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COMMON FIXED POINT THEOREMS FOR SIX SELF-MAPPINGS ON $S-$ METRIC SPACES

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Abstract. In this paper, we introduce the concepts of common property $-(E.A)$ and common limit range property for six self-mappings and prove common fixed point theorems of such mappings satisfying $(\psi,\varphi)-$ weak contraction on an $S-$metric space. Examples are given to illustrate our results.

1. Introduction and Preliminaries

In 2006, Mustafa and Sims [21] introduced $G-$ metric space to overcome fundamental flaws in B. C. Dhage's theory of generalized metric spaces ([10–12]) and discussed the topological properties of $G-$ metric spaces. In 2012, Sedghi et al. [26] introduced the concept of $S-$ metric space as a modification of D^*- metric space [27] and $G-$ metric space [21]. But, in 2014, Dung et al. [14] showed by giving examples that the class of $S-$ metric spaces and the class of $G-$ metric spaces are distinct.

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Before going to our main work, let us recall some basic definitions, lemmas, and preliminaries that will be used in this paper.

Definition 1.1. [26] Let X be a non-empty set. A function $S : X \times X \times X \to [0, \infty)$ is said to be an $S-$ metric on X if it satisfies the following properties:

(S_1) $S(x, y, z) = 0$ if and only if $x = y = z$;

(S_2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, for all $x, y, z, a \in X$.

The pair (X, S) is called an $S-$ metric space.

Example 1.1. [26] Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X. Define $S(x, y, z) = \|2x - y - z\| + \|y - z\|$, for all $x, y, z \in X$. Then (X, S) is an $S-$ metric space.

Example 1.2. [26] Let $X = \mathbb{R}$. Define $S(x, y, z) = |x - z| + |y - z|$, for all $x, y, z \in X$. Then (X, S) is an $S-$ metric space.

Definition 1.2. [26] Let (X, S) be an $S-$ metric space.

(i) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

(ii) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(iii) The $S-$ metric space (X, S) is said to be complete if every Cauchy sequence in it is convergent.

Lemma 1.1. [26] In an $S-$ metric space, we have $S(x, x, y) = S(y, y, x)$.

Lemma 1.2. [26] Let (X, S) be an $S-$ metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 1.3. [26] Let (X, S) be an $S-$ metric space. If sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.4. [26] Let (X, S) be an $S-$ metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Definition 1.3. [3] Let $X \neq \emptyset$ and $\mathcal{P}, \mathcal{Q} : X \to X$ be two self-mappings. If $u = \mathcal{P}x = \mathcal{Q}x$, for some $x \in X$, then x is called a coincidence point of $\mathcal{P}$ and $\mathcal{Q}$, and u is called a point of coincidence (briefly, *poc*) of $\mathcal{P}$ and $\mathcal{Q}$.

Lemma 1.5. [3] Suppose that $\mathcal{P}$ and $\mathcal{Q}$ be weakly compatible self-mappings on a non-empty set X. If $\mathcal{P}$ and $\mathcal{Q}$ have a unique point of coincidence $u = \mathcal{P}x = \mathcal{Q}x$, then u is the unique common fixed point $\mathcal{P}$ and $\mathcal{Q}$.

In 1997, Alber and Guere-Delabriere [5] introduced the concept of weak contraction, wherein the authors introduced the following notion for mappings defined on a Hilbert space X.

Consider the following set of real functions $\Phi = \big\{\varphi : [0,\infty) \to [0,\infty) : \varphi$ is a lower semi-continuous and $\varphi(t) = 0$ if and only if $t = 0\big\}$.

A mapping $\mathcal{T} : X \to X$ is called a $\varphi-$ weak contraction if there exists a function $\varphi \in \Phi$ such that

$$d\big(\mathcal{T}x, \mathcal{T}y\big) \leq d(x,y) - \varphi\big(d(x,y)\big), \text{ for all } x, y \in X.$$

Dutta and Choudhury [15] proved a fixed point theorem for a self-mapping satisfying $(\psi,\varphi)-$weak contractive condition as follows.

Theorem 1.1. Let (X, d) be a complete metric space and $\mathcal{T} : X \to X$ be a self-mapping satisfying

$$\psi\big(d(\mathcal{T}x, \mathcal{T}y)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big), \text{ for some } \varphi \in \Phi \text{ and}$$

$\psi \in \Psi = \big\{\psi : [0,\infty) \to [0,\infty) : \psi$ is continuous non-decreasing and $\psi(0) = 0\big\}$.

Then, $\mathcal{T}$ has a common fixed point in X.

Many researchers utilized $(\psi,\varphi)-$ weak contractive conditions to prove a number of metrical fixed point theorems (e.g., [2, 4–9, 13], [20], [30]). Recently, Singh and Bimol Singh [29] proved some coincidence and common fixed point theorems involving $\psi \in \Psi$ and $\varphi \in \Phi$ in $S-$ metric spaces.

Definition 1.4. [28] A pair $(\mathcal{A}, \mathcal{B})$ of self-mappings of an $S-$ metric space (X, S) is said to be compatible if $\lim_{n\to\infty} S(\mathcal{AB}x_n, \mathcal{AB}x_n, \mathcal{BA}x_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{B}x_n = t$, for some $t \in X$.

In 1998, Jungck and Rhoades [18] introduced the following concept of weakly compatibility.

Definition 1.5. A pair $(\mathcal{A}, \mathcal{B})$ of self-mappings of an $S-$ metric space (X, S) is said to be weakly compatible if they commute at each coincidence point (i.e., $\mathcal{AB}x = \mathcal{BA}x,\ x \in X$ whenever $\mathcal{A}x = \mathcal{B}x$).

In 2002, Aamri and Moutawakil [1] introduced the concept of property $-(E.A)$ in metric spaces. In the same line, we use this concept in $S-$ metric space as follows.

Definition 1.6. A pair $(\mathcal{A}, \mathcal{P})$ of self-mappings of an $S-$ metric space (X, S) is said to satisfy the property $-(E.A)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = t, \text{ for some } t \in X.$$

Any pair of compatible as well as non-compatible self-mappings of an $S-$ metric space (X, S) satisfy the property $-(E.A)$, but a pair of mappings satisfying the property $-(E.A)$ need not be non-compatible (see Example 1 of [16]).

In 2005, Liu et al. [19] introduced the notion of common property $-(E.A)$ for hybrid pairs of mappings, which contain the property $-(E.A)$. For more details on various type of compatible mappings and their relation, one may refer to ([8], [22–25], [31], [32]) and references therein.

Definition 1.7. Two pairs $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{Q})$ of self-mappings of an $S-$ metric space (X, S) are said to satisfy the common property $-(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = t, \text{ for some } t \in X.$$

In a similar way, we define the notion of common property $-(E.A)$ for six self-mappings on $S-$metric space.

Definition 1.8. Three pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ of self-mappings of an $S-$ metric space (X, S) are said to satisfy the common property $-(E.A)$ if there exist three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{C}z_n = \lim_{n\to\infty} \mathcal{R}z_n = t,$$

for some $t \in X$.

It can be observed that the fixed point results usually require closeness of the underlying subspaces for the existence of common fixed points under the property $-(E.A)$ and common property $-(E.A)$. In 2011, Sintunavarat and Kumam [33] coined the idea of 'common limit range property'. In 2012, Imdad et al. [17] extended the notion of common limit range property to two pairs of self-mappings of a metric space which relax the closeness requirements of the underlying subspaces.

Definition 1.9. A pair $(\mathcal{A}, \mathcal{P})$ of self-mappings of an $S-$ metric space (X, S) is said to satisfy the common limit range property with respect to $\mathcal{P}$, (briefly, $(CLR_{\mathcal{P}})-$ property), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = t, \text{ where } t \in \mathcal{P}X.$$

Thus, one can infer that a pair $(\mathcal{A}, \mathcal{P})$ satisfying the property $-(E.A)$ along with the closeness of the subspace $\mathcal{P}X$ always enjoys the $(CLR_{\mathcal{P}})-$ property with respect to the mapping $\mathcal{P}$ (see Examples 2.16–2.17 of [17]).

Definition 1.10. Two pairs $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{Q})$ of self-mappings of an $S-$ metric space (X, S) are said to satisfy the common limit range property (briefly, $(CLR_{\mathcal{P}\mathcal{Q}})-$ property) with respect to mappings $\mathcal{P}$ and $\mathcal{Q}$, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = t, \text{ where } t \in \mathcal{P}X \cap \mathcal{Q}X.$$

Example 1.3. [20] Let $X = [0, 12)$ endow with $S-$ metric $S(x, y, z) = |x-z|+|y-z|$. Define self-mappings $\mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{Q} : X \to X$ by

$$\mathcal{A}x = \begin{cases} 6, & 0 \le x \le 6 \\ 9, & 6 < x < 12 \end{cases} \quad ; \quad \mathcal{B}x = \begin{cases} 0, & 0 \le x < 6 \\ 6, & 6 \le x < 12 \end{cases} ;$$

$$\mathcal{P}x = \begin{cases} 6, & 0 \le x \le 6 \\ 3, & 6 < x < 12 \end{cases} \quad ; \quad \mathcal{Q}x = \begin{cases} 4, & 0 \le x < 6 \\ 12 - x, & 6 \le x < 12. \end{cases}$$

Consider two sequences $\{x_n\}$ and $\{y_n\}$ of X such that $x_n = \frac{1}{n}$ and $y_n = 6+\frac{1}{n}$, $n \in \mathbb{N}$. Note that $\mathcal{P}X = \{3, 6\}$ and $\mathcal{Q}X = (0, 6]$. Also, we have

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = 6 \in X \text{ and } \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = 6 \in \mathcal{Q}X.$$

It follows that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = t, \text{ where } t = 6 \in \mathcal{P}X \cap \mathcal{Q}X.$$

Therefore the pairs $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{Q})$ satisfy $(CLR_{\mathcal{P}\mathcal{Q}})-$ property.

In a similar mode, we give the concept of the common limit range property for six self-mappings as follows.

Definition 1.11. Three pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ of self-mappings of an $S-$metric space (X, S) are said to satisfy the common limit range property with respect to mappings $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ (briefly, $(CLR_{\mathcal{P}\mathcal{Q}\mathcal{R}})-$ property), if there exist three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{C}z_n = \lim_{n\to\infty} \mathcal{R}z_n = t,$$

where $t \in \mathcal{P}X \cap \mathcal{Q}X \cap \mathcal{R}X$, for some $t \in X$.

Example 1.4. Let $X = [0, 5]$. Define a mapping $S : X^3 \to [0, \infty)$ by $S(x, y, z) = |x - y| + |y - z|$, $\forall x, y, z \in X$. Clearly, (X, S) is an $S-$metric space.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q}, \mathcal{R} : X \to X$ be six self-mappings defined by

$$\mathcal{A}x = \begin{cases} 1, \text{ if } x = [0, 1] \\ 2, \text{ if } x \in (1, 5] \end{cases} ; \ \mathcal{B}x = \begin{cases} 0, \text{ if } x = [0, 1) \\ 1, \text{ if } x \in [1, 5] \end{cases} ; \ \mathcal{C}x = \begin{cases} 1, \text{ if } x = [0, 1] \\ 5, \text{ if } x \in (1, 5] \end{cases} ;$$

$$\mathcal{P}x = \begin{cases} 1, \text{ if } x = [0, 1] \\ 3, \text{ if } x \in (1, 5] \end{cases} ; \ \mathcal{Q}x = \begin{cases} \frac{1}{2}, \text{ if } x = [0, 1) \\ 1, \text{ if } x \in [1, 5] \end{cases} ; \ \mathcal{R}x = \begin{cases} 1, \text{ if } x = [0, 1] \\ 4, \text{ if } x \in (1, 5]. \end{cases}$$

Consider the three sequences $\{x_n\} = \left\{\dfrac{1}{n}\right\}$, $\{y_n\} = \left\{1 + \dfrac{1}{2n}\right\}$, $\{z_n\} = \left\{1 - \frac{1}{n}\right\}, \forall n \in \mathbb{N}$. Now, we have $\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{C}z_n = \lim_{n\to\infty} \mathcal{R}z_n = 1 \in \mathcal{P}X \cap \mathcal{Q}X \cap \mathcal{R}X$. The pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ satisfy the $(CLR_{\mathcal{P}\mathcal{Q}\mathcal{R}})-$property.

Definition 1.12. Let (X,S) be an $S-$ metric space and $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{P},\mathcal{Q},\mathcal{R}: X\to X$ be six self-mappings. Then the mappings $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{P},\mathcal{Q}$ and $\mathcal{R}$ are called an $(\mathcal{A},\mathcal{B},\mathcal{C})_{(\psi,\varphi)}-$ weak contraction with respect to $(\mathcal{P},\mathcal{Q},\mathcal{R})$ if there exist two functions $\psi\in\Psi$ and $\varphi\in\Phi$ such that

$$\psi\big(M(x,y,z)\big)\le\psi\big(\Delta(x,y,z)\big)-\varphi\big(\Delta(x,y,z)\big), \tag{1.1}$$

for all $x,y,z\in X$, where

$$M(x,y,z)=\max\Big\{S(\mathcal{A}x,\mathcal{A}x,\mathcal{B}y),S(\mathcal{B}y,\mathcal{B}y,\mathcal{C}z)\Big\}$$

and

$$\Delta(x,y,z)=\max\Big\{S(\mathcal{P}x,\mathcal{P}x,\mathcal{Q}y),S(\mathcal{A}x,\mathcal{A}x,\mathcal{R}z),S(\mathcal{P}x,\mathcal{P}x,\mathcal{B}y),S(\mathcal{Q}y,\mathcal{Q}y,\mathcal{C}z)\Big\}.$$

In the present paper, we discuss some common fixed point theorems for three pairs of self-mappings employing the common property $-(E.A)$ and common limit range property in $S-$metric spaces.

2. Main results

Before we start to prove our main theorems, we discuss the following lemmas.

Lemma 2.1. Let (X,S) be an $S-$ metric space and $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{P},\mathcal{Q},\mathcal{R}: X\to X$ be an $(\mathcal{A},\mathcal{B},\mathcal{C})_{(\psi,\varphi)}-$ weak contraction with respect to $(\mathcal{P},\mathcal{Q},\mathcal{R})$ satisfying the following conditions:
(i) $\mathcal{B}X\subset\mathcal{R}X$ (resp. $\mathcal{A}X\subset\mathcal{R}X$);
(ii) the pairs $(\mathcal{A},\mathcal{P})$ and $(\mathcal{B},\mathcal{Q})$ satisfy the common property $-(E.A)$.
Then the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ share the common property $-(E.A)$.

Proof. Suppose the pair $(\mathcal{A},\mathcal{P})$ and $(\mathcal{B},\mathcal{Q})$ satisfy the common property $-(E.A)$, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty}\mathcal{A}x_n=\lim_{n\to\infty}\mathcal{P}x_n=\lim_{n\to\infty}\mathcal{B}y_n=\lim_{n\to\infty}\mathcal{Q}y_n=t,$$

for some $t\in X$. Since $\mathcal{B}X\subset\mathcal{R}X$ and $\lim_{n\to\infty}\mathcal{B}y_n=t$, then there exist $n_0\in\mathbb{N}\cup\{0\}$ and a sequence $\{z_n\}$ in $\mathcal{R}X$ such that $\mathcal{B}y_n=\mathcal{R}z_n$, for all $n\ge n_0$. Therefore $\lim_{n\to\infty}\mathcal{B}y_n=\lim_{n\to\infty}\mathcal{R}z_n=t$. Now we claim that $\lim_{n\to\infty}\mathcal{C}z_n=t$. On contrary, we suppose that $\lim_{n\to\infty}\mathcal{C}z_n\ne t$, then there exists $\varepsilon>0$ and $k\ge n_0$ for all $k\in\mathbb{N}\cup\{0\}$ such that $\lim_{k\to\infty}S(t,t,\mathcal{C}z_{n_k})=\varepsilon$. For this, from (1.1), we obtain

$$\psi\big(M(x_{n_k},y_{n_k},z_{n_k})\big)\le\psi\big(\Delta(x_{n_k},y_{n_k},z_{n_k})\big)-\varphi\big(\Delta(x_{n_k},y_{n_k},z_{n_k})\big),$$

where

$$M(x_{n_k},y_{n_k},z_{n_k})=\max\Big\{S(\mathcal{A}x_{n_k},\mathcal{A}x_{n_k},\mathcal{B}y_{n_k}),S(\mathcal{B}y_{n_k},\mathcal{B}y_{n_k},\mathcal{C}z_{n_k})\Big\}$$

and

$$\Delta(x_{n_k}, y_{n_k}, z_{n_k}) = \max\Big\{S(\mathcal{P}x_{n_k}, \mathcal{P}x_{n_k}, \mathcal{Q}y_{n_k}), S(\mathcal{A}x_{n_k}, \mathcal{A}x_{n_k}, \mathcal{R}z_{n_k}), S(\mathcal{P}x_{n_k}, \mathcal{P}x_{n_k}, \mathcal{B}y_{n_k}), S(\mathcal{Q}y_{n_k}, \mathcal{Q}y_{n_k}, \mathcal{C}z_{n_k})\Big\}$$

Taking limit as $n \to \infty$, we obtain

$$\lim_{k\to\infty} \psi\big(M(x_{n_k}, y_{n_k}, z_{n_k})\big) \le \lim_{k\to\infty} \psi\big(\Delta(x_{n_k}, y_{n_k}, z_{n_k})\big) - \lim_{k\to\infty} \varphi\big(\Delta(x_{n_k}, y_{n_k}, z_{n_k})\big),$$

where

$$\lim_{k\to\infty} M(x_{n_k}, y_{n_k}, z_{n_k}) = \lim_{k\to\infty} \max\{S(t,t,t), S(t,t,\mathcal{C}z_{n_k})\} = \lim_{k\to\infty} S(t,t,\mathcal{C}z_{n_k}) = \varepsilon$$

and

$$\lim_{k\to\infty} \Delta(x_{n_k}, y_{n_k}, z_{n_k}) \quad = \quad \max\{0,0,0,\varepsilon\} = \varepsilon.$$

Since φ is lower semi-continuous function, so we obtain

$$\varphi(\varepsilon) \le \lim_{k\to\infty} \inf \varphi\big(\Delta(x_{n_k}, y_{n_k}, z_{n_k})\big).$$

Consequently, we obtain

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon)),$$

gives $\varphi(\varepsilon)) = 0$ implies $\varepsilon = 0$. This is a contradiction. □

Lemma 2.2. Let (X, S) be an $S-$ metric space and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q}, \mathcal{R} : X \to X$ be an $(\mathcal{A}, \mathcal{B}, \mathcal{C})_{(\psi,\varphi)}-$ weak contraction with respect to $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ satisfying the following conditions:

(i) $\mathcal{B}X \subset \mathcal{R}X$ and $\mathcal{R}X$ is closed;

(ii) the pairs $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{Q})$ satisfy the $(CLR_{\mathcal{P}\mathcal{Q}})-$ property.

Then the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ share the common property $-(E.A)$.

Proof. By Lemma 2.1, the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ satisfy the common property $-(E.A)$. Then there exist three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{C}z_n = \lim_{n\to\infty} \mathcal{R}z_n = t,$$

for some $t \in \mathcal{P}X \cap \mathcal{Q}X$. Also by (ii), we obtain $t \in \mathcal{R}X$. This completes the proof. □

Theorem 2.1. Let (X, S) be an $S-$ metric space and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q}, \mathcal{R} : X \to X$ be six self-mappings. Suppose the mappings $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q},$ and $\mathcal{R}$ be $(\mathcal{A}, \mathcal{B}, \mathcal{C})_{(\psi,\varphi)}-$ weak contraction with respect to $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ satisfying the following conditions:

(i) the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ share the common property $-(E.A)$;

(ii) $\mathcal{P}X$, $\mathcal{Q}X$ and $\mathcal{R}X$ are closed subsets of X.

Then the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have their coincidence points in X. Further, $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{P},\mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point, provided the pairs $(\mathcal{A},\mathcal{P})$ $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are weakly compatible.

Proof. From (i), the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ share the common property $-(E.A)$, then there exist three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty}\mathcal{A}x_n=\lim_{n\to\infty}\mathcal{P}x_n=\lim_{n\to\infty}\mathcal{B}y_n=\lim_{n\to\infty}\mathcal{Q}y_n=\lim_{n\to\infty}\mathcal{C}z_n=\lim_{n\to\infty}\mathcal{R}z_n=t,$$

for some $t\in X$. Since $\mathcal{P}X$ is a closed subset of X and $\lim_{n\to\infty}\mathcal{P}x_n=t$, then there exists a point $u\in X$ such that $\mathcal{P}u=t$. Now, we assert that $\mathcal{A}u=\mathcal{P}u$. Using inequality (1.1) with $x=u$, $y=y_n$ and $z=z_n$, we get

$$\psi\big(M(u,y_n,z_n)\big)\le\psi\big(\Delta(u,y_n,z_n)\big)-\varphi\big(\Delta(u,y_n,z_n)\big), \tag{2.1}$$

where

$$M(u,y_n,z_n)=\max\{S(\mathcal{A}u,\mathcal{A}u,\mathcal{B}y_n),S(\mathcal{B}y_n,\mathcal{B}y_n,\mathcal{C}z_n)\}$$

and

$$\begin{aligned}\Delta(u,y_n,z_n)=\max\Big\{&S(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}y_n),S(\mathcal{A}u,\mathcal{A}u,\mathcal{R}z_n),S(\mathcal{P}u,\mathcal{P}u,\mathcal{B}y_n),\\&S(\mathcal{Q}y_n,\mathcal{Q}y_n,\mathcal{C}z_n)\Big\}.\end{aligned}$$

Taking the limit as $n\to\infty$ in (2.1), we obtain

$$\psi\big(S(\mathcal{A}u,\mathcal{A}u,t)\big)\le\lim_{n\to\infty}\psi\big(\Delta(u,y_n,z_n)\big)-\lim_{n\to\infty}\varphi\big(\Delta(u,y_n,z_n)\big), \tag{2.2}$$

where

$$\lim_{n\to\infty}M(u,y_n,z_n)=\max\Big\{S(\mathcal{A}u,\mathcal{A}u,t),S(t,t,t)\Big\}=S(\mathcal{A}u,\mathcal{A}u,t)$$

and

$$\begin{aligned}\lim_{n\to\infty}\Delta(u,y_n,z_n)&=\max\Big\{S(\mathcal{P}u,\mathcal{P}u,t),S(\mathcal{A}u,\mathcal{A}u,t),S(\mathcal{P}u,\mathcal{P}u,t),S(t,t,t)\Big\}\\&=\max\Big\{0,S(\mathcal{A}u,\mathcal{A}u,t),0,0\Big\}\\&=S(\mathcal{A}u,\mathcal{A}u,t).\end{aligned} \tag{2.3}$$

Since φ is lower semi-continuous, we obtain

$$\varphi\Big(S(\mathcal{A}u,\mathcal{A}u,t)\Big)\le\lim_{n\to\infty}\inf\varphi\big(\Delta(u,y_n,z_n)\big). \tag{2.4}$$

From (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned}\psi\big(S(\mathcal{A}u,\mathcal{A}u,t)\big)&\le\psi\Big(S(\mathcal{A}u,\mathcal{A}u,t)\Big)-\lim_{n\to\infty}\inf\varphi\big(\Delta(u,y_n,z_n)\big)\\&\le\psi\Big(S(\mathcal{A}u,\mathcal{A}u,t)\Big)-\varphi\Big(S(\mathcal{A}u,\mathcal{A}u,t)\Big).\end{aligned} \tag{2.5}$$

Consequently, $\varphi\big(S(\mathcal{A}u,\mathcal{A}u,t)\big)=0$ implies $S(\mathcal{A}u,\mathcal{A}u,t)=0$. Hence $\mathcal{A}u=t=\mathcal{P}u$. This shows that the pair $(\mathcal{A},\mathcal{P})$ has a coincidence point in X. Since $\mathcal{Q}X$ is a closed subset of X, then $\lim_{n\to\infty}\mathcal{Q}y_n=t\in\mathcal{Q}X$. Then there exists a point $v\in X$ such that $\mathcal{Q}v=t$. Now, we assert that $\mathcal{B}v=\mathcal{Q}v$. Otherwise from (1.1) with $x=u,\ y=v$ and $z=z_n$, we obtain

$$\psi\big(M(u,v,z_n)\big)\le\psi\big(\Delta(u,v,z_n)\big)-\varphi\big(\Delta(u,v,z_n)\big) \tag{2.6}$$

where

$$M(u,v,z_n)=\max\Big\{S(\mathcal{A}u,\mathcal{A}u,\mathcal{B}v),S(\mathcal{B}v,\mathcal{B}v,\mathcal{C}z_n)\Big\}$$

and

$$\begin{aligned}\Delta(u,v,z_n)\;&=\;\max\Big\{S(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v),S(\mathcal{A}u,\mathcal{A}u,\mathcal{R}z_n),S(\mathcal{P}u,\mathcal{P}u,\mathcal{B}v),\\&\qquad S(\mathcal{Q}v,\mathcal{Q}v,\mathcal{C}z_n)\Big\}\end{aligned}$$

Taking the limit as $n\to\infty$ in (2.6), we get

$$\lim_{n\to\infty}\psi\big(M(u,v,z_n)\big)\le\lim_{n\to\infty}\psi\big(\Delta(u,v,z_n)\big)-\lim_{n\to\infty}\varphi\big(\Delta(u,v,z_n)\big) \tag{2.7}$$

where

$$\lim_{n\to\infty}M(u,v,z_n)=\max\Big\{S(t,t,\mathcal{B}v),S(\mathcal{B}v,\mathcal{B}v,t)\Big\}=S(t,t,\mathcal{B}v)$$

and

$$\begin{aligned}\lim_{n\to\infty}\Delta(u,v,z_n)\;&=\;\max\Big\{S(t,t,t),S(t,t,t),S(t,t,\mathcal{B}v),S(t,t,t)\Big\}\\&=\;S(t,t,\mathcal{B}v)\end{aligned} \tag{2.8}$$

Moreover, lower semi-continuity of φ, we have

$$\varphi\big(S(t,t,\mathcal{B}v)\big)\le\lim_{n\to\infty}\varphi\big(\Delta(u,v,z_n)\big) \tag{2.9}$$

From (2.7), (2.8) and (2.9), we obtain

$$\psi\big(S(t,t,\mathcal{B}v)\big)\le\psi\big(S(t,t,\mathcal{B}v)\big)-\varphi\big(S(t,t,\mathcal{B}v)\big),$$

so $\varphi(S(t,t,\mathcal{B}v))=0$ and it implies $S(t,t,\mathcal{B}v)=0$. Hence $\mathcal{B}v=\mathcal{Q}v=t$. This shows that v is a coincidence point of the pair $(\mathcal{B},\mathcal{Q})$ in X.

Also since $\mathcal{R}X$ is a closed subset of X and $\lim_{n\to\infty}\mathcal{R}z_n=t$. Then there exists a point $w\in X$ such that $\mathcal{R}w=t$. We show that $\mathcal{R}w=\mathcal{C}w$. Using inequality (1.1) with $x=u,\ y=v$ and $z=w$, we get

$$\psi\big(M(u,v,w)\big)\le\psi\big(\Delta(u,v,w)\big)-\varphi\big(\Delta(u,v,w)\big),$$

where

$$\begin{aligned} M(u,v,w) &= \max\Big\{S(\mathcal{A}u,\mathcal{A}u,\mathcal{B}v),S(\mathcal{B}v,\mathcal{B}v,\mathcal{C}w)\Big\} \\ &= \max\Big\{S(t,t,t),S(t,t,\mathcal{C}w)\Big\} = S(t,t,\mathcal{C}w) \end{aligned}$$

and

$$\begin{aligned} \Delta(u,v,w) &= \max\Big\{S(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v),S(\mathcal{A}u,\mathcal{A}u,\mathcal{R}w),S(\mathcal{P}u,\mathcal{P}u,\mathcal{B}v),S(\mathcal{Q}v,\mathcal{Q}v,\mathcal{C}w)\Big\} \\ &= \max\Big\{S(t,t,t),S(t,t,t),S(t,t,t),S(t,t,\mathcal{C}w)\Big\} \\ &= S(t,t,\mathcal{C}w). \end{aligned}$$

From the above inequality, we obtain

$$\psi\big(S(t,t,\mathcal{C}w)\big) \le \psi\big(S(t,t,\mathcal{C}w)\big) - \varphi\big(S(t,t,\mathcal{C}w)\big).$$

So $\varphi\big(S(t,t,\mathcal{C}w)\big) = 0$, then $S(t,t,\mathcal{C}w) = 0$. Hence $\mathcal{C}w = t = \mathcal{R}w$. This shows that w is a coincidence point of the pair $(\mathcal{C},\mathcal{R})$.

Thus the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have their coincidence points in X.

It remains to prove that the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have a unique common fixed point in X.

Since the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are weakly compatible. Then $\mathcal{A}u = \mathcal{P}u = t$ implies $\mathcal{A}t = \mathcal{A}\mathcal{P}u = \mathcal{P}\mathcal{A}u = \mathcal{P}t$. Similarly, $\mathcal{B}t = \mathcal{B}\mathcal{Q}v = \mathcal{Q}\mathcal{B}v = \mathcal{Q}t$ and $\mathcal{C}t = \mathcal{C}\mathcal{R}w = \mathcal{R}\mathcal{C}w = \mathcal{R}t$. Therefore, t is a coincidence point of the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$. One can show that $\mathcal{A}t = \mathcal{P}t = t$ by taking $x = t, y = v$ and $z = w$ in (1.1). Also $\mathcal{A}t = \mathcal{B}t$, this can be proved by putting $x = y = t$ and $z = w$ in (1.1). Similarly, by putting $x = u, y = v$ and $z = t$ in (1.1), we obtain $\mathcal{B}t = \mathcal{C}t$. Thus, $\mathcal{A}t = \mathcal{B}t = \mathcal{C}t = \mathcal{P}t = \mathcal{Q}t = \mathcal{R}t$. Now, we show that the point of coincidence of the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ is unique.

If the point of coincidence of the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ is not unique, then there exist $\xi,\xi^* \in X, \xi \neq \xi^*$ such that $\mathcal{A}t = \mathcal{P}t = \mathcal{B}t = \mathcal{Q}t = \xi$ and $\mathcal{C}t = \mathcal{R}t = \xi^*$. Using inequality (1.1), we obtain

$$\psi\big(\mathcal{M}(t,t,t)\big) \le \psi\big(\Delta(t,t,t)\big) - \varphi\big(\Delta(t,t,t)\big).$$

where

$$\begin{aligned} \mathcal{M}(t,t,t) &= \max\Big\{S(\mathcal{A}t,\mathcal{A}t,\mathcal{B}t),S(\mathcal{B}t,\mathcal{B}t,\mathcal{C}t)\Big\} \\ &= \max\Big\{S(\xi,\xi,\xi),S(\xi,\xi,\xi^*)\Big\} = S(\xi,\xi,\xi^*) \end{aligned}$$

and

$$\begin{aligned}\Delta(t,t,t) =&\max\Big\{S(\mathcal{P}t,\mathcal{P}t,\mathcal{Q}t),S(\mathcal{A}t,\mathcal{A}t,\mathcal{R}t),S(\mathcal{P}t,\mathcal{P}t,\mathcal{B}t),S(\mathcal{Q}t,\mathcal{Q}t,\mathcal{C}t)\Big\}\\ =&\max\Big\{S(\xi,\xi,\xi),S(\xi,\xi,\xi^*),S(\xi,\xi,\xi),S(\xi,\xi,\xi^*)\Big\}\\ =&S(\xi,\xi,\xi^*)\end{aligned}$$

Therefore, the above inequality becomes

$$\psi\big(S(\xi,\xi,\xi^*)\big) \le \psi\big(S(\xi,\xi,\xi^*)\big) - \varphi\big(S(\xi,\xi,\xi^*)\big),$$

so $\varphi\big(S(\xi,\xi,\xi^*)\big) = 0$ i.e., $S(\xi,\xi,\xi^*) = 0$ which implies $\xi = \xi^*$. Therefore, the point of coincidence of the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ is unique and hence by Lemma 1.5, the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have a unique common fixed point in X. □

Example 2.1. Let $X = [0,1]$. Define a mapping $S : X^3 \to [0,\infty)$ by

$$S(x,y,z) = \begin{cases} 0, & \text{if } x = y = z \\ \max\{x,y,z\}, & \text{otherwise}\end{cases}$$

for all $x,y,z \in X$. Clearly, (X,S) is an $S-$ metric space. Consider the self-mappings $\mathcal{A}x = \frac{x}{4}$, $\mathcal{B}x = \frac{x}{4}$, $\mathcal{C}x = \frac{x}{4}$, $\mathcal{P}x = x$, $\mathcal{Q}x = \mathcal{R}x = \frac{x}{2}$, for all $x \in X$. Setting $\psi(t) = t$ and $\varphi(t) = \frac{t}{4}$ for $t \in [0,\infty)$.

(a) In order to check the inequality (1.1), consider the following four cases:

(i) $x = y = z$, (ii) $x \le y < z$, (iii) $x \le z < y$, (iv) $y \le z < x$.

Case (i): If $x = y = z$, we get $M(x,y,z) = 0$, so the condition is trivially satisfied.

Case (ii): If $x \le y < z$. Then, we have

$$M(x,y,z) = \max\left\{S\left(\frac{x}{4},\frac{x}{4},\frac{y}{4}\right),S\left(\frac{y}{4},\frac{y}{4},\frac{z}{4}\right)\right\} = \frac{z}{4}$$

and

$$\begin{aligned}\Delta(x,y,z) &= \max\left\{S\left(x,x,\frac{y}{2}\right),S\left(\frac{x}{4},\frac{x}{4},\frac{z}{2}\right),S\left(x,x,\frac{y}{4}\right),S\left(\frac{y}{2},\frac{y}{2},\frac{z}{4}\right)\right\}\\ &= x \text{ or } \frac{z}{2}\end{aligned}$$

If $x < \frac{z}{2}$, then $\psi\left(\frac{z}{4}\right) = \frac{z}{4} \le \frac{3z}{8} = \psi\left(\frac{z}{2}\right) - \varphi\left(\frac{z}{2}\right)$

If $\frac{z}{2} < x \implies \frac{z}{4} < \frac{x}{2}$, so $\psi\left(\frac{z}{4}\right) < \psi\left(\frac{x}{2}\right) \le \frac{3x}{4} = \psi(x) - \varphi(x)$.

Similarly, the inequality (1.1) is also satisfied for case (iii).

Case (iv): If $y \le z < x$, we have $M(x,y,z) = \frac{x}{4}$ and $\Delta(x,y,z) = x$, so the inequality (1.1) reduces to

$$\psi\left(\frac{x}{4}\right) = \frac{x}{4} \le \frac{3x}{4} = \psi(x) - \varphi(x).$$

Thus, for all $x, y, z \in X$, we obtain

$$\psi\big(M(x,y,z)\big) \leq \psi\big(\Delta(x,y,z)\big) - \varphi\big(\Delta(x,y,z)\big).$$

(b) Now, let us show that the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are weakly compatible. For this, let $\mathcal{A}x = \mathcal{P}x \implies \frac{x}{4} = x \implies x = 0$. Now, $\mathcal{A}\mathcal{P}0 = \mathcal{A}0 = 0 = \mathcal{P}0 = \mathcal{P}\mathcal{A}0$. Therefore, $(\mathcal{A},\mathcal{P})$ is weakly compatible. Similarly, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are also weakly compatible mappings.

(c) Now, we show that the pairs $(\mathcal{A},\mathcal{P}),(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ share the common property $-(E.A)$. For this, let $x_n = \frac{1}{n},\ y_n = \frac{1}{n+2}$ and $z_n = \frac{1}{2n+3}$ for $n \in \mathbb{N}$. Clearly, $\{x_n\},\ \{y_n\}$ and $\{z_n\}$ are in X. Then, we have

$$S(\mathcal{A}x_n, \mathcal{A}x_n, 0) = S\left(\frac{1}{4n}, \frac{1}{4n}, 0\right) = \max\left\{\frac{1}{4n}, \frac{1}{4n}, 0\right\} = \frac{1}{4n} \to 0 \text{ as } n \to \infty.$$

Also,

$$S(\mathcal{P}x_n, \mathcal{P}x_n, 0) = S\left(\frac{1}{n}, \frac{1}{n}, 0\right) = \max\left\{\frac{1}{n}, \frac{1}{n}, 0\right\} = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Similarly, we get that $\mathcal{B}y_n,\ \mathcal{Q}y_n,\ \mathcal{C}z_n$ and $\mathcal{R}z_n \to 0$ as $n \to \infty$.

Therefore, there exist three sequences $\{x_n\},\ \{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{C}z_n = \lim_{n\to\infty} \mathcal{R}z_n = t,$$

Therefore, $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ share the common property $-(E.A)$.

(d) As $\mathcal{P}X = [0,1],\ \mathcal{Q}X = \mathcal{R}X = [0, \frac{1}{2}]$, then $\mathcal{P}X,\ \mathcal{Q}X$ and $\mathcal{R}X$ are closed subsets of X.

Therefore, all the conditions of Theorem 2.1 are satisfied and 0 is the unique common fixed point of the self-mappings.

Theorem 2.2. *Let (X,S) be an $S-$ metric space and $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{P},\mathcal{Q},\mathcal{R} : X \to X$ be an $(\mathcal{A},\mathcal{B},\mathcal{C})_{(\psi,\varphi)}-$ weak contraction with respect to $(\mathcal{P},\mathcal{Q},\mathcal{R})$. If the pairs $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ satisfy the $(CLR_{\mathcal{P}\mathcal{Q}\mathcal{R}})-$ property, then $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have their coincidence points.*

Moreover, $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{P},\mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point provided the pairs $(\mathcal{A},\mathcal{P}),(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are weakly compatible.

Proof. Suppose the pairs $(\mathcal{A},\mathcal{P}),(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ satisfy the $(CLR_{\mathcal{P}\mathcal{Q}\mathcal{R}})-$ property, then there exist three sequences $\{x_n\},\ \{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{A}x_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{B}y_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{C}z_n = \lim_{n\to\infty} \mathcal{R}z_n = t,$$

for some $t \in \mathcal{P}X \cap \mathcal{Q}X \cap \mathcal{R}X$. It follows that $t \in \mathcal{P}X$ and there exists $u \in X$ such that $\mathcal{P}u = t$. Now we assert that $\mathcal{A}u = \mathcal{P}u$. Using inequality (1.1) with $x = u,\ y = y_n,\ z = z_n$, we get

$$\psi\big(M(u,y_n,z_n)\big) \leq \psi\big(\Delta(u,y_n,z_n)\big) - \varphi\big(\Delta(u,y_n,z_n)\big), \tag{2.10}$$

where

$$M(u, y_n, z_n) = \max\Big\{S(\mathcal{A}u, \mathcal{A}u, \mathcal{B}y_n), S(\mathcal{B}y_n, \mathcal{B}y_n, \mathcal{C}z_n)\Big\}$$

$$\begin{aligned}\Delta(u, y_n, z_n) &= \max\Big\{S(\mathcal{P}u, \mathcal{P}u, \mathcal{Q}y_n), S(\mathcal{A}u, \mathcal{A}u, \mathcal{R}z_n), S(\mathcal{P}u, \mathcal{P}u, \mathcal{B}y_n),\\ &\quad S(\mathcal{Q}y_n, \mathcal{Q}y_n, \mathcal{C}z_n)\Big\}.\end{aligned}$$

Taking the limit as $n \to \infty$ in (2.10), we get

$$\lim_{n\to\infty} \psi\big(M(u, y_n, z_n)\big) \le \lim_{n\to\infty} \psi\big(\Delta(u, y_n, z_n)\big) - \lim_{n\to\infty} \varphi\big(\Delta(u, y_n, z_n)\big)$$

where

$$\lim_{n\to\infty} M(u, y_n, z_n) = \max\Big\{S(\mathcal{A}u, \mathcal{A}u, t), S(t, t, t)\Big\} = S(\mathcal{A}u, \mathcal{A}u, t)$$

$$\begin{aligned}\lim_{n\to\infty} \Delta(u, y_n, z_n) &= \max\Big\{S(t, t, t), S(\mathcal{A}u, \mathcal{A}u, t), S(t, t, t), S(t, t, t),\Big\}\\ &= S(\mathcal{A}u, \mathcal{A}u, t).\end{aligned}$$

From the above inequality, we obtain

$$\psi\big(S(\mathcal{A}u, \mathcal{A}u, t)\big) \le \psi\big(S(\mathcal{A}u, \mathcal{A}u, t)\big) - \varphi\big(S(\mathcal{A}u, \mathcal{A}u, t)\big),$$

so $\varphi\big(S(\mathcal{A}u, \mathcal{A}u, t)\big) = 0$, i.e., $S(\mathcal{A}u, \mathcal{A}u, t) = 0$. Hence $\mathcal{A}u = t = \mathcal{P}u$, which shows that u is a coincidence point of the pair $(\mathcal{A}, \mathcal{P})$. As $t \in \mathcal{Q}X$, there exists a point $v \in X$ such that $\mathcal{Q}v = t$. We show that $\mathcal{B}v = \mathcal{Q}v$. Using inequality (1.1) with $x = u,\ y = v$ and $z = z_n$, we have

$$\psi\big(M(u, v, z_n)\big) \le \psi\big(\Delta(u, v, z_n)\big) - \varphi\big(\Delta(u, v, z_n)\big) \tag{2.11}$$

where

$$\begin{aligned}M(u, v, z_n) &= \max\Big\{S(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v), S(\mathcal{B}v, \mathcal{B}v, \mathcal{C}z_n)\Big\}\\ &= \max\Big\{S(t, t, \mathcal{B}v), S(\mathcal{B}v, \mathcal{B}v, \mathcal{C}z_n)\Big\}\end{aligned}$$

and

$$\begin{aligned}\Delta(u, v, z_n) &= \max\Big\{S(\mathcal{P}u, \mathcal{P}u, \mathcal{Q}v), S(\mathcal{A}u, \mathcal{A}u, \mathcal{R}z_n), S(\mathcal{P}u, \mathcal{P}u, \mathcal{B}v),\\ &\quad S(\mathcal{Q}v, \mathcal{Q}v, \mathcal{C}z_n)\Big\}\\ &= \max\Big\{S(t, t, t), S(t, t, \mathcal{R}z_n), S(t, t, \mathcal{B}v), S(t, t, \mathcal{C}z_n)\Big\}\end{aligned}$$

Taking the limit as $n \to \infty$ in (2.11), we get

$$\lim_{n\to\infty} \psi\big(M(u, v, z_n)\big) \le \lim_{n\to\infty} \psi\big(\Delta(u, v, z_n)\big) - \lim_{n\to\infty} \varphi\big(\Delta(u, v, z_n)\big)$$

where

$$\lim_{n\to\infty} M(u,v,z_n) = \max\Big\{S(t,t,\mathcal{B}v), S(\mathcal{B}v,\mathcal{B}v,t)\Big\} = S(\mathcal{B}v,\mathcal{B}v,t)$$

and

$$\begin{aligned}\lim_{n\to\infty} \Delta(u,v,z_n) &= \max\Big\{S(t,t,t), S(t,t,t), S(t,\mathcal{B}v,\mathcal{B}v), S(t,t,t)\Big\}\\ &= S(\mathcal{B}v,\mathcal{B}v,t),\end{aligned}$$

The above equation gives

$$\psi\big(S(\mathcal{B}v,\mathcal{B}v,t)\big) \le \psi\big(S(\mathcal{B}v,\mathcal{B}v,t)\big) - \varphi\big(S(\mathcal{B}v,\mathcal{B}v,t)\big),$$

so $\varphi(S(\mathcal{B}v,\mathcal{B}v,t)) = 0$, i.e., $S(\mathcal{B}v,\mathcal{B}v,t) = 0$. Hence, $\mathcal{B}v = Qv = t$, which shows that v is a coincidence point of the pair $(\mathcal{B},\mathcal{Q})$.

As $t \in \mathcal{R}X$, there exists a point $w \in X$ such that $\mathcal{R}w = t$. We show that $\mathcal{R}w = \mathcal{C}w$. Using inequality (1.1) with $x = u,\ y = v$ and $z = w$, we get

$$\psi\big(M(u,v,w)\big) \le \psi\big(\Delta(u,v,w)\big) - \varphi\big(\Delta(u,v,w)\big)$$

where

$$M(u,v,w) = \max\Big\{S(\mathcal{A}u,\mathcal{A}u,\mathcal{B}v), S(\mathcal{B}v,\mathcal{B}v,\mathcal{C}w)\Big\} = S(t,t,\mathcal{C}w)$$

and

$$\begin{aligned}\Delta(u,v,w) &= \max\Big\{S(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v), S(\mathcal{A}u,\mathcal{A}u,\mathcal{R}w), S(\mathcal{P}u,\mathcal{P}u,\mathcal{B}v), S(\mathcal{Q}v,\mathcal{Q}v,\mathcal{C}w)\Big\}\\ &= \max\Big\{S(t,t,t), S(t,t,t), S(t,t,t), S(t,t,\mathcal{C}w)\Big\}\\ &= S(t,t,\mathcal{C}w).\end{aligned}$$

Follows from the above inequality, we obtain

$$\psi\big(S(t,t,\mathcal{C}w)\big) \le \psi\big(S(t,t,\mathcal{C}w)\big) - \varphi\big(S(t,t,\mathcal{C}w)\big),$$

so $\varphi\big(S(t,t,\mathcal{C}w)\big) = 0$, i.e., $S(t,t,\mathcal{C}w) = 0$. Hence, $\mathcal{C}w = t = \mathcal{R}w$, which shows that w is a point of coincidence of the pair $(\mathcal{C},\mathcal{R})$. Thus the pairs $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have their coincidence points in X.

It remains to prove that the pairs $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have a unique common fixed point in X.

Since the pairs $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are weakly compatible. Then $\mathcal{A}u = \mathcal{P}u = t$ implies $\mathcal{A}t = \mathcal{A}\mathcal{P}u = \mathcal{P}\mathcal{A}u = \mathcal{P}t$. Similarly, $\mathcal{B}t = \mathcal{B}\mathcal{Q}v = \mathcal{Q}\mathcal{B}v = \mathcal{Q}t$ and $\mathcal{C}t = \mathcal{C}\mathcal{R}w = \mathcal{R}\mathcal{C}w = \mathcal{R}t$. Therefore, t is a coincidence point of the pairs $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$. Following the same steps as in Theorem 2.1, one can show that $\mathcal{A}t = \mathcal{B}t = \mathcal{C}t = \mathcal{P}t = \mathcal{Q}t = \mathcal{R}t$. Now, we show that the point of coincidence of the pairs $(\mathcal{A},\mathcal{P}),\ (\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ is unique.

If the point of coincidence of the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ is not unique, then there exist $\xi,\xi^*\in X,\xi\neq\xi^*$ such that $\mathcal{A}t=\mathcal{P}t=\mathcal{B}t=\mathcal{Q}t=\xi$ and $\mathcal{C}t=\mathcal{R}t=\xi^*$. Using inequality (1.1), we obtain

$$\psi\big(\mathcal{M}(t,t,t)\big)\leq\psi\big(\Delta(t,t,t)\big)-\varphi\big(\Delta(t,t,t)\big),$$

where

$$\begin{aligned}\mathcal{M}(t,t,t)&=\max\Big\{S(\mathcal{A}t,\mathcal{A}t,\mathcal{B}t),S(\mathcal{B}t,\mathcal{B}t,\mathcal{C}t)\Big\}\\&=\max\Big\{S(\xi,\xi,\xi),S(\xi,\xi,\xi^*)\Big\}=S(\xi,\xi,\xi^*)\end{aligned}$$

and

$$\begin{aligned}\Delta(t,t,t)&=\max\Big\{S(\mathcal{P}t,\mathcal{P}t,\mathcal{Q}t),S(\mathcal{A}t,\mathcal{A}t,\mathcal{R}t),S(\mathcal{P}t,\mathcal{P}t,\mathcal{B}t),S(\mathcal{Q}t,\mathcal{Q}t,\mathcal{C}t)\Big\}\\&=\max\Big\{S(\xi,\xi,\xi),S(\xi,\xi,\xi^*),S(\xi,\xi,\xi),S(\xi,\xi,\xi^*)\Big\}\\&=S(\xi,\xi,\xi^*)\end{aligned}$$

Therefore, the above inequality becomes

$$\psi\big(S(\xi,\xi,\xi^*)\big)\leq\psi\big(S(\xi,\xi,\xi^*)\big)-\varphi\big(S(\xi,\xi,\xi^*)\big),$$

so $\varphi\big(S(\xi,\xi,\xi^*)\big)=0$ i.e., $S(\xi,\xi,\xi^*)=0$ which implies $\xi=\xi^*$. Therefore, the point of coincidence of the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ is unique and hence by Lemma 1.5, the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ have a unique common fixed point in X.

□

Example 2.2. Let $X=[0,20]$. Define a mapping $S:X^3\to[0,\infty)$ by $S(x,y,z)=|x-y|+|y-z|$, $\forall x,y,z\in X$. Clearly, (X,S) is an $S-$metric space.

Let $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{P},\mathcal{Q},\mathcal{R}:X\to X$ be six self-mappings defined by

$$Ax=\begin{cases}2, \text{ if } x\in[0,2]\\3, \text{ if } x\in(2,20]\end{cases};\ Bx=\begin{cases}1, \text{ if } x\in[0,2)\\2, \text{ if } x\in[2,20]\end{cases};\ Cx=\begin{cases}2, \text{ if } x\in[0,2]\\1, \text{ if } x\in(2,20]\end{cases}$$

$$Px=\begin{cases}2, \text{ if } x\in[0,2]\\6, \text{ if } x\in(2,20]\end{cases},\ Qx=\begin{cases}4, \text{ if } x\in[0,2)\\2, \text{ if } x\in[2,20]\end{cases};\ Rx=\begin{cases}2, \text{ if } x\in[0,2]\\8, \text{ if } x\in(2,20].\end{cases}$$

Consider three sequences $\{x_n\}=\{2-\frac{1}{n}\}$, $\{y_n\}=\{2+\frac{1}{n+1}\}$, $\{z_n\}=\{\frac{1}{n}\},\forall n\in\mathbb{N}$.

$$\lim_{n\to\infty}\mathcal{A}x_n=\lim_{n\to\infty}\mathcal{P}x_n=\lim_{n\to\infty}\mathcal{B}y_n=\lim_{n\to\infty}\mathcal{Q}y_n=\lim_{n\to\infty}\mathcal{C}z_n=\lim_{n\to\infty}\mathcal{R}z_n=2,$$

where $2\in\mathcal{P}X\cap\mathcal{Q}X\cap\mathcal{R}X$. Therefore, the pairs $(\mathcal{A},\mathcal{P})$, $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ satisfy $(CLR_{\mathcal{PQR}})-$ property.

Consider $\psi(t)=t$ and $\varphi(t)=\frac{t}{4}$.

In order to check the inequality (1.1), we have the following eight cases:

(i) $x, z \in [0,2], y \in [0,2)$, (ii) $x \in [0,2], y \in [0,2)$, $z \in (2,20]$, (iii) $x \in [0,2]$, $y \in [2,20]$, $z \in [0,2]$, (iv) $x \in [0,2]$, $y \in [2,20]$, $z \in (2,20]$, (v) $x \in (2,20], y \in [0,2), z \in [0,2]$, (vi) $x \in (2,20] y \in [0,2), z \in (2,20]$, (vii) $x \in (2,20], y \in [2,20], z \in [0,2]$, $(viii)$ $x \in (2,20], y \in [2,20], z \in (2,20]$,

In case (i), we have $M(x,y,z) = 1$ and $\Delta(x,y,z) = 2$, so the inequality (1.1) reduces to

$$\psi(1) = 1 \leq \frac{3}{2} = \psi(2) - \varphi(2)$$

In case (ii) and (vi), we have $M(x,y,z) = 1$ and $\Delta(x,y,z) = 6$, so (1.1) reduces to

$$\psi(1) = 1 \leq \frac{9}{2} = \psi(6) - \varphi(6).$$

In case (iii), we have $M(x,y,z) = 0$, so the inequality (1.1) is trivially satisfied. In case (v) and (vi), we have $M(x,y,z) = 2$ and $\Delta(x,y,z) = 5$, so the inequality (1.1) reduces to

$$\psi(2) = 2 \leq \frac{15}{4} = \psi(5) - \varphi(5)$$

In case (vii), we have $M(x,y,z) = 1$ and $\Delta(x,y,z) = 4$, so the inequality (1.1) reduces to

$$\psi(1) = 1 \leq 3 = \psi(4) - \varphi(4)$$

In case $(viii)$, we have $M(x,y,z) = 1$ and $\Delta(x,y,z) = 5$, so the inequality (1.1) reduces to

$$\psi(1) = 1 \leq \frac{15}{4} = \psi(5) - \varphi(5)$$

Thus, the inequality (1.1) holds true for all $x, y, z \in X$.

Hence, all the conditions of Theorem 2.2 are satisfied, and 2 is a unique common fixed point of the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ which also remains a point of coincidence. Here, one may notice that all the involved mappings are discontinuous at their unique common fixed point 2.

Theorem 2.3. Let (X, S) be an $S-$ metric space and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q}, \mathcal{R} : X \to X$ be an $(\mathcal{A}, \mathcal{B}, \mathcal{C})_{(\psi,\varphi)}-$ weak contraction with respect to $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ satisfying the following conditions:

(i) $\mathcal{B}X \subset \mathcal{R}X$ (resp. $\mathcal{A}X \subset \mathcal{R}X$);

(ii) the pairs $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{Q})$ satisfy the common property $-(E.A)$;

(iii) $\mathcal{P}X$, $\mathcal{Q}X$ and $\mathcal{R}X$ are closed subsets of X.

Then the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ have their coincidence points in X. Further, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point, provided the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ are weakly compatible.

Proof. It follows from Lemma 2.1 and Theorem 2.1. □

Theorem 2.4. Let (X, S) be an $S-$ metric space and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q}, \mathcal{R} : X \to X$ be an $(\mathcal{A}, \mathcal{B}, \mathcal{C})_{(\psi,\varphi)}-$ weak contraction with respect to $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ satisfying the following conditions:

(i) $\mathcal{B}X \subset \mathcal{R}X$ and $\mathcal{R}X$ is closed;

(ii) the pairs $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{Q})$ satisfy the $(CLR_{\mathcal{PQ}})-$ property.

Then the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ have their coincidence points in X. Further, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point, provided the pairs $(\mathcal{A}, \mathcal{P})$, $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ are weakly compatible.

Proof. It follows from Lemma 2.2 and Theorem 2.2. □

2.1. **Conclusion.** The concepts of the property $-(E.A)$ and the common limit range property for six self-mappings are discussed to obtain common fixed point theorems of $(\psi, \varphi)-$ weak contraction with illustrative examples on $S-$metric space. The main advantages of this work are, the mappings and the space used in our results do not require continuity and completeness to obtain the fixed point.

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