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ON THE EQUIFORM DIFFERENTIAL GEOMETRY OF AW(k)-TYPE CURVES IN PSEUDO-GALILEAN 3-SPACE

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Abstract. The aim of this paper is to study AW(k)-type ($1 \leq k \leq 3$) curves according to the equiform differential geometry of the pseudo-Galilean space G_3^1. We give some geometric properties of AW(k) and weak AW(k)-type curves. Moreover, we give some relations between the equiform curvatures of these curves. Finally, examples of some special curves are given and plotted to support our main results.

1. Introduction

The geometry of space is associated with mathematical group. The idea of invariance of geometry under transformation group may imply that, on some spacetimes of maximum symmetry there should be a principle of relativity which requires the invariance of physical laws without gravity under transformations among inertial systems [1]. The theory of curves and the curves of constant curvature in the equiform differential geometry of the isotropic spaces I_3^1 , I_3^2 and the Galilean space G_3 are described in [2] and [3], respectively. The pseudo-Galilean space is one of the real Cayley-Klein spaces. It has projective signature $(0, 0, +, -)$ according to [2]. The absolute of the pseudo-Galilean space is an ordered triple $\{w, f, I\}$ where w is the ideal plane, f a line in w and I is the fixed hyperbolic involution of the points of f. In [4], from the differential

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geometric point of view, K. Arslan and A. West defined the notion of AW(k)-type submanifolds. Since then, many works have been done related to AW(k)-type submanifolds (see, for example, [5–10]). In [9], Özgür and Gezgin studied a Bertrand curve of AW(k)-type and furthermore, they showed that there is no such Bertrand curve of AW(1) and AW(3)-types if and only if it is a right circular helix. In addition, they studied weak AW(2)-type and AW(3)-type conical geodesic curves in Euclidean 3-space E^3. Besides, In 3-dimensional Galilean space and Lorentz space, the curves of AW(k)-type were investigated in [6, 8]. In [7], the authors gave curvature conditions and characterizations related to AW(k)-type curves in E^n and in [10], the authors investigated curves of AW(k)-type in the 3-dimensional null cone.

This paper is organized as follows. In section 2, the basic notions and properties of a pseudo-Galilean geometry are reviewed. In section 3, properties of the equiform geometry of the pseudo-Galilean space G_3^1 are given. Section 4 contains a study of AW(k)-type equiform Frenet curves. Finally, some examples of special curves in G_3^1 are included in section 5.

2. Basic concepts

In this section, we recall some basic notions from pseudo-Galilean geometry [11,12]. In the inhomogeneous affine coordinates for points and vectors (point pairs) the similarity group H_8 of G_3^1 has the following form

$$\begin{aligned} \bar{x} &= a + b.x, \\ \bar{y} &= c + d.x + r.\cosh\theta.y + r.\sinh\theta.z, \\ \bar{z} &= e + f.x + r.\sinh\theta.y + r.\cosh\theta.z, \end{aligned} \tag{2.1}$$

where a, b, c, d, e, f, r and θ are real numbers. Particularly, for $b = r = 1$, the group (2.1) becomes the group $B_6 \subset H_8$ of isometries (proper motions) of the pseudo-Galilean space G_3^1. The motion group leaves invariant the absolute figure and defines the other invariants of this geometry. It has the following form

$$\begin{aligned} \bar{x} &= a + x, \\ \bar{y} &= c + d.x + \cosh\theta.y + \sinh\theta.z, \\ \bar{z} &= e + f.x + \sinh\theta.y + \cosh\theta.z. \end{aligned} \tag{2.2}$$

According to the motion group in the pseudo-Galilean space, there are non-isotropic vectors $A(A_1, A_2, A_3)$ (for which holds $A_1 \neq 0$) and four types of isotropic vectors: spacelike ($A_1 = 0,\ A_2^2 - A_3^2 > 0$), timelike ($A_1 = 0,\ A_2^2 - A_3^2 < 0$) and two types of lightlike vectors ($A_1 = 0, A_2 = \pm A_3$). The scalar product of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in G_3^1 is defined by

$$\langle u, v\rangle = \begin{cases} u_1 v_1, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0, \\ u_2 v_2 - u_3 v_3 & \text{if } u_1 = 0 \text{ and } v_1 = 0. \end{cases}$$

We introduce a pseudo-Galilean cross product in the following way

$$u \times_{G_3^1} v = \begin{vmatrix} 0 & -j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $j = (0, 1, 0)$ and $k = (0, 0, 1)$ are unit spacelike and timelike vectors, respectively. Let us recall basic facts about curves in G_3^1, that were introduced in [13–15].

A curve $\gamma(s) = (x(s), y(s), z(s))$ is called an admissible curve if it has no inflection points ($\dot{\gamma} \times \ddot{\gamma} \neq 0$) and no isotropic tangents ($\dot{x} \neq 0$) or normals whose projections on the absolute plane would be lightlike vectors ($\dot{y} \neq \pm \dot{z}$). An admissible curve in G_3^1 is an analogue of a regular curve in Euclidean space [12].

For an admissible curve $\gamma(s) : I \subseteq \mathbb{R} \to G_3^1$, the curvature $\kappa(s)$ and torsion $\tau(s)$ are defined by

$$\kappa(s) = \frac{\sqrt{|\ddot{y}(s)^2 - \ddot{z}(s)^2|}}{(\dot{x}(s))^2},\ \tau(s) = \frac{\ddot{y}(s)\dddot{z}(s) - \dddot{y}(s)\ddot{z}(s)}{|\dot{x}(s)|^5 \cdot \kappa^2(s)}, \tag{2.3}$$

expressed in components. Hence, for an admissible curve $\gamma : I \subseteq \mathbb{R} \to G_3^1$ parameterized by the arc length s with differential form $ds = dx$ is given by

$$\gamma(x) = (x, y(x), z(x)). \tag{2.4}$$

The formulas (2.3) have the following form

$$\kappa(x) = \sqrt{|y''(x)^2 - z''(x)^2|},\ \tau(x) = \frac{y''(x)z'''(x) - y'''(x)z''(x)}{\kappa^2(x)}. \tag{2.5}$$

The associated trihedron is given by

$$\begin{aligned} \mathbf{e}_1 &= \gamma'(x) = (1, y'(x), z'(x)), \\ \mathbf{e}_2 &= \frac{1}{\kappa(x)}\gamma''(x) = \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \\ \mathbf{e}_3 &= \frac{1}{\kappa(x)}(0, \epsilon z''(x), \epsilon y''(x)), \end{aligned} \tag{2.6}$$

where $\epsilon = +1$ or $\epsilon = -1$, chosen by criterion $\det(e_1, e_2, e_3) = 1$, that means

$$\left| y''(x)^2 - z''(x)^2 \right| = \epsilon(y''(x)^2 - z''(x)^2).$$

The curve γ given by (2.4) is timelike (resp. spacelike) if $\mathbf{e}_2(s)$ is a spacelike (resp. timelike) vector. The principal normal vector or simply normal is spacelike if $\epsilon = +1$ and timelike if $\epsilon = -1$. For derivatives of the tangent $\mathbf{e}_1$, normal $\mathbf{e}_2$ and binormal $\mathbf{e}_3$ vector fields, the following Frenet formulas in G_3^1 hold:

$$\begin{aligned} \mathbf{e}_1'(x) &= \kappa(x)\mathbf{e}_2(x), \\ \mathbf{e}_2'(x) &= \tau(x)\mathbf{e}_3(x), \\ \mathbf{e}_3'(x) &= \tau(x)\mathbf{e}_2(x). \end{aligned} \tag{2.7}$$

3. Frenet equations according to the equiform geometry of G_3^1

This section contains some important facts about equiform geometry. The equiform differential geometry of curves in the pseudo-Galilean space G_3^1 has been described in [11]. In the equiform geometry a few specific terms will be introduced. So, let $\gamma(s) : I \to G_3^1$ be an admissible curve in the pseudo-Galilean space G_3^1, the equiform parameter of γ is defined by

$$\sigma := \int \frac{1}{\rho} ds = \int \kappa ds,$$

where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve γ. Then, we have

$$\frac{ds}{d\sigma} = \rho. \tag{3.1}$$

Let h be a homothety with center at origin and the coefficient μ. If we put $\bar{\gamma} = h(\gamma)$, then it follows

$$\bar{s} = \mu s \quad \text{and} \quad \bar{\rho} = \mu\rho,$$

where $\bar{s}$ is the arc-length parameter of $\bar{\gamma}$ and $\bar{\rho}$ is the radius of curvature of this curve. Therefore, σ is an equiform invariant parameter of γ (see [11]).

Notation 3.1. *The functions κ and τ are not invariants of the homothety group, then from* (2.3) *it follows that $\bar{\kappa} = \frac{1}{\mu}\kappa$ and $\bar{\tau} = \frac{1}{\mu}\tau$.*

Now we define the Frenet formulas of the curve γ with respect to its equiform invariant parameter σ in G_3^1. The vector

$$\mathbf{T} = \frac{d\gamma}{d\sigma},$$

is called a tangent vector of the curve γ. From (2.6) and (3.1), we get

$$\mathbf{T} = \frac{d\gamma}{ds}\frac{ds}{d\sigma} = \rho \cdot \frac{d\gamma}{ds} = \rho \cdot \mathbf{e}_1. \tag{3.2}$$

Also, the principal normal and the binormal vectors are respectively, given by

$$\mathbf{N} = \rho \cdot \mathbf{e}_2, \quad \mathbf{B} = \rho \cdot \mathbf{e}_3. \tag{3.3}$$

It is easy to show that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an equiform invariant frame of γ. On the other hand, the derivatives of these vectors with respect to σ are given by

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} \dot{\rho} & 1 & 0 \\ 0 & \dot{\rho} & \rho\tau \\ 0 & \rho\tau & \dot{\rho} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{3.4}$$

The functions $\mathcal{K} : I \to \mathbb{R}$ defined by $\mathcal{K} = \dot{\rho}$ is called the equiform curvature of the curve γ and $\mathcal{T} : I \to \mathbb{R}$ defined by $\mathcal{T} = \rho\tau = \frac{\tau}{\kappa}$ is called the equiform torsion of this curve. In the light of this, the formulas (3.4) analogous to the Frenet formulas in the equiform geometry of the pseudo-Galilean space G_3^1 can be written as

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} \mathcal{K} & 1 & 0 \\ 0 & \mathcal{K} & \mathcal{T} \\ 0 & \mathcal{T} & \mathcal{K} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{3.5}$$

The equiform parameter $\sigma = \int \kappa(s)ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of Euclidean space. Also, the function $\frac{\tau}{\kappa}$ has been already known as a conical curvature and it also has interesting geometric interpretation.

Notation 3.2. *Let $\gamma : I \to G_3^1$ be a Frenet curve in the equiform geometry of G_3^1, the following statements are true (for more details, see [11, 13]):*

(1) If $\gamma(s)$ is an isotropic logarithmic spiral in G_3^1. Then, $\mathcal{K}$ =const. $\neq 0$ and $\mathcal{T} = 0$,
(2) If $\gamma(s)$ is a circular helix in G_3^1. Then, $\mathcal{K}$ =0 and $\mathcal{T}$ =const. $\neq 0$,
(3) If $\gamma(s)$ is an isotropic circle in G_3^1. Then, $\mathcal{K}$ =0 and $\mathcal{T} = 0$.

4. AW(k)-type curves in the equiform geometry of G_3^1

Let $\gamma(s) : I \to G_3^1$ be a curve in the equiform geometry of the pseudo-Galilean space G_3^1. The curve γ is called a Frenet curve of osculating order l if its derivatives:

$$\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(l)}(s),$$

are linearly dependent and

$$\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(l+1)}(s),$$

are no longer linearly independent for all $s \in I$.

To each Frenet curve of order 3, one can associate an orthonormal 3-frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along γ, such that $\gamma'(s) = \frac{1}{\rho}\mathbf{T}$, called the equiform Frenet frame (Eqs. (3.5)).

Now, we consider equiform Frenet curves of osculating order 3 in G_3^1 and discuss some important results.

Let $\gamma(s) : I \to G_3^1$ be a Frenet curve in the equiform geometry of the pseudo-Galilean space. By the use of Frenet formulas (3.5), we obtain the higher order derivatives of γ as follows

$$\begin{aligned}
\gamma'(s) &= \frac{d\gamma}{d\sigma}\frac{d\sigma}{ds} = \frac{1}{\rho}\mathbf{T}, \\
\gamma''(s) &= \frac{1}{\rho^2}\mathbf{N}, \\
\gamma'''(s) &= \frac{1}{\rho^3}\left(-\mathcal{K}\mathbf{N}+\mathcal{T}\mathbf{B}\right), \\
\gamma''''(s) &= \frac{1}{\rho^4}[(2\mathcal{K}^2+\mathcal{T}^2 - \mathcal{K}')\mathbf{N} + (\mathcal{T}' - 3\mathcal{K}\mathcal{T})\mathbf{B}].
\end{aligned}$$

Notation 4.1. *Let us write*

$$Q_1 = \frac{1}{\rho^2}\mathbf{N}, \tag{4.1}$$

$$Q_2 = \frac{1}{\rho^3}\left(-\mathcal{K}\mathbf{N}+\mathcal{T}\mathbf{B}\right), \tag{4.2}$$

$$Q_3 = \frac{1}{\rho^4}[(2\mathcal{K}^2+\mathcal{T}^2-\mathcal{K}')\mathbf{N}+(\mathcal{T}'-3\mathcal{K}\mathcal{T})\mathbf{B}]. \tag{4.3}$$

Notation 4.2. $\gamma'(s), \gamma''(s), \gamma'''(s)$ *and* $\gamma''''(s)$ *are linearly dependent if and only if* Q_1, Q_2 *and* Q_3 *are linearly dependent.*

Definition 4.1. *[5] Frenet curves (of osculating order 3) in the equiform geometry of the pseudo-Galilean space* G_3^1 *are called curves of type:*

(1) *equiform* $AW(1)$ *if they satisfy* $Q_3 = 0$,

(2) *equiform* $AW(2)$ *if they satisfy* $\|Q_2\|^2\ Q_3 = \langle Q_3, Q_2\rangle Q_2$,

(3) *equiform* $AW(3)$ *if they satisfy* $\|Q_1\|^2\ Q_3 = \langle Q_3, Q_1\rangle Q_1$,

(4) *weak equiform* $AW(2)$ *if they satisfy*

$$Q_3 = \langle Q_3, Q_2^*\rangle\, Q_2^*, \tag{4.4}$$

(5) *weak equiform* $AW(3)$ *if they satisfy*

$$Q_3 = \langle Q_3, Q_1^*\rangle\, Q_1^*, \tag{4.5}$$

where

$$\begin{aligned} Q_1^* &= \frac{Q_1}{\|Q_1\|}, \\ Q_2^* &= \frac{Q_2 - \langle Q_2, Q_1^*\rangle\, Q_1^*}{\|Q_2 - \langle Q_2, Q_1^*\rangle\, Q_1^*\|}. \end{aligned} \tag{4.6}$$

Proposition 4.1. *Let* $\gamma : I \to G_3^1$ *be a Frenet curve* (*of osculating order* 3) *in the equiform geometry of the pseudo-Galilean space* G_3^1*, therefore*

(i) γ *is of type weak equiform* $AW(2)$ *if and only if*

$$2\mathcal{K}^2 + \mathcal{T}^2 - \mathcal{K}' = 0, \tag{4.7}$$

(ii) γ *is of type weak equiform* $AW(3)$ *if and only if*

$$\mathcal{T}' - 3\mathcal{K}\mathcal{T}(s) = 0. \tag{4.8}$$

Proof. Using Definition 4.1 and Notation 4.1, the proof will be obvious. □

Theorem 4.1. *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space G_3^1. Then γ is of type equiform $AW(1)$ if and only if*

$$-\mathcal{K}' + 2\mathcal{K}^2 + \mathcal{T}^2 = 0,$$

$$3\mathcal{K}\mathcal{T} - \mathcal{T}' = 0. \tag{4.9}$$

Proof. Since γ is of type equiform AW(1), then from (4.3), we obtain

$$\frac{1}{\rho^4}[(2\mathcal{K}^2+\mathcal{T}^2(s) - \mathcal{K}')\mathbf{N} + (\mathcal{T}' - 3\mathcal{K}\mathcal{T})\mathbf{B}] = 0.$$

As we know, the vectors **N** and **B** are linearly independent, so we can write

$$2\mathcal{K}^2+\mathcal{T}^2 - \mathcal{K}' = 0 \text{ and } \mathcal{T}' - 3\mathcal{K}\mathcal{T} = 0.$$

The converse statement is straightforward and therefore, the proof is completed. □

Theorem 4.2. *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space G_3^1. Then γ is of type equiform $AW(2)$ if*

$$\mathcal{K}^2\mathcal{T} - \mathcal{K}\mathcal{T}' + \mathcal{T}\mathcal{K}' - \mathcal{T}^3 = 0. \tag{4.10}$$

Proof. Assuming that γ is a Frenet curve in the equiform geometry of G_3^1 , then from (4.2) and (4.3), one can write

$$Q_2 = a_{11}\mathbf{N} + a_{12}\mathbf{B},$$

$$Q_3 = a_{21}\mathbf{N} + a_{22}\mathbf{B},$$

where a_{11},a_{12}, a_{21} and a_{22} are differentiable functions. Since Q_2 and Q_3 are linearly dependent, hence coefficients determinant equals zero, that is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \tag{4.11}$$

where

$$\begin{aligned} a_{11} &= \frac{-1}{\rho^3}\mathcal{K},\ a_{12} = \frac{1}{\rho^3}\mathcal{T}, \\ a_{21} &= \frac{1}{\rho^4}[-\mathcal{K}' + 2\mathcal{K}^2 + \mathcal{T}^2], \\ a_{22} &= \frac{1}{\rho^4}[-3\mathcal{K}\mathcal{T} + \mathcal{T}']. \end{aligned} \tag{4.12}$$

From (4.11) and (4.12), we obtain (4.10). □

Theorem 4.3. *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order 3) in the equiform geometry of G_3^1. Then γ is of equiform $AW(3)$-type if*

$$\mathcal{T}' - 3\mathcal{K}\mathcal{T} = 0. \tag{4.13}$$

Proof. Using Definition 4.1 and Eqs. (4.1) and (4.3), we obtain (4.13). □

5. Computational examples

We consider some examples (timelike and spacelike curves [11, 12]) which characterize equiform general (circular) helices with respect to the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in the equiform geometry of G_3^1 which satisfy some conditions of equiform curvatures (i)$\mathcal{K} = \mathcal{K}(s), \mathcal{T} = \mathcal{T}(s)$ (ii)$\mathcal{K} =$*const.* $\neq 0, \mathcal{T} =$*const.* $\neq 0$ (iii)$\mathcal{K} =$*const.* $\neq 0, \mathcal{T} =0$.

Example 5.1. *Consider the equiform* ***timelike*** *general helix $\mathbf{r} : I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ which parameterized by the arc length s with differential form $ds = dx$ is given by*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{e^{-as}}{\left(a^2 - b^2\right)^2} \left(\left(a^2 + b^2\right) \cosh(bs) + 2ab \sinh(bs)\right), \\ z(s) &= \frac{e^{-as}}{\left(a^2 - b^2\right)^2} \left(2ab \cosh(bs) + \left(a^2 + b^2\right) \sinh(bs)\right); \\ a, b &\in \mathbb{R} - \{0\}. \end{aligned}$$

The corresponding derivatives of $\mathbf{r}$ *are as follows*

$$\begin{aligned} \mathbf{r}' &= \left(1, \frac{-e^{-as}}{(a^2 - b^2)} \left(a \cosh(bs) + b \sinh(bs)\right), \frac{e^{-as}}{(b^2 - a^2)} \left(b \cosh(bs) + a \sinh(bs)\right)\right), \\ \mathbf{r}'' &= \left(0, e^{-as} \cosh(bs), e^{-as} \sinh(bs)\right), \\ \mathbf{r}''' &= \left(0, e^{-as} \left(-a \cosh(bs) + b \sinh(bs)\right), e^{-as} \left(b \cosh(bs) - a \sinh(bs)\right)\right). \end{aligned}$$

The tangent vector of $\mathbf{r}$ *has the form*

$$\begin{aligned} \mathbf{e}_1 &= (x', y', z') \\ &= \left(1, \frac{-e^{-as}}{(a^2 - b^2)} \left(a \cosh(bs) + b \sinh(bs)\right), \frac{e^{-as}}{(b^2 - a^2)} \left(b \cosh(bs) + a \sinh(bs)\right)\right), \end{aligned}$$

and the two normals (normal and binormal) of the curve are, respectively

$$\begin{aligned} \mathbf{e}_2 &= \left(0, \cosh(bs), \sinh(bs)\right), \\ \mathbf{e}_3 &= \left(0, \sinh(bs), \cosh(bs)\right); \ \det[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1. \end{aligned}$$

Therefore, the curvature and torsion of $\mathbf{r}$ *are respectively, given by*

$$\kappa = e^{-as},\ \tau = b\ .$$

From the equiform Frenet formulas, we can express the vector fields $\mathbf{T}, \mathbf{N}, \mathbf{B}$ *as follows*

$$\begin{aligned}
\mathbf{T} &= \left(e^{as}, \frac{-1}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right), \frac{1}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right)\right),\\
\mathbf{N} &= \left(0, e^{as}\cosh(bs), e^{as}\sinh(bs)\right),\\
\mathbf{B} &= \left(0, e^{as}\sinh(bs), e^{as}\cosh(bs)\right),
\end{aligned}$$

respectively. In the light of this, the equiform curvatures are given by

$$\mathcal{K} = ae^{as}, \mathcal{T} = -be^{as}.$$

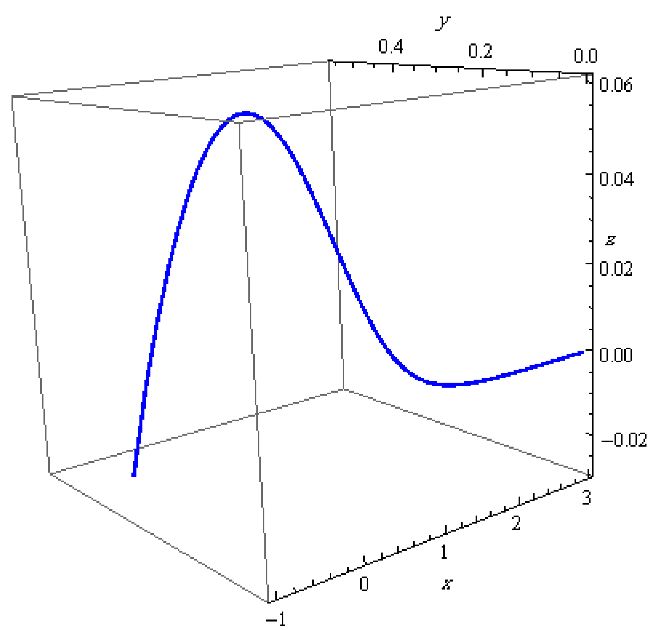

FIGURE 1. Equiform timelike general helix with $\mathcal{K} = 5e^{5s}, \mathcal{T} = -2e^{5s}$.

Example 5.2. *Let* $\mathbf{r} : I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ *be the equiform spacelike general helix, and it is given by*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{e^{-as}}{\left(a^2-b^2\right)^2}\left(2ab\cosh(bs)+\left(a^2+b^2\right)\sinh(bs)\right), \\ z(s) &= \frac{e^{-as}}{\left(a^2-b^2\right)^2}\left(\left(a^2+b^2\right)\cosh(bs)+2ab\sinh(bs)\right); \\ a,b &\in \mathbb{R}-\{0\}. \end{aligned}$$

For the coordinate functions of $\mathbf{r}$*, we have*

$$\begin{aligned} \mathbf{r}' &= \left(1, \frac{e^{-as}}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right), \frac{-e^{-as}}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right)\right), \\ \mathbf{r}'' &= \left(0, e^{-as}\sinh(bs), e^{-as}\cosh(bs)\right), \\ \mathbf{r}''' &= \left(0, e^{-as}\left(b\cosh(bs)-a\sinh(bs)\right), e^{-as}\left(b\sinh(bs)-a\cosh(bs)\right)\right). \end{aligned}$$

Also, the associated trihedron is given by

$$\begin{aligned} \mathbf{e}_1 &= \left(1, \frac{e^{-as}}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right), \frac{-e^{-as}}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right)\right), \\ \mathbf{e}_2 &= \left(0, \sinh(bs), \cosh(bs)\right), \\ \mathbf{e}_3 &= \left(0, -\cosh(bs), -\sinh(bs)\right). \end{aligned}$$

The curvature and torsion of this curve are

$$\kappa = e^{-as},\ \tau = -b\ .$$

Furthermore, the tangent, normal and binormal vector fields in the equiform geometry of G_3^1 *are obtained as follows*

$$\begin{aligned} \mathbf{T} &= \left(e^{as}, \frac{1}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right), \frac{-1}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right)\right), \\ \mathbf{N} &= \left(0, e^{as}\sinh(bs), e^{as}\cosh(bs)\right), \\ \mathbf{B} &= \left(0, -e^{as}\cosh(bs), -e^{as}\sinh(bs)\right), \end{aligned}$$

respectively.

The equiform curvatures of $\mathbf{r}$ *are*

$$\mathcal{K} = ae^{as}, \mathcal{T} = -be^{as}.$$

Example 5.3. *Consider the equiform timelike circular helix* $\mathbf{r} : I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ *is given by*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

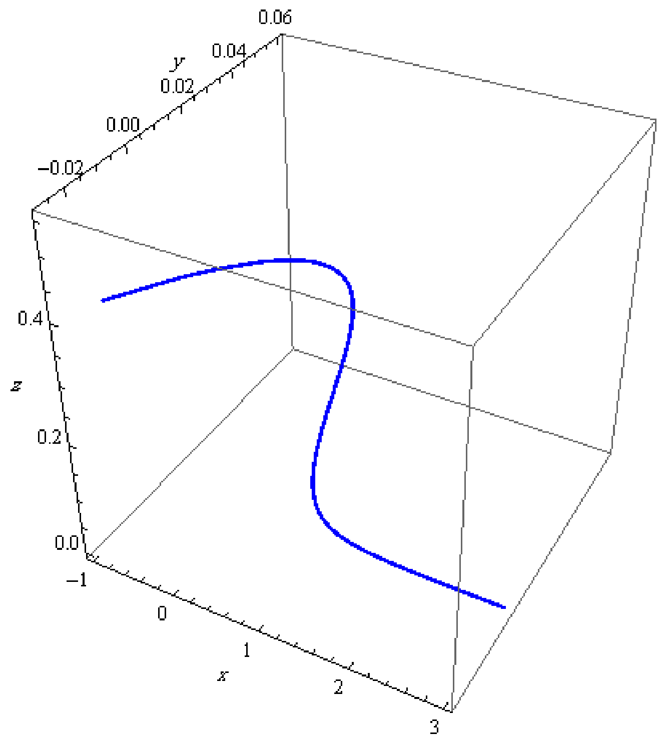

FIGURE 2. Equiform spacelike general helix with $\mathcal{K} = 5e^{5s}, \mathcal{T} = -2e^{5s}$.

where

$$
\begin{aligned}
x(s) &= s,\\
y(s) &= \frac{a^3 s}{b\left(b^2-a^2\right)}\left(b\sinh\left(\frac{b}{a}\ln(as)\right) - a\cosh\left(\frac{b}{a}\ln(as)\right)\right),\\
z(s) &= \frac{a^3 s}{b\left(b^2-a^2\right)}\left(b\cosh\left(\frac{b}{a}\ln(as)\right) - a\sinh\left(\frac{b}{a}\ln(as)\right)\right);\\
a, b &\in \mathbb{R} - \{0\}.
\end{aligned}
$$

For this curve, the equiform vector fields are obtained as follows

$$
\begin{aligned}
\mathbf{T} &= \left(\frac{s}{a}, \frac{as}{b}\cosh\left(\frac{b}{a}\ln(as)\right), \frac{as}{b}\sinh\left(\frac{b}{a}\ln(as)\right)\right),\\
\mathbf{N} &= \left(0, \frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right), \frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right)\right),\\
\mathbf{B} &= \left(0, \frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right), \frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right)\right),
\end{aligned}
$$

respectively.

It follows that

$$\mathcal{K} = \frac{1}{a}, \mathcal{T} = \frac{-b}{a^2}.$$

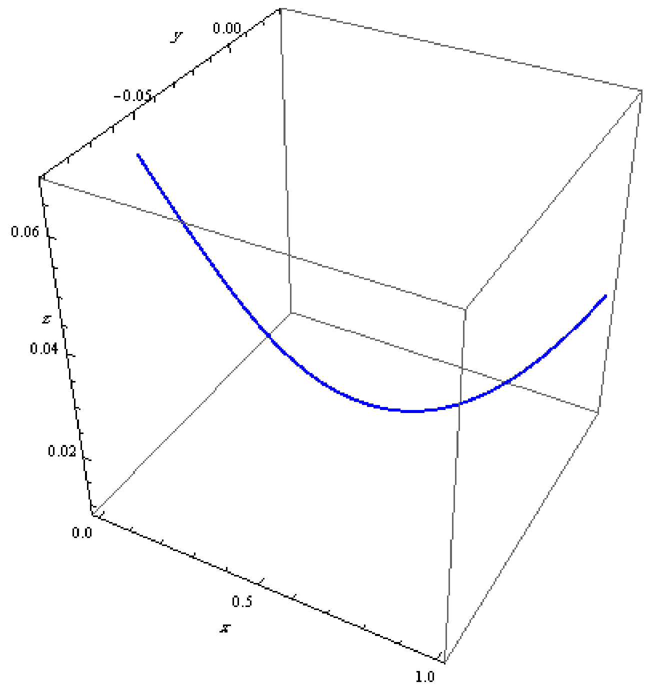

FIGURE 3. Equiform timelike circular helix with $\mathcal{K}=\frac{1}{2},\mathcal{T}=\frac{-5}{4}$.

Example 5.4. *Let the equiform spacelike circular helix $\mathbf{r}: I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ be*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{a^3 s}{b\,(b^2-a^2)}\left(b\cosh\left(\frac{b}{a}\ln(as)\right) - a\sinh\left(\frac{b}{a}\ln(as)\right)\right), \\ z(s) &= \frac{a^3 s}{b\,(b^2-a^2)}\left(b\sinh\left(\frac{b}{a}\ln(as)\right) - a\cosh\left(\frac{b}{a}\ln(as)\right)\right); \\ a, b &\in \mathbb{R}-\{0\}. \end{aligned}$$

Here, the equiform differential vectors respectively, are as follows

$$\begin{aligned} \mathbf{T} &= \left(\frac{s}{a}, \frac{as}{b}\sinh\left(\frac{b}{a}\ln(as)\right), \frac{as}{b}\cosh\left(\frac{b}{a}\ln(as)\right)\right), \\ \mathbf{N} &= \left(0, \frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right), \frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right)\right), \\ \mathbf{B} &= \left(0, -\frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right), -\frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right)\right). \end{aligned}$$

Equiform curvature and equiform torsion are calculated as follows

$$\mathcal{K} = \frac{1}{a}, \mathcal{T} = \frac{b}{a^2}.$$

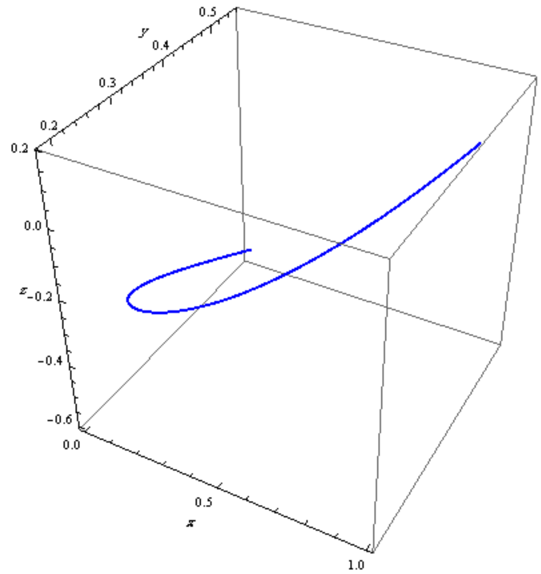

FIGURE 4. Equiform spacelike circular helix with $\mathcal{K} = \frac{1}{3}, \mathcal{T} = \frac{4}{9}$.

Example 5.5. *Let* $\mathbf{r} : I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ *be a equiform timelike isotropic logarithmic spiral which parameterized by the arc length* s *with differential form* $ds = dx$, *and is given by*

$$\mathbf{r}(x) = (x, y(x), 0),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{as+b}{a^2}\left(\ln(as+b) - 1\right), \\ z(s) &= 0; \\ a, b &\in \mathbb{R} - \{0\}. \end{aligned}$$

For this curve, we get

$$\begin{aligned} \mathbf{r}' &= \left(1, \frac{\ln(as+b)}{a}, 0\right), \\ \mathbf{r}'' &= \left(0, \frac{1}{as+b}, 0\right), \\ \mathbf{r}''' &= \left(0, \frac{-a}{(as+b)^2}, 0\right), \end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_1 &= \left(1, \frac{\ln(as+b)}{a}, 0\right), \\
\mathbf{e}_2 &= (0,1,0), \\
\mathbf{e}_3 &= (0,0,1); \ \kappa = \frac{1}{as+b}, \ \tau = 0.
\end{aligned}$$

In this case, equiform Frenet vectors and equiform curvatures are as follows

$$\begin{aligned}
\mathbf{T} &= \left(as+b, \frac{(as+b)\ln(as+b)}{a}, 0\right), \\
\mathbf{N} &= (0, as+b, 0), \\
\mathbf{B} &= (0,0,as+b), \ \mathcal{K} = a, \mathcal{T} = 0.
\end{aligned}$$

respectively.

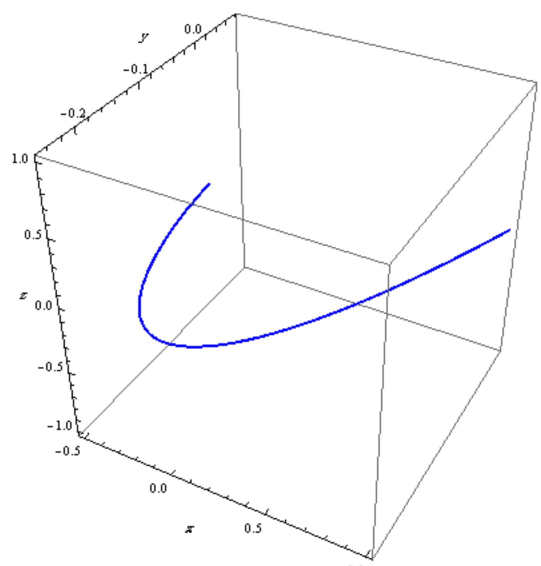

FIGURE 5. Equiform timelike isotropic logarithmic spiral with $\mathcal{K} = 2, \mathcal{T} = 0$.

From aforementioned calculations, according to (Proposition 4.2 and Theorems $4.1 - 4.3$), the first four examples are not characterize curves of equiform AW(1), weak equiform AW(2) or weak equiform AW(3)-types. On the other hand, the last example shows that the curve is of equiform AW(2) and AW(3)-types and it is not of equiform AW(1)-type. Also, this curve is of weak equiform AW(2) and not of weak equiform AW(3)-types.

6. Conclusion

In this paper, we have considered some special curves of equiform AW(k)-type of the pseudo-Galilean 3-space. Also, using the equiform curvature conditions of these curves, the necessary and sufficient conditions for them to be equiform AW(k) and weak equiform AW(k)-types are obtained. Furthermore, some examples to support our main results are given and plotted.

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