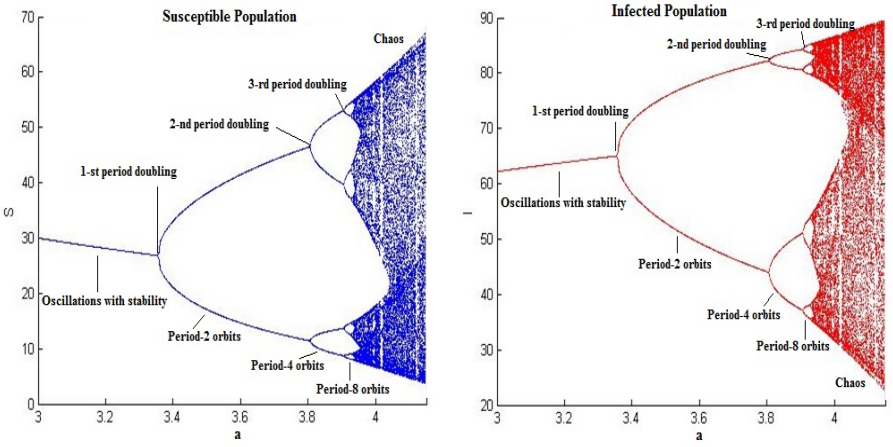

FIGURE 5. Bifurcation diagrams for susceptible and infected populations with $a \in (3.0, 4.15)$

Case (ii): Fixing parameters $N = 100$, $p = 0.0005$, $\beta = 0.8$, $a = 3.5$ and varying γ. The bifurcation diagrams of model (3.2) are plotted with recovery rate $\gamma \in (0, 0.4)$, as the bifurcation parameter. When $\gamma \in (0, 0.023)$ there appears chaos. In the range $\gamma \in (0.023, 0.03)$ periodic-8 orbits, for $\gamma \in (0.03, 0.05)$ periodic-4 orbits, for $\gamma \in (0.05, 0.12)$ periodic-2 orbits occur which is called as periodic half bifurcation. Finally for the range $\gamma \in (0.12, 0.4)$ there appears stability (see Figure-7).

Case (iii): Fixing parameters $N = 100$, $\beta = 0.8$, $a = 3.5$, $\gamma = 0.1$ and varying p.

The bifurcation diagrams of model (3.2) are plotted in the particular range of $p \in (0, 0.2)$, with proportion

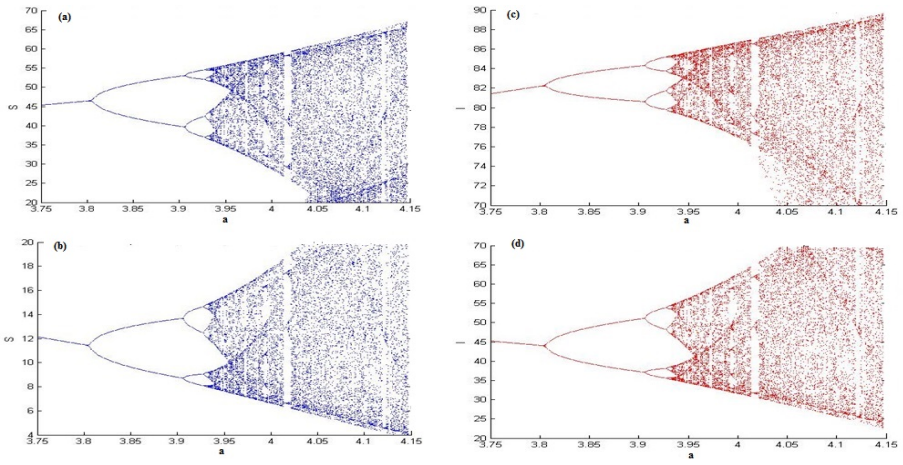

FIGURE 6. Local amplification corresponding to figure (5) for $a \in (3.75, 4.15)$

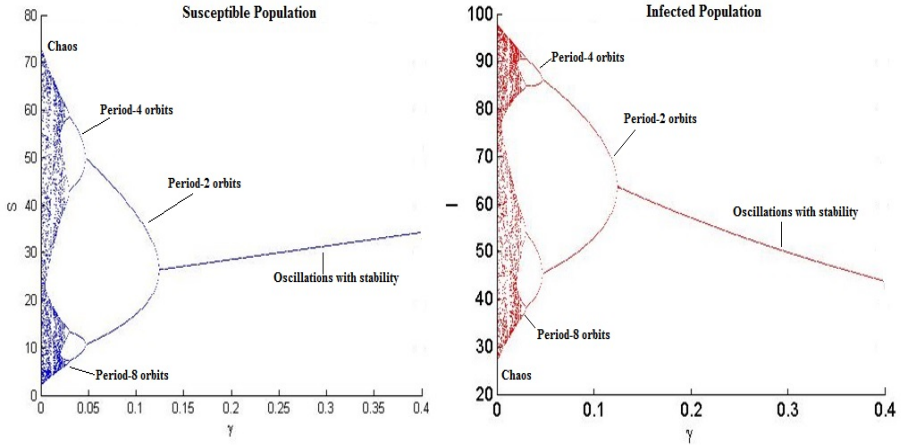

FIGURE 7. Bifurcation diagrams for susceptible and infected populations with $\gamma \in (0, 0.4)$

vaccinated rate as the bifurcation parameter. When $p \in (0, 0.02)$ there appears chaos. In the range $p \in (0.02, 0.056)$ there appears stability, in the range $p \in (0.056, 0.127)$ there appears periodic-2 orbits, in the range $p \in (0.127, 0.15)$ periodic-4 orbits, in the range $p \in (0.15, 0.16)$ periodic-8 orbits and in the range $p \in (0.16, 0.185)$ there is chaos (see Figure-8).

Case (iv): Fixing parameters $N = 100$, $p = 0.0005$, $a = 4.2$, $\gamma = 0.1$ and varying β. The bifurcation diagrams of the model (3.2) are plotted in the particular range of $\beta \in (1.0, 2.5)$, with birth rate as the bifurcation parameter. Local amplification corresponding to figure (9) for $\beta \in (1.5, 2.5)$ can be seen in Figure (10).

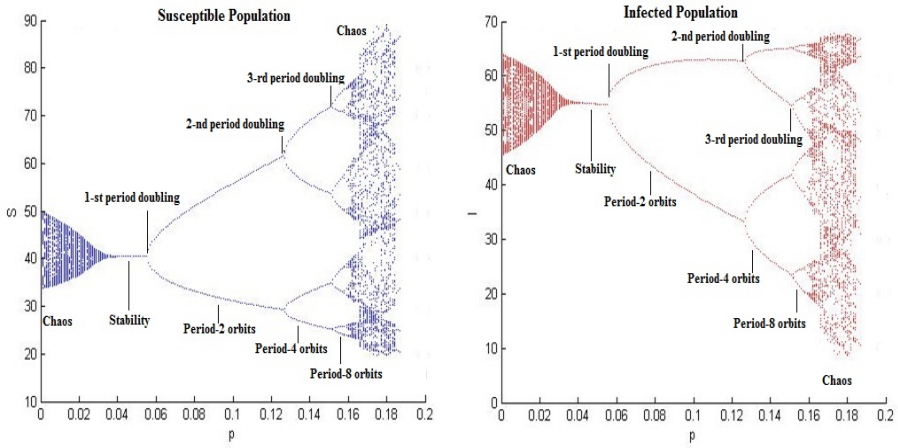

FIGURE 8. Bifurcation diagrams for susceptible and infected populations with $p \in (0, 0.2)$

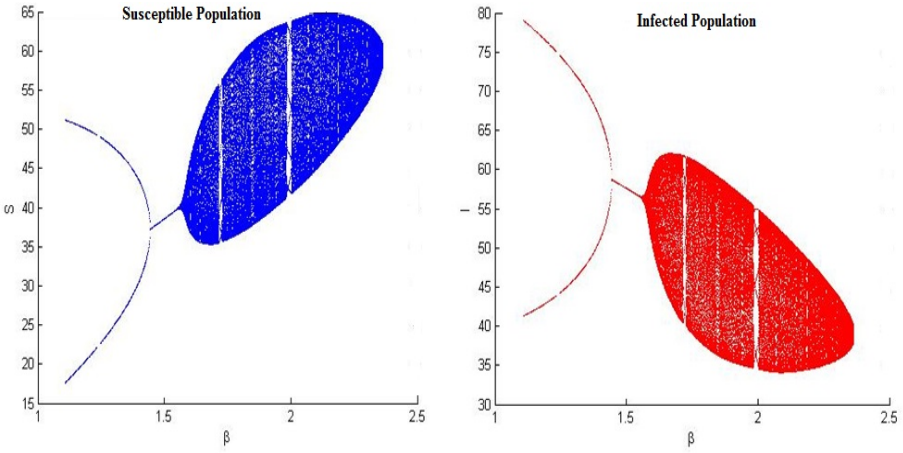

FIGURE 9. Bifurcation diagrams for susceptible and infected population with $\beta \in (1.0, 2.5)$

6. Conclusion

In this paper, we consider an discrete time SIR epidemic model with vaccination and obtained the conditions for the existence of the equilibrium points and discussed the stability of the system at DFE and EE points. Also the numerical examples ascertain the theoretical findings. Time plots and phase portraits are presented for the susceptible and infected populations for biological feasible parameters. Bifurcation diagrams and local amplifications of the same are presented. The discrete model exhibits varied and rich dynamical behavior.

Estimates on R_0 have been obtained to determine the emergence of diseases such as measles, chickenpox and smallpox [24]. We present the dynamics of the model with the effect of vaccine ([1], [2]). In Example 4.1-(a) and in Example 4.2-(a), we observe that the diseases free equilibrium is local asymptotic stable since

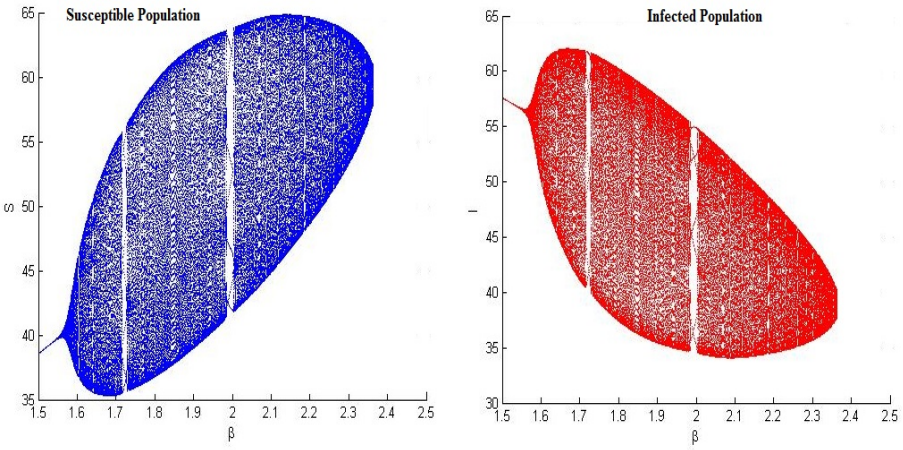

FIGURE 10. Local amplification corresponding to figure (9) for $\beta \in (1.5, 2.5)$

$R_0 < 1$ (see Figure-1) and the endemic equilibrium point is local asymptotic stable since $R_0 > 1$ (see Figure-3) by taking $p = 0.0005$ and $N = 100$. Example 4.1-(b) shows that there is a decrease in the number of susceptible persons even if the vaccination rate increases when the rate of recovery decreases and the rate of contact increases (see Figure-2). If the rate of contact increases further, Example 4.2-(b) demonstrates an increase in the number of diseases (see Figure-4). Figure 5 points the bifurcation diagrams for susceptible and infected populations with changing values of a. In Figure 6, we exhibit the local amplification corresponding to Figure 5. Figure 7 shows the bifurcation diagrams for susceptible and infected populations with changing values γ. Figure 8 displays the bifurcation diagrams for susceptible and infected populations with changing values p. Lastly, for the particular range of β, local amplification corresponding to Figure 9 which shows bifurcation diagrams are presented in Figure 10. Consequently, the lower contact rate of a has an effect of reducing the disease ([1], [24]). Also increasing of the rate of vaccination has a reinforcing effect on the reduction of the disease as other parameters remain constant.

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