International Journal of Analysis and Applications
Volume 18, Number 4 (2020), 633-643
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-18-2020-633

PAIR $(\mathcal{F}, h)$ UPPER CLASS ON SOME FIXED POINT RESULTS IN PROBABILISTIC MENGER SPACE

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Abstract. In this paper, we define the concept of $(\mathcal{F}, h, \alpha, \beta, \psi)$- contractive mappings in a probabilistic Menger space, which generalizes some previous related concepts. Also, we investigate the existence of fixed points for such mappings. Some examples are given to support the obtained results.

1. Introduction and mathematical preliminaries

The study of fixed points of mappings in a Menger PM-space satisfying certain contractive conditions has been at the center of vigorous research activity. Menger PM-space were introduced in 1942 by Menger [9]. Afterwards the study of these spaces was performed by Schweizer and Sklar [8] and many others, [3–6]. In 1984 Khan et al. introduced the concept of altering distance function [15]. A φ-function is the extension of altering distance function and has been worked by many authors, [16], [17].The concepts of $\alpha - \psi-$type contractive and $\alpha-$ admissible mappings were introduced by Gopal et. al. [7], who also established some fixed point theorems for these mappings in complete Menger spaces. After that, Shams and Jafari generalized this concept to (α, β, ψ)-contractive and $\alpha - \beta-$admissible mappings and proved some fixed point theorems for such maps [14].

Received February 10th, 2020; accepted March 5th, 2020; published May 26th, 2020.

2010 *Mathematics Subject Classification.* 47H10, 47H09.

Key words and phrases. Fixed point; Contractive mapping; Probabilistic Menger space; T-norm.

In this paper, we give a generalization of the concepts discussed in [1,14]. Thus we introduce the concept of $(\mathcal{F}, h, \alpha, \beta, \psi)$- contractive mapping in Menger PM-space, and establish corresponding fixed point theorems for such contractive mappings which are based on the mentioned generalized notion of such a contractive mapping. In particular, the presented theorems extend, generalize and improve the results in [1,14]. Also, some examples are given to support the obtained results.

We first bring notion, definitions and known results, which are related to our work. For more details, we refer the reader to [2].

We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of non-negative real numbers and $\mathbb{N}$ the set of positive integers

Definition 1.1. *A distribution function is a function $F : (-\infty, \infty) \to [0,1]$, that is non-decreasing and left continuous on $\mathbb{R}$. Moreover, $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. The set of all the distribution functions is denoted by D, and the set of those distribution functions such that $F(0) = 0$ is denoted by D^+. We will denote the specific Heaviside distribution function by:*

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Definition 1.2. *A binary operation $T : [0,1]\times[0,1] \to [0,1]$ is a continuous t-norm if the following conditions hold:*

(a) T is commutative and associative,

(b) T is continuous,

(c) $T(a,1) = a$ for all $a \in [0,1]$,

(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$, for $a,b,c,d \in [0,1]$.

The following are three basic continuous t-norms.

(i) The minimum t-norm, say T_M, defined by $T_M(a,b) = \min\{a,b\}$.

(ii) The product t-norm, say T_p, defined by $T_p(a,b) = a.b$.

(iii)The Lukasiewicz t-norm, say T_L, defined by $T_L(a,b) = \max\{a+b-1,0\}$.

These t- norms are related in the following way: $T_L \leq T_P \leq T_M$.

Definition 1.3. *A Menger PM-space is a triple (X,F,T), where X is a nonempty set, T is a continuous t-norm, and F is a mapping from $X \times X$ into D^+ such that the following conditions hold:*

(PM1) $F_{x,y}(t) = H(t)$ *if and only if* $x = y$,

(PM2) $F_{x,y}(t) = F_{y,x}(t)$

(PM3) $F_{x,y}(t+s) \geq T(F_{x,z}(t), F_{z,y}(s))$ *for all* $x,y,z \in X$ *and* $s,t \geq 0$

Definition 1.4. *Let (X, F, T) be a Menger PM-space. Then*

(i) *A sequence x_n in X is said to be convergent to x if, for every $\epsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer N such that $F_{x_n x}(\epsilon) > 1 - \lambda$, whenever $n \geq N$.*

(ii) *A sequence x_n in X is called Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N scuh that $F_{x_n x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$.*

(iii) *A Menger PM-space is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.*

(iv) *A sequence x_n is called G-Cauchy if $\lim_{n\to\infty} F_{x_n x_{n+m}}(t) = 1$, for each $m \in \mathbb{N}$ and $t > 0$.*

(v) *The space (X, F, T) is called G-complete if every G-Cauchy sequence in X is convergent.*

It follows immediately that a Cauchy sequence is a G-Cauchy sequence. The converse is not always true. This has been established by an example in [11].

According to [8], the (ϵ, λ)-topology in Menger PM-space (X, F, T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by

$$N_x = N_x(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1),$$

where

$$N_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The (ϵ, λ) -topology is a Hausdorff topology. In this topology, a function f is continuous in $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$, for every sequence $x_n \to x_0$, as $n \to \infty$.

Definition 1.5. *[10] A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Φ-function if it satisfies the following conditions:*

(i) *$\phi(t) = 0$ if and only if $t = 0$,*

(ii) *$\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,*

(iii) *ϕ is left continuous in $(0, \infty)$,*

(iv) *ϕ is continuous at 0.*

In the sequel, the class of all Φ-functions will be denoted by Φ. Also we denote by Ψ the class of all continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$ and $\psi^n(a_n) \to 0$, whenever $a_n \to 0$ as $n \to \infty$.

Theorem 1.1. *[1] Let (X, F, T) be a G-complete Menger PM-space and $f : X \to X$ be a mapping satisfying the following inequality*

$$\frac{1}{F_{fx,fy}(\varphi(ct))} - 1 \leq \psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right) \tag{1.1}$$

where $x, y \in X, c \in (0, 1)$, $\varphi \in \Phi, \psi \in \Psi$ and $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$. Then f has a unique fixed point.

Definition 1.6. *[14] Let (X, F, T) be a Menger PM-space and $f : X \to X$ be a given mapping and $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$, be two functions, we say that f is $\alpha - \beta$-admissible if*

(i) *For all $x, y \in X$ and for all $t > 0$, $\alpha(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1$,*

(ii) *For all $x, y \in X$ and for all $t > 0$, $\beta(x, y, t) \leq 1 \Rightarrow \beta(fx, fy, t) \leq 1$.*

2. Fixed point theorems for $(\mathcal{F}, h, \alpha, \beta, \psi)$-contractive mappings

In this section, we state some allied definitions and results which are needed for the development of the present topic. Also we introduce the notion of $(\mathcal{F}, h, \alpha, \beta, \psi)$-contractive mappings in Menger PM-spaces and prove some fixed point theorems for such a contractive mappings.

Definition 2.1. *[12, 13] We say that the function $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if $x \geq 1 \Longrightarrow h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.*

Example 2.1. *[12, 13] Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x, y) = (y + l)^x, l > 1$;

(b) $h(x, y) = (x + l)^y, l > 1$;

(c) $h(x, y) = x^n y,\ n \in \mathbb{N}$;

(d) $h(x, y) = y$;

(e) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y,\ n \in \mathbb{N}$;

(f) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1,\ n \in \mathbb{N}$

for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 2.2. *[12, 13] Let $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type I, if h is a function of subclass of type I and:*

(i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$,

(ii) $h(1, y) \leq \mathcal{F}(1, t) \Longrightarrow y \leq t$ *for all $t, y \in \mathbb{R}^+$.*

Example 2.2. *[12, 13] Define $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x, y) = (y + l)^x, l > 1$ *and* $\mathcal{F}(s, t) = st + l$;

(b) $h(x, y) = (x + l)^y, l > 1$ *and* $\mathcal{F}(s, t) = (1 + l)^{st}$;

(c) $h(x, y) = x^m y,\ m \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;

(d) $h(x, y) = y$ *and* $\mathcal{F}(s, t) = t$;

(d) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y,\ n \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;

(e) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1,\ n \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 2.3. *[12, 13] We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II,*

if $x, y \geq 1 \Longrightarrow h(1,1,z) \leq h(x,y,z)$ for all $z \in \mathbb{R}^+$.

Example 2.3. *[12, 13] Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x,y,z) = (z+l)^{xy}, l > 1;$

(b) $h(x,y,z) = (xy+l)^{z}, l > 1;$

(c) $h(x,y,z) = z;$

(d) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N};$

(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}$

for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 2.4. *[12, 13] Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type II, if h is a subclass of type II and:*

(i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s,t) \leq \mathcal{F}(1,t),$

(ii) $h(1,1,z) \leq \mathcal{F}(s,t) \Longrightarrow z \leq st$ *for all* $s,t,z \in \mathbb{R}^+$.

Example 2.4. *[12, 13] Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x,y,z) = (z+l)^{xy}, l > 1, \mathcal{F}(s,t) = st + l;$

(b) $h(x,y,z) = (xy+l)^{z}, l > 1, \mathcal{F}(s,t) = (1+l)^{st};$

(c) $h(x,y,z) = z, F(s,t) = st;$

(d) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N}, \mathcal{F}(s,t) = s^p t^p$

(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}, \mathcal{F}(s,t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type II.

Now we introduce the following definition that is a generalization of inequality (1.1) introduced in [1], see example 2.6.

Definition 2.5. *Let (X, F, T) be a PM-space and $f : X \to X$ be a given mapping. We say that f is an $(\mathcal{F}, h, \alpha, \beta, \psi)$- contractive mapping if there exist two functions $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ and $\psi \in \Psi$ satisfying the following inequality*

$$\mathcal{F}\left(\beta(x,y,t), \psi(\frac{1}{F_{x,y}(\varphi(t))} - 1)\right) \geq h\left(\alpha(x,y,t), \left(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1)\right)\right) \tag{2.1}$$

for all $x, y \in X$ and for all $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$, where $c \in (0,1)$ and $\varphi \in \Phi$.

Remark 1 If $\alpha(x,y,t) = 1$ and $\beta(x,y,t) = 1$ for all $x, y \in X$ and for all $t > 0$, the condition (2.1) reduce to condition (1.1), but the converse is not necessarily true, (see example 2.6).

The following Theorem shows that a $(\mathcal{F}, h, \alpha, \beta, \psi)$ contractive mapping under which conditions has fixed points. Also, examples 2.5 and 2.6 show that this theorem extends the previous results in [1, 14].

Theorem 2.1. *Let (X, F, T) be a G-complete Menger PM-space and $f : X \to X$ be a $(\mathcal{F}, h, \alpha, \beta, \psi)$ contractive mapping satisfying the following conditions:*

(i) *f is $\alpha - \beta$-admissible.*

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$, for all $t > 0$.*

(iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, t) \geq 1$ and $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.*

Then f has a fixed point, i.e, there exists a point $u \in X$ such that $fu = u$.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$. Define a sequence $\{x_n\}$ in X so that $x_{n+1} = fx_n$, for all $n \in \mathbb{N}$. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then, by using the fact that f is $\alpha - \beta$-admissible, we write

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \Rightarrow \alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \geq 1.$$

Similarly we write

$$\beta(x_0, fx_0, t) = \beta(x_0, x_1, t) \leq 1 \Rightarrow \beta(x_1, x_2, t) = \beta(fx_0, fx_1, t) \leq 1.$$

By induction, it follows that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$, for all $t > 0$. From the properties of function φ, we can find $t > 0$ such that $F_{x_0 x_1}(\varphi(t)) > 0$. Thuse by applying (2.1), we have

$$\begin{aligned} \mathcal{F}\left(1, \psi(\frac{1}{F_{x_0,x_1}(\varphi(t))} - 1)\right) &\geq \mathcal{F}\left(\beta(x_0, x_1, t), \psi(\frac{1}{F_{x_0,x_1}(\varphi(t))} - 1)\right) \\ &\geq h\left(\alpha(x_0, x_1, t), (\frac{1}{F_{fx_0,fx_1}(\varphi(ct))} - 1\right) \\ &\geq h\left(1, (\frac{1}{F_{fx_0,fx_1}(\varphi(ct))} - 1)\right) \\ &\Longrightarrow \end{aligned}$$

$$\frac{1}{F_{fx_0,fx_1}(\varphi(ct))} - 1 \leq \psi(\frac{1}{F_{x_0,x_1}(\varphi(t))} - 1) \tag{2.2}$$

Repeating the above procedure successively n times, we obtain

$$\mathcal{F}\left(1, \psi^n(\frac{1}{F_{x_0,x_1}(\varphi(\frac{ct}{c^{n-1}}))} - 1)\right) \geq h\left(1, (\frac{1}{F_{x_n,x_{n+1}}(\varphi(ct))} - 1)\right).$$

Hence we have

$$\frac{1}{F_{x_n,x_{n+1}}(\varphi(ct))} - 1 \leq \psi^{n-1}\left(\frac{1}{F_{x_1,x_2}(\varphi(\frac{ct}{c^{n-1}}))} - 1\right).$$

Rewrite this sentence for $n > r$,

$$\frac{1}{F_{x_n,x_{n+1}}(\varphi(c^r t))} - 1 \leq \psi^{n-r}\left(\frac{1}{F_{x_n,x_{n+1}}(\varphi(\frac{c^r t}{c^{n-r}}))} - 1\right). \tag{2.3}$$

Since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$, we have from (2.3), for all $r > 0$

$$F_{x_n,x_{n+1}}(\varphi(c^r t)) \to 1. \tag{2.4}$$

Now let $\epsilon > 0$ be given, then by virtue of the properties of φ, we can find $r > 0$ such that $\varphi(c^r t) < \epsilon$. Then it follows from (2.4) that $F_{x_n,x_{n+1}}(\epsilon) \to 1$, as $n \to \infty$ for every $\epsilon > 0$. On the other hand, we know that

$$F_{x_n,x_{n+p}}(\epsilon) \geq T(F_{x_n,x_{n+1}}(\frac{\epsilon}{p}), T(F_{x_{n+1},x_{n+2}}(\frac{\epsilon}{p}), ..., (F_{x_{n+p-1},x_{n+p}}(\frac{\epsilon}{p}))...).$$

Thus, letting $n \to \infty$, we have for any integer p, $F_{x_n,x_{n+p}}(\epsilon) \to 1$, as $n \to \infty$ for every $\epsilon > 0$. Hence $\{x_n\}$ is a G-cauchy sequence

As (X, F, T) is G-complete, $\{x_n\}$ is convergent and hence $x_n \to u$ as $n \to \infty$ for some $u \in X$. Again

$$F_{fu,u}(\epsilon) \geq T\left(F_{fu,x_{n+1}}(\frac{\epsilon}{2}), F_{x_{n+1},u}(\frac{\epsilon}{2})\right). \tag{2.5}$$

Using the properties of φ-function, we can find a $t_2 > 0$ such that $\varphi(t_2) < \frac{\epsilon}{2}$. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $F_{x_n,u}(\varphi(t_2)) > 0$. Then, for $n > n_0$, we write

$$\begin{aligned}
\mathcal{F}\left(1, \psi(\frac{1}{F_{x_n,u}(\varphi(\frac{t_2}{c}))} - 1)\right) \quad &\geq \quad \mathcal{F}\left(\beta(x_n, u, \varphi(t_2)), \psi(\frac{1}{F_{x_n,u}(\varphi(\frac{t_2}{c}))} - 1)\right) \\
&\geq \quad h\left(\alpha(x_n, u, \varphi(t_2)), (\frac{1}{F_{fx_n,fu}(\varphi(t_2))} - 1)\right) \\
&\geq \quad h\left(1, \frac{1}{F_{fx_n,fu}(\varphi(t_2))} - 1\right) \\
&\Longrightarrow
\end{aligned}$$

$$\frac{1}{F_{x_{n+1},fu}(\frac{\epsilon}{2})} - 1 \leq \frac{1}{F_{fx_n,fu}(\varphi(t_2))} - 1 \leq \psi(\frac{1}{F_{x_n,u}(\varphi(\frac{t_2}{c}))} - 1).$$

Making $n \to \infty$, utilizing $\psi(0) = 0$ and the continuity of ψ, we obtain

$$F_{x_{n+1},fu}(\frac{\epsilon}{2}) \to 1 \quad as\ n \to \infty. \tag{2.6}$$

Passing to the limit for $n \to \infty$ in (2.5), from (2.6), the continuity of T and the fact that $x_n \to u$ as $n \to \infty$, we have $F_{fu,u}(\epsilon) = 1$ for every $\epsilon > 0$. Hence $u = fu$. □

The uniqueness of the fixed point is proved in the next theorem.

Theorem 2.2. *With the same hypotheses of Theorem 2.1, if for all $x \in X$ and for all $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\beta(x, z, t) \leq 1$, then f has a unique fixed point.*

Proof. Let $u, v \in X$ be such that $fu = u$ and $fv = v$. From hypotheses there exists $z \in Y$ such that $\alpha(u, z, t) \geq 1$ and $\alpha(v, z, t) \geq 1$, $\beta(u, z, t) \leq 1$ and $\beta(v, z, t) \leq 1$. Since f is $\alpha - \beta$-admissible, we get

$$\alpha(u, f^n z, t) \geq 1,\ \alpha(v, f^n z, t) \geq 1,\ \beta(u, f^n z, t) \leq 1,\ \beta(v, f^n z, t) \leq 1, \tag{2.7}$$

for all $t > 0$ and $n \in \mathbb{N}$. So by using (2.1) and (2.7), we obtain

$$\begin{aligned} \mathcal{F}\left(1, \psi(\frac{1}{F_{u,f^{n-1}z}(\varphi(t))} - 1\right) &\geq \mathcal{F}\left(\beta(u, f^{n-1}z, t), \psi(\frac{1}{F_{u,f^{n-1}z}(\varphi(t))} - 1)\right) \\ &\geq h\left(\alpha(u, f^{n-1}z, t), \frac{1}{F_{fu,f(f^{n-1}z)}(\varphi(ct))} - 1\right) \\ &\geq h\left(1, \frac{1}{F_{fu,f(f^{n-1}z)}(\varphi(ct))} - 1\right) \end{aligned}$$

This implies that

$$(\frac{1}{F_{u,f^n z}(\varphi(ct))} - 1) \leq \psi^n(\frac{1}{F_{u,z}(\varphi(\frac{t}{c^n}))} - 1).$$

Finally, making $n \to \infty$, we obtain $f^n z \to u$. A similar argument shows that for all $n \in \mathbb{N}$, $f^n z \to v$ as $n \to \infty$. Now, the uniqueness of the limit gives us $u = v$ and hence the proof is complete. □

Corollary 2.1. *[14] Let (X, F, T) be a G-complete Menger PM-space and $f : X \to X$ be a mapping satisfying the following conditions:*

(i) *for all $x, y \in X$ and for all $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$, where $c \in (0, 1)$ and $\varphi \in \Phi$*

$$\alpha(x, y, t)(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1) \leq \beta(x, y, t)\psi(\frac{1}{F_{x,y}(\varphi(t))} - 1) \tag{2.8}$$

(ii) *f is $\alpha - \beta$-admissible.*

(iii) *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$, for all $t > 0$.*

(iv) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, t) \geq 1$ and $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.*

Then f has a fixed point, i.e, there exists a point $u \in X$ such that $fu = u$

The following examples show the usefulness of definition 2.5 proposed in this paper to extend previous results in [1, 14].

Example 2.5. *Let $X = \mathbb{R}, T(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x,y \in X$ and for all $t > 0$. Clearly (X,F,T) is a G-complete Menger PM-space. Define the mapping $f : X \to X$ by*

$$fx = \begin{cases} \frac{1}{2} & x \in [0,1) \\ 1 & x = 1 \\ 2 & otherwise \end{cases}$$

and the functions $\alpha, \beta : X \times X \times (0,\infty) \to [0,\infty)$ by

$$\alpha(x,y,t) = \begin{cases} 1 & x,y \in [0,1] \\ \frac{1}{2} & otherwise, \end{cases}$$

$$\beta(x,y,t) = \begin{cases} 4 & x,y \in [0,1) \\ 1 & x = y = 1 \\ \frac{1}{3} & otherwise. \end{cases}$$

We define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $h(x,y) = (y+l)^x$ and $\mathcal{F}(s,t) = st + l$. Let $c = \frac{1}{2}$. If we define $\varphi, \psi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \psi(t) = t$, then the mapping f satisfies the hypotheses of theorem 2.1. Let $x,y \in X$ be such that $\alpha(x,y,t) \geq 1$ and $\beta(x,y,t) \leq 1$ for all $t > 0$, then by definition of f and α and β, we have $\alpha(fx,fy,t) = 1$ and $\beta(fx,fy,t) \leq 1$, that is, f is a $\alpha - \beta$-admissible. Also for $x_0 = 1$ we have $\alpha(1,f(1),t) = 1$ and $\beta(1,f(1),t) = 1$. Suppose both x,y are in $[0,1)$, then $\alpha(x,y,t) = 1$ and $\beta(x,y,t) = 4$ and so by definition of $\mathcal{F}$ and h we have $4t + l \geq l$, so inequality (2.1) holds. If $x = y = 1$, then $t + l \geq l$, so inequality (2.1) holds. Also in other cases inequality (2.1) hold. Next, let $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$, for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \to x$ as $n \to \infty$, this implies $x_n = 1$ and so $\alpha(x_n, x, t) = 1$ and $\beta(x_n, x, t) = 1$, for all $n \in \mathbb{N}$ and for all $t > 0$. Hence we conclude that all the conditions of theorem 2.1 hold and so, f has three fixed points $x = 1$, $x = \frac{1}{2}$ and $x = 2$. We show that corollary 2.1 is not applicable in this case. Consider $x = 1$ and $y = 2$, then by applying inequality (2.8) we have $9 \leq c$, which gives a contradiction to the fact that $c \in (0,1)$.

Example 2.6. *Let $X = \mathbb{R}, T(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x,y \in X$ and for all $t > 0$. Clearly (X,F,T) is a G-complete Menger PM-space. Define the mapping $f : X \to X$ by*

$$fx = \begin{cases} \frac{1}{5} & x \in [0, \frac{1}{4}] \\ \frac{x^2}{3} & otherwise \end{cases}$$

and the functions $\alpha, \beta : X \times X \times (0,\infty) \to [0,\infty)$ by

$$\alpha(x,y,t)=\begin{cases}1 & x,y\in[0,\frac{1}{4}]\\ \frac{2}{3} & otherwise,\end{cases}$$

$$\beta(x,y,t)=\begin{cases}\frac{1}{2} & x,y\in[0,\frac{1}{4}]\\ \\ 2 & otherwise.\end{cases}$$

We define $h\colon \mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$, $\mathcal{F}\colon \mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ *by* $h(x,y)=xy$ *and* $\mathcal{F}(s,t)=st$. *Let* $c=\frac{1}{2}$. *If we define* $\varphi,\psi:[0,\infty)\to[0,\infty)$ *by* $\varphi(t)=\psi(t)=t$, *then the mapping* f *satisfies the hypotheses of theorem 2.1. It is easy to show that* f *is* $\alpha-\beta$*-admissible. Also for* $x_0=\frac{1}{4}$, *we have* $\alpha(\frac{1}{4},f(\frac{1}{4}),t)=1$ *and* $\beta(\frac{1}{4},f(\frac{1}{4}),t)=\frac{1}{2}$.

Next, let $\{x_n\}$ *is a sequence such that* $\alpha(x_n,x_{n+1},t)\geq 1$ *and* $\beta(x_n,x_{n+1},t)\leq 1$ *for all* $n\in\mathbb{N}$, $t>0$ *and* $x_n\to x$ *as* $n\to\infty$, *this implies that* $\{x_n\},x\in[0,\frac{1}{4}]$. *So by definition of* α,β *this implies that* $\beta(x_n,x,t)\leq 1$ *and* $\alpha(x_n,x,t)\geq 1$ *for all* $n\in\mathbb{N}$ *and for all* $t>0$. *The other conditions are the same as example 2.5. Hence we conclude that all the conditions of theorem 2.1 hold and so,* f *has two fixed points* $x=3$ *and* $x=\frac{1}{5}$. *We show that theorem 1.1 is not applicable in this case. Consider* $x=1$, *and* $y=2$, *then by applying inequality* (1.1) *we givea a contradiction to the fact that* $c\in(0,1)$.

Acknowledgements: The fourth author thanks the Basque Government for Grant IT1207-19.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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