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$*-$CONFORMAL $\eta-$RICCI SOLITONS ON $\alpha-$COSYMPLECTIC MANIFOLDS

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Abstract. The object of this paper is to study $*-$conformal $\eta-$Ricci solitons on $\alpha-$cosymplectic manifolds. First, $\alpha-$cosymplectic manifolds admitting $*-$conformal $\eta-$Ricci solitons satisfying the conditions $R(\xi,.)\cdot S$ and $S(\xi,.)\cdot R=0$ are being studied. Further, $\alpha-$cosymplectic manifolds admitting $*-$conformal $\eta-$Ricci solitons satisfying certain conditions on the $\mathcal{M}-$projective curvature tensor are being considered and obtained several interesting results. Among others it is proved that a $\phi-\mathcal{M}-$projeectively semisymmetric $\alpha-$cosymplectic manifold admitting a $*-$conformal $\eta-$Ricci soliton is an Einstein manifold. Finally, the existence of $*-$conformal $\eta-$Ricci soliton in an $\alpha-$cosymplectic manifolds has been proved by a concrete example.

1. Introduction

In recent years, Ricci solitons and their generalizations are enjoying rapid growth by providing new techniques in understanding the geometry and topology of arbitrary Riemannian manifolds. Ricci soliton is a natural generalization of Einstein metric, and is also a self-similar solution to Hamilton's Ricci flow [20,21]. It plays a specific role in the study of singularities of the Ricci flow. A solution $g(t)$ of the non-linear evolution PDE: $\frac{\partial}{\partial t}g(t)=-2S(g(t)),\ \ t\in[0,I]$ is called the Ricci flow [30], where S is the Ricci tensor field associated

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to the metric g. A Riemannian manifold (M, g) is called a *Ricci soliton* (g, V, λ) if there are a smooth vector field V and a scalar $\lambda \in R$ such that

$$S + \pounds_V g = \lambda g \tag{1.1}$$

on M, where S is the Ricci tensor and $\pounds_V g$ is the Lie derivative of the metric g. If the potential vector field V vanishes identically, then the Ricci soliton becomes trivial, and in this case manifold is an Einstein one. As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [10]. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants, and g is a Riemannian metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where S is the Ricci tensor associated to g.

The notion of $*$-Ricci soliton has been studied by Kaimakamis and Panagiotidou [22] within the framework of real hypersurfaces of complex space forms. They essentially modified the definition of Ricci soliton by replacing the Ricci tensor S in (1.1) with the $*$-Ricci tensor S^*. Here, it is mentioned that the notion of $*$-Ricci tensor was first introduced by Tachibana [39] on almost Hermitian manifolds and further studied by Hamada [19] on real hypersurfaces of non-flat complex space forms. A Riemannian metric g on a smooth manifold M is called a $*$-Ricci soliton if there exists a smooth vector field V and a real number λ, such that

$$(\pounds_V g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0 \tag{1.3}$$

where

$$S^*(X, Y) = g(Q^*X, Y) = \text{Trace}\,\{\phi \circ R(X, \phi Y)\} \tag{1.4}$$

for all vector fields X, Y on M. Here, ϕ is a tensor field of type (0, 2). In this connection, we recommend the papers [5, 6, 11, 15, 24, 31, 32, 36, 40] and the references therein for more details about the study of Ricci solitons, η-Ricci solitons and $*$-Ricci solitons in the context of contact Riemannian geometry.

In 2004, Fischer [14] introduced a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the conformal Ricci flow equations and are given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \text{ and } r = -1,$$

for a dynamically evolving metric g, Ricci tensor S, scalar curvature r and a scalar non-dynamical field p. Since, these equations are the vector field sum of a conformal flow equation and a Ricci flow equation, they play an important role in conformal geometry. In the Riemannian setting, the notion of conformal Ricci soliton was introduced by Basu and Bhattacharyya [3] on a Kenmotsu manifold of dimension n as

$$\pounds_V g + 2S = (2\lambda - (p + \frac{2}{n}))g, \tag{1.5}$$

where λ is a constant and $\pounds_V$ is the Lie derivative along the vector field V. This notion was also studied by several authors on various kinds of almost contact metric manifolds (see, [13, 25, 37]). Further, Siddiqi [38] introduced the notion of conformal η-Ricci soliton as

$$\pounds_V g + 2S + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0, \tag{1.6}$$

where λ and μ are constants. Recently, Roy et al. [37] introduced and studied the notion of $*-$conformal $\eta-$Ricci soliton on an n-dimensional Sasakian manifold. A Riemannian metric g on M is called $*-$**conformal** $\eta-$**Ricci soliton**, if

$$\pounds_\xi g + 2S^* + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0, \tag{1.7}$$

where $\pounds_\xi$ is the Lie derivative along the vector field ξ, S^* is the $*-$Ricci tensor and λ, μ are constants.

On the other hand, the geometry of contact Riemannian manifolds and related topics have also drawn a great deal of interest in the last years. An important class of almost contact manifolds is given by cosymplectic manifolds. They were introduced by Goldberg and Yano [16] in 1969. A cosymplectic manifold is a (2n+1)-dimensional smooth manifold equipped with closed 1-form η and closed 2-form ω such that $\eta \wedge \omega^n$ is a volume form. The products of almost Kaehlerian manifolds and the real line $\mathbb{R}$ or the S^1 circle are the simplest examples of almost cosymplectic manifolds [28]. We refer to [9] for a nice overview on cosymplectic geometry and its connection with other areas of mathematics (especially, geometric mechanics) as well as with physics.

In this paper we undertake the study of $\alpha-$cosymplectic manifolds admitting $*$-conformal η-Ricci solitons. The present paper is organized as follows: Section 2 is concerned about preliminaries on $\alpha-$cosymplectic manifolds. In section 3, $\alpha-$cosymplectic manifolds admitting a $*-$conformal $\eta-$Ricci solitons is being studied. Section 4 is devoted to the study of $\mathcal{M}-$projective curvature tensor on $\alpha-$cosymplectic manifolds admitting $*$-conformal η-Ricci solitons. In the last section, we construct an example of a 5-dimensional manifold which verifies existence of $*$-conformal η-Ricci soliton on a $\alpha-$cosymplectic manifold.

2. **Preliminaries**

Let M be an n-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a (1, 1)-tensor field, ξ is a vector field, η is a 1-form on M such that [7]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

which implies

$$\phi\xi = 0, \quad \eta \cdot \phi = 0, \quad rank(\phi) = n-1. \tag{2.2}$$

If M admits a Riemannian metric g, such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(\xi) \tag{2.3}$$

then M is said to admit almost contact structure (ϕ, ξ, η, g) [7]. On such a manifold the 2-form Φ of M is defined as

$$\Phi(X,Y) = g(\phi X, Y) \tag{2.4}$$

for all $X, Y \in \chi(M)$; where $\chi(M)$ denotes the collection of all smooth vector fields of M. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be almost cosymplectic [16] if $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. An almost contact manifold (M, ϕ, ξ, η, g) is said to be normal if the Nijenhuis torsion

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y] + 2d\eta(X,Y)\xi$$

vanishes for any vector fields X and Y. A normal almost cosymplectic manifolds is called a cosymplectic manifolds.

An almost contact metric manifold M is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant.

Kim and Pak [23] combined almost α-Kenmotsu and almost cosymplectic manifolds into a new class called almost α-cosymplectic manifolds, where α is a scalar. If we join these two classes, we obtain a new notion of an almost α-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

for any real number α. A normal almost α-cosymplectic manifold is called an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu under the condition $\alpha \neq 0$, for $\alpha \in \mathbb{R}$. For detailed study of $\alpha-$cosymplectic manifolds we refer to the readers ([1, 2, 4, 8, 17, 29]) and many others.

In an $\alpha-$cosymplectic manifold, we have [18]

$$(\nabla_X \phi)Y = \alpha(g(\phi X, Y)\xi - \eta(Y)\phi X). \tag{2.5}$$

Let M be a n-dimensional α-cosymplectic manifold. From Eq. (2.5), it is easy to see that

$$\nabla_X \xi = -\alpha\phi^2 X = \alpha[X - \eta(X)\xi]. \tag{2.6}$$

where ∇ denotes the Riemannian connection. On an α-cosymplectic manifold M, the following relations are hold:

$$\eta(R(X,Y)Z) = \alpha^2(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)), \tag{2.7}$$

$$R(\xi,X)Y = \alpha^2(\eta(Y)X - g(X,Y)\xi), \tag{2.8}$$

$$R(X,Y)\xi = \alpha^2(\eta(X)Y - \eta(Y)X), \tag{2.9}$$

$$R(\xi,X)\xi = \alpha^2(X - \eta(X)\xi), \tag{2.10}$$

$$S(X,\xi) = -\alpha^2(n-1)\eta(X) \tag{2.11}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Definition 2.1. *An $\alpha-$cosymplectic manifold is said to be an η-Einstein manifold if the Ricci tensor S is of the form [41]*

$$S(X,Y)Y = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold. If $b = 0$, then the manifold is said to be an Einstein manifold.

Lemma 2.1. *In an $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), we have*

$$\begin{aligned}\bar{R}(X,Y,\phi Z,\phi W) &= \bar{R}(X,Y,Z,W) \\ &+\alpha^2[\Phi(X,Z)\Phi(Y,W) - \Phi(Y,Z)\Phi(X,W) \\ &+g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{2.12}$$

for any X, Y, Z, W on M, where $\bar{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and Φ is the fundamental 2-form of M defined by $\Phi(X,Y) = g(\phi X, Y)$.

Proof. In view of

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} \tag{2.13}$$

we can write

$$\begin{aligned} & \bar{R}(X,Y,\phi Z,\phi W) \\ = & \; g(\nabla_X\nabla_Y\phi Z,\phi W)-g(\nabla_Y\nabla_X\phi Z,\phi W)-g(\nabla_{[X,Y]}\phi Z,\phi W). \end{aligned} \tag{2.14}$$

By making use of (2.1), (2.3), (2.6) and (2.5), the Eq.(2.14) takes the form

$$\begin{aligned} \bar{R}(X,Y,\phi Z,\phi W) &= g(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,W) \\ &+ \alpha^2[g(\phi Y,Z)g(X,\phi W)-g(\phi X,Z)g(Y,\phi W) \\ &- g(X,Z)g(Y,W)+g(Y,Z)g(X,W) \\ &+ g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)] \\ &- \eta(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z)\eta(W) \end{aligned}$$

which in view of (2.7) and (2.13) turns to

$$\begin{aligned} \bar{R}(X,Y,\phi Z,\phi W) &= \bar{R}(X,Y,Z,W) \\ &\quad +\alpha^2[g(\phi Y,Z)g(X,\phi W)-g(\phi X,Z)g(Y,\phi W) \\ &\quad -g(X,Z)g(Y,W)+g(Y,Z)g(X,W)]. \end{aligned} \tag{2.15}$$

This completes the proof. □

Lemma 2.2. *In an $n-$dimensional $\alpha-$cosymplectic manifold (M,ϕ,ξ,η,g), the $*-$Ricci tensor is given by*

$$S^*(Y,Z) = S(Y,Z)+\alpha^2(n-2)g(Y,Z)+\alpha^2\eta(Y)\eta(Z) \tag{2.16}$$

for any $Y,Z\in\chi(M)$, where S and S^* are the Ricci tensor and the $*-$Ricci tensor of type $(0,2)$, respectively on M.

Proof. Let $\{e_i\}, i=1,2,3.....n$ be an orthonormal basis of the tangent space at each point of the manifold. From the equations (2.12) and (1.4), we have

$$\begin{aligned} S^*(Y,Z) &= \sum_{i=1}^n \bar{R}(e_i,Y,\phi Z,\phi e_i) \\ &= \sum_{i=1}^n \bar{R}(e_i,Y,Z,e_i)+\alpha^2\sum_{i=1}^n[\Phi(e_i,Z)\Phi(Y,e_i)-\Phi(Y,Z)\Phi(e_i,e_i) \\ &+ g(Y,Z)g(e_i,e_i)-g(e_i,Z)g(Y,e_i)]. \end{aligned}$$

By using (2.3) and $\Phi(X,Y)=g(\phi X,Y)$ in the above equation, Lemma 2.2 follows. □

3. $\alpha-$cosymplectic manifolds admitting $*-$conformal $\eta-$Ricci solitons

In this section, first let us consider an n-dimensional $\alpha-$cosymplectic manifold M admitting a $*-$conformal $\eta-$Ricci soliton. Then, from (1.7) we have

$$(3.1)\qquad (\pounds_\xi g)(Y,Z)+2S^*(Y,Z)+(2\lambda-(p+\frac{2}{n}))g(Y,Z)+2\mu\eta(Y)\eta(Z)=0.$$

In an $\alpha-$cosymplectic manifold, from (2.6)we write

$$(3.2)\qquad (\pounds_\xi g)(Y,Z)=g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)=2\alpha(g(Y,Z)-\eta(Y)\eta(Z)).$$

By making use of (3.2) in (3.1), we find

$$(3.3)\qquad S^*(Y,Z)=-(\lambda+\alpha-\frac{1}{2}(p+\frac{2}{n}))g(Y,Z)+(\alpha-\mu)\eta(Y)\eta(Z).$$

Further, by plugging (3.3) in (2.16) we get

$$(3.4)\qquad S(Y,Z) \;=\; Ag(Y,Z)+B\eta(Y)\eta(Z).$$

where $A=-\{\lambda+\alpha+\alpha^2(n-2)-\frac{1}{2}(p+\frac{2}{n})\}$ and $B=-\{\mu+\alpha^2-\alpha\}$. Therefore, the manifold M is an η-Einstein manifold. Next, taking $Z=\xi$ in (3.4), we find

$$(3.5)\qquad S(Y,\xi)=(A+B)\eta(Y).$$

It yields

$$(3.6)\qquad Q\xi=(A+B)\xi.$$

In view of (2.11) we obtain from (3.5) that

$$(3.7)\qquad \lambda+\mu=\frac{1}{2n}(np+2).$$

Hence, we are able to state the following:

Proposition 3.1. *Let M be an $n-$dimensional $\alpha-$cosymplectic manifold. If the manifold M admits a $*-$conformal $\eta-$Ricci soliton, then M is an $\eta-$Einstein manifold and the soliton constants λ and μ are related by $\lambda+\mu=\frac{1}{2n}(np+2)$.*

Next, assume that an $n-$dimensional $\alpha-$cosymplectic manifold M admitting $*-$conformal $\eta-$Ricci soliton satisfies the condition

$$(3.8)\qquad R(\xi,X)\cdot S=0.$$

The condition (3.8) implies that

$$(3.9)\qquad S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0$$

for any vector fields $X, Y, Z \in \chi(M)$. By using (3.4) in (3.9), we find

$$B[\eta(R(\xi, X)Y)\eta(Z) + \eta(R(\xi, X)Z)\eta(Y)] = 0,$$

which in view of (2.7) takes the form

$$\alpha^2 B[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{3.10}$$

Contracting (3.10) over X and Y we get

$$(n-1)\alpha^2 B\eta(Z) = 0. \tag{3.11}$$

In general $\eta(Z) \neq 0$, therefore, we have either $\alpha^2 = 0$ or $B = 0$. Hence, for $\alpha^2 = 0$, i.e., $\alpha = 0$ the manifold M reduces to cosymplectic manifiold. And, for the latter case, from (3.7) we find $\lambda = \frac{1}{2}(p + \frac{2}{n}) + \alpha^2 - \alpha$. Thus, we state the following:

Theorem 3.1. *Let M be an $n-$dimensional $\alpha-$cosymplectic manifold admitting a $*-$conformal $\eta-$Ricci soliton. If the manifold M satisfies the condition $R(\xi, X) \cdot S = 0$, then the manifold M is either a cosymplectic or $\lambda = \frac{1}{2}(p + \frac{2}{n}) + \alpha^2 - \alpha$ and $\mu = \alpha - \alpha^2$.*

Further, consider that the manifold M admitting a $*-$conformal $\eta-$Ricci soliton satisfies the condition

$$S(\xi, Y) \cdot R = 0. \tag{3.12}$$

The condition (3.12) implies that

$$\begin{aligned} & S(Y, R(U, V)W)\xi - S(\xi, R(U, V)W)Y + S(Y, U)R(\xi, V)W \\ - \; & S(\xi, U)R(Y, V)W + S(Y, V)R(U, \xi)W - S(\xi, V)R(U, Y)W \\ + \; & S(Y, W)R(U, V)\xi - S(\xi, W)R(U, V)Y = 0 \end{aligned} \tag{3.13}$$

for any vector fields $Y, U, V, W \in \chi(M)$. Taking the inner product of (3.13) with ξ, we have

$$\begin{aligned} & S(Y, R(U, V)W) - S(\xi, R(U, V)W)\eta(Y) + S(Y, U)\eta(R(\xi, V)W) \\ - \; & S(\xi, U)\eta(R(Y, V)W) + S(Y, V)\eta(R(U, \xi)W) - S(\xi, V)\eta(R(U, Y)W) \\ + \; & S(Y, W)\eta(R(U, V)\xi) - S(\xi, W)\eta(R(U, V)Y) = 0. \end{aligned} \tag{3.14}$$

By putting $U = W = \xi$ in (3.14) and then using (2.7), (2.8) and (3.5) we have

$$\alpha^2[S(Y, V) - (n-1)\alpha^2\{g(Y, V) - 2\eta(Y)\eta(V)\}] = 0. \tag{3.15}$$

Thus, we have either $\alpha = 0$, or

$$S(Y, V) = (n-1)\alpha^2 g(Y, V) - 2(n-1)\alpha^2\eta(Y)\eta(V).$$

This can be stated in the following form:

Theorem 3.2. *Let M be an $n-$dimensional $\alpha-$cosymplectic manifold admitting a $*-$conformal $\eta-$Ricci soliton. If the manifold M satisfies the condition $S(\xi, Y)\cdot R = 0$, then the manifold M is either a cosymplectic or an η-Einstein manifold.*

4. $\mathcal{M}-$Projective Curvature Tensor on $\alpha-$Cosymplectic Manifolds Admitting $*-$Conformal $\eta-$Ricci Solitons

In Riemannian geometry, one of the basic interest is curvature properties and to what extend these determine the manifold itself. The $\mathcal{M}$-projective curvature tensor differs from the Riemannian curvature tensor and is an important tensor field from the differential geometric point of view because it bridges the gap between the conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one side and the $\mathcal{H}$-projective curvature tensor on the other. It is known that, $\mathcal{M}$-projectively flat Riemannian manifold is an Einstein manifold.

The $\mathcal{M}-$projective curvature tensor in an $n-$dimensional $\alpha-$cosymplectic manifold is defined by [26]

$$\begin{aligned}\mathcal{M}(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{4.1}$$

where $R(X,Y)Z$ and $S(X,Y)$ are the curvature tensor and the Ricci tensor of M, respectively; and Q is the Ricci operator defined as $S(X,Y) = g(QX,Y)$. In 1985, Ojha showed some properties of $\mathcal{M}$-projective curvature tensor in a Sasakian manifold [27]. Subsequently, many geometers have studied this curvature tensor and obtained important properties of various kinds of Riemannian and pseudo-Riemannian manifolds (see, for instance, [2,33,34,42]). In this section, we study $\alpha-$cosymplectic manifolds admitting $*-$conformal $\eta-$Ricci solitons satisfying certain conditions on the $\mathcal{M}-$projective curvature tensor.

First, let us consider an $n-$dimensional $\alpha-$cosymplectic manifold M admitting $*-$conformal $\eta-$Ricci soliton, which is $\xi - \mathcal{M}-$projectively flat, i.e., $\mathcal{M}(X,Y)\xi = 0$. Then, from (4.1) it follows that

$$R(X,Y)\xi = \frac{1}{2(n-1)}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY]. \tag{4.2}$$

Making use of (2.9) and (3.5) in the above equation, we have

$$\eta(Y)QX - \eta(X)QY = \{(A+B) + 2(n-1)\alpha^2\}[\eta(X)Y - \eta(Y)X]. \tag{4.3}$$

Again, taking $Y = \xi$ in (4.3) and then using (3.5) we obtain

$$QX = -\{(A+B) + 2(n-1)\alpha^2\}X + 2\{(A+B) + (n-1)\alpha^2\}\eta(X)\xi.$$

Taking the inner product of the above equation with W, we obtain

$$\begin{aligned} S(X,W) &= -\{(A+B)+2(n-1)\alpha^2\}g(X,W) \\ &+ 2\{(A+B)+(n-1)\alpha^2\}\eta(X)\eta(W). \end{aligned} \tag{4.4}$$

In M, by virtue of (3.7) and with the values of A and B, it follows that

$$A+B=-(n-1)\alpha^2. \tag{4.5}$$

By using (4.5), the Eq. (4.4) reduce to

$$S(X,W)=-(n-1)\alpha^2 g(X,W). \tag{4.6}$$

Thus, the manifold M is an Einstein. Hence, we have the following:

Theorem 4.1. *Let M be an $n-$dimensional $\alpha-$cosymplectic manifold admitting a $*-$conformal $\eta-$Ricci soliton. If the manifold M is $\xi-\mathcal{M}-$projectively flat, then the manifold M is an Einstein manifold.*

Definition 4.1. *An $\alpha-$cosymplectic manifold is said to be $\phi-\mathcal{M}-$projectively semisymmetric if [12, 35]*

$$\mathcal{M}(X,Y)\cdot\phi=0$$

for all X,Y on M.

Next, let us consider an $n-$dimensional $\alpha-$cosymplectic manifold M admitting $*-$conformal $\eta-$Ricci soliton, which is $\phi-\mathcal{M}-$projectively semisymmetric, i.e., $\mathcal{M}(X,Y)\cdot\phi=0$. Then, it follows that

$$(\mathcal{M}(X,Y)\cdot\phi)W=\mathcal{M}(X,Y)\phi W-\phi\mathcal{M}(X,Y)W=0. \tag{4.7}$$

By virtue of (4.1), (4.7) takes the form

$$\begin{aligned} & R(X,Y)\phi W-\phi R(X,Y)W \\ &+ \frac{1}{2(n-1)}[S(Y,W)\phi X-S(X,W)\phi Y-S(Y,\phi WZ)X \\ &+ S(X,\phi W)Y+g(Y,W)\phi QX-g(X,W)\phi QY \\ &- g(Y,\phi W)QX+g(X,\phi W)QY] \\ &= 0. \end{aligned} \tag{4.8}$$

Putting $Y=\xi$ in (4.8) and using (2.1), (2.2), (2.8) and (3.5), we find

$$[A+B+2(n-1)\alpha^2](g(X,\phi W)\xi+\eta(W)\phi X)+S(X,\phi WZ)\xi+\eta(W)\phi QX=0$$

whcih by taking the inner product with ξ reduces to

$$S(X,\phi W)=-[A+B+2(n-1)\alpha^2]g(X,\phi W). \tag{4.9}$$

Replacing W by ϕW in (4.9) and using (2.1) and (3.5), we obtain

$$\begin{aligned} S(X,W) &= -[A+B+2(n-1)\alpha^2)]g(X,W) \\ &+ 2[A+B+(n-1)\alpha^2]\eta(X)\eta(W). \end{aligned} \tag{4.10}$$

By using (4.5), the Eq. (4.10) turns to

$$S(X,W) = -(n-1)\alpha^2 g(X,W). \tag{4.11}$$

Thus, we have the following:

Theorem 4.2. *Let M be an $n-$dimensional $\alpha-$cosymplectic manifold admitting a $*-$conformal $\eta-$Ricci soliton. If the manifold M is $\phi-\mathcal{M}-$projectively semisymmetric, then the manifold M is an Einstein manifold.*

Finally, consider an $n-$dimensional $\alpha-$cosymplectic manifold M admitting $*-$conformal $\eta-$Ricci soliton, which satisfies the condition

$$R(\xi,X)\cdot\mathcal{M} = 0. \tag{4.12}$$

From (4.12) it follows that

$$\begin{aligned} &R(\xi,X)\mathcal{M}(U,V)W - \mathcal{M}(R(\xi,X)U,V)W \\ -\; &\mathcal{M}(U,R(\xi,X)V)W - \mathcal{M}(U,V)R(\xi,X)W = 0. \end{aligned} \tag{4.13}$$

In view of (2.8), (4.13) takes the form

$$\begin{aligned} &\alpha^2[\eta(\mathcal{M}(U,V)W)X - g(X,\mathcal{M}(U,V)W)\xi - \eta(U)\mathcal{M}(X,V)W \\ +\; &g(X,U)\mathcal{M}(\xi,V)W - \eta(V)\mathcal{M}(U,X)W + g(X,V)\mathcal{M}(U,\xi)W \\ -\; &\eta(W)\mathcal{M}(U,V)X + g(X,W)\mathcal{M}(U,V)\xi] = 0. \end{aligned} \tag{4.14}$$

Taking the inner product of (4.14) with ξ, we have

$$\begin{aligned} &\alpha^2[\eta(\mathcal{M}(U,V)W)\eta(X) - g(X,\mathcal{M}(U,V)W) - \eta(U)\eta(\mathcal{M}(X,V)W) \\ +\; &g(X,U)\eta(\mathcal{M}(\xi,V)W) - \eta(V)\eta(\mathcal{M}(U,X)W) + g(X,V)\eta(\mathcal{M}(U,\xi)W) \\ -\; &\eta(W)\eta(\mathcal{M}(U,V)X) + g(X,W)\eta(\mathcal{M}(U,V)\xi)] = 0. \end{aligned} \tag{4.15}$$

From (4.1), we find

$$\eta(\mathcal{M}(U,V)W) = (\alpha^2 + \frac{2A+B}{2(n-1)})(g(U,W)\eta(V) - g(V,W)\eta(U)), \tag{4.16}$$

$$\eta(\mathcal{M}(\xi,V)W) = (\alpha^2 + \frac{2A+B}{2(n-1)})(\eta(V)\eta(W) - g(V,W)), \tag{4.17}$$

$$\eta(\mathcal{M}(U,V)\xi) = 0. \tag{4.18}$$

In view of (4.16)-(4.18), (4.15) reduces to

$$\alpha^2[g(X,\mathcal{M}(U,V)W) - (\alpha^2 + \frac{2A+B}{2(n-1)})(g(U,W)g(X,V) - g(X,U)g(V,W))] = 0. \tag{4.19}$$

Thus we have either $\alpha = 0$, or

$$g(X,\mathcal{M}(U,V)W) = (\alpha^2 + \frac{2A+B}{2(n-1)})(g(U,W)g(X,V) - g(X,U)g(V,W)). \tag{4.20}$$

By using (4.1), (4.20) takes the form

$$\begin{aligned} & g(R(U,V)W,X) \\ = \; & \frac{1}{2(n-1)}[S(V,W)g(X,U) - S(U,W)g(X,V) \\ + \; & S(X,U)g(V,W) - S(X,V)g(U,W)] \\ = \; & [\alpha^2 + \frac{2A+B}{2(n-1)}](g(U,W)g(X,V) - g(X,U)g(V,W)). \end{aligned} \tag{4.21}$$

Let $\{e_i\}$, $i = 1,2,3....n$ be an orthonormal basis of the tangent space at any point of the manifold. If we put $X = U = e_i$ in (4.21) and taking summation with respect to $i(1 \leq i \leq n)$, then we get

$$S(V,W) = \frac{1}{2n}\{r - (n-1)(2(n-1)\alpha^2 - (2A+B))\}g(V,W), \tag{4.22}$$

where A and B are given in (3.4). Thus we can state the following:

Theorem 4.3. *Let M be an $n-$dimensional $\alpha-$cosymplectic manifold admitting a $*-$conformal $\eta-$Ricci soliton. If the manifold M satisfies the condition $R(\xi,X)\cdot\mathcal{M} = 0$, then the manifold M is an Einstein manifold.*

5. **Example**

We consider the 5-dimensional manifold $M = \{(x_1,x_2,y_1,y_2,z) \in \mathbb{R}^5\}$, where (x_1,x_2,y_1,y_2,z) are the standard coordinates in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M given by

$$e_1 = e^{\alpha z}\frac{\partial}{\partial x_1},\; e_2 = e^{\alpha z}\frac{\partial}{\partial x_2},\; e_3 = e^{\alpha z}\frac{\partial}{\partial y_1},\; e_4 = e^{\alpha z}\frac{\partial}{\partial y_2},\; e_5 = -\frac{\partial}{\partial z} = \xi.$$

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = 0, \ \ i \neq j, \ \ i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

Let η be the 1-form on M defined by $\eta(X) = g(X, e_5) = g(X, \xi)$ for all $X \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field on M defined by

$$\phi e_1 = -e_2, \ \phi e_2 = e_1, \ \phi e_3 = -e_4, \ \ \phi e_4 = e_3, \ \phi e_5 = 0.$$

By applying the linearity of ϕ and g, we have

$$\eta(\xi) = 1, \ \phi^2 X = -X + \eta(X)\xi, \ \eta(\phi X) = 0,$$

$$g(X, \xi) = \eta(X), \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0,$$

$$[e_1, e_5] = \alpha e_1, [e_2, e_5] = \alpha e_2, [e_3, e_5] = \alpha e_3, [e_4, e_5] = \alpha e_4.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -\alpha e_5, \ \ \nabla_{e_1} e_2 = 0, \ \ \nabla_{e_1} e_3 = 0, \ \ \nabla_{e_1} e_4 = 0, \ \ \nabla_{e_1} e_5 = \alpha e_1,$$

$$\nabla_{e_2} e_1 = 0, \ \ \nabla_{e_2} e_2 = -\alpha e_5, \ \ \nabla_{e_2} e_3 = 0, \ \ \nabla_{e_2} e_4 = 0, \ \ \nabla_{e_2} e_5 = \alpha e_2,$$

$$\nabla_{e_3} e_1 = 0, \ \ \nabla_{e_3} e_2 = 0, \ \ \nabla_{e_3} e_3 = -\alpha e_5, \ \ \nabla_{e_3} e_4 = 0, \ \ \nabla_{e_3} e_5 = \alpha e_3,$$

$$\nabla_{e_4} e_1 = 0, \ \ \nabla_{e_4} e_2 = 0, \ \ \nabla_{e_4} e_3 = 0, \ \ \nabla_{e_4} e_4 = -\alpha e_5, \ \ \nabla_{e_4} e_5 = \alpha e_4,$$

$$\nabla_{e_5} e_1 = 0, \ \ \nabla_{e_5} e_2 = 0, \ \ \nabla_{e_5} e_3 = 0, \ \ \nabla_{e_5} e_4 = 0, \ \ \nabla_{e_5} e_5 = 0.$$

It can be easily verified that the manifold satisfy

$$\nabla_X \xi = \alpha[X - \eta(X)\xi] \quad \text{and} \quad (\nabla_X \phi)Y = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] \text{ for } \xi = e_5.$$

Thus the manifold M is an $\alpha-$cosymplectic manifold. By using (2.13) we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = R(e_1, e_5)e_5 = -\alpha^2 e_1,$$

$$R(e_1, e_2)e_1 = \alpha^2 e_2, \ R(e_1, e_3)e_1 = R(e_2, e_3)e_2 = R(e_5, e_3)e_5 = \alpha^2 e_3,$$

$$R(e_2, e_3)e_3 = R(e_2, e_4)e_4 = R(e_2, e_5)e_5 = -\alpha^2 e_2, R(e_3, e_4)e_4 = -\alpha^2 e_3,$$

$$R(e_1, e_5)e_2 = R(e_1, e_5)e_1 = R(e_4, e_5)e_4 = R(e_3, e_5)e_3 = \alpha^2 e_5,$$

$$R(e_1, e_4)e_1 = R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = R(e_5, e_4)e_5 = \alpha^2 e_4.$$

With the help of the above results we get the components of the Ricci tensor as follows:

(5.1) $$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = S(e_5, e_5) = -4\alpha^2.$$

From (3.4), we have $S(e_5, e_5) = -4\alpha^2 - \lambda - \mu + \frac{1}{2}(p + \frac{2}{5})$. By equating both the values of $S(e_5, e_5)$, we obtain

$$\lambda + \mu = \frac{1}{2}(p + \frac{2}{5}).$$

Hence, λ and μ satisfies the equation (3.7) for $n = 5$ and, so g defines a $*-$conformal $\eta-$Ricci soliton on a 5-dimensional $\alpha-$cosymplectic manifold.

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