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SOME RESULTS BY USING CLR's-PROPERTY IN PROBABILISTIC 2-METRIC SPACE

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ABSTRACT. The aim of this paper is to generate two fixed point theorems in probabilistic 2-metric space by applying CLR'_S-property and occasionally weakly compatible mappings (OWC), these two results generalize the theorem proved by V. K. Gupta, Arihant Jain and Rajesh Kumar. Further these results are justified with suitable examples.

1. INTRODUCTION

Menger [1] pioneered the statistical metric(SM) space theory. One of the major achievements was the translation of probabilistic concepts into geometry. Menger used the notation of new distance distribution function from p to q by a Fpq. B. Schweizer, and A. Sklar [2] introduced a new notion of a probabilistic-norm. This norm naturally generates topology, convergence ,continuity and completeness in SM-space. Mishra [3] used compatible mappings and generated some fixed points in Menger space. Altumn Turkoglu [4] proved some more results of SM-space by utilizing the implicit relation in multivalued mappings. Zhang, Xiaohong, Huacan He, and Yang Xu [5] employed the Schweizer-Sklar t-norm established fuzzy

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logic system to contribute in development of SM-space. Sehgal, V. M., and A. T. Bharucha-Reid [6] used classical Banach contraction to establish the first result of Menger space for coincidence points. Weakly compatible mappings were generalized by Al-Thagafi and Shahzad [7], by introducing occasionally weakly compatible mappings. Futher Chauhan, Sunny, Wutiphol Sintunavarat, and Poom Kumam[9] proved some more theorems by using CLR'S-property in fuzzy metric space. Further some more results can be witnessed by using the concepts of sub sequentially continuous and semi compatible mappings in Menger space [10].

2. PRELIMINARIES

Definition 2.1 [8] A continuously t-norm is mapping t: [0, 1] × [0, 1] → [0, 1] and it satisfies the following properties

(t_1) $t\ is\ abelian$ & $associative$

(t_2) $t(\gamma, 1) = \gamma,\ \forall\, \gamma \in [0,1]$

(t_3) $t\,(\gamma, \omega) \leq t\,(\alpha, \vartheta)\ for\, \gamma \leq \alpha\ and\ \omega \leq \vartheta\ \forall\, \gamma, \alpha, \vartheta, \omega \in [0,1]$.

Definition 2.2 [8] The pair (X, F) named as Probabilistic 2-metric space (2-PM space) where $X \neq \emptyset$ and $F : X \times X \times X \to L$ here L is the set of all distribution functions and the F value at $(u, v, w) \in X \times X \times X$ is represented by $F_{u,v,w}$ and obeys properties as under

a) $F_{u,v,w}$ (0) = 0,
b) For all distinct u, v in X ∃ w ∈ X with $F_{u,v,w}$ (t) < 1 for some t > 0,
c) $F_{u,v,w}$ (t) = 1 ∀ t > 0, If any two of the three points have to be the same,
d) $F_{u,v,w}$ (t) = $F_{v,w,u}$ (t) = $F_{w,u,v}$ (t),
e) $F_{u,v,w}$ (t_a) = $F_{v,w,u}$ (t_b) = $F_{w,u,v}$ (t_c) = 1 then $F_{u,v,w}$ ($t_a + t_b + t_c$) = 1.

Definition 2.3 [8] A sequence $\langle x_n \rangle$ in 2-Menger space (X, F, t) is

(i) **Converges** to β if for each $\epsilon^* > 0$, $t > 0$, ∃ N(ϵ^*) ∈ N implies $F_{x_n,\beta,a}$ (ϵ^*) > 1 –t, $\forall\, a \epsilon\, X\ and\ n \geq N\,(\epsilon^*)$.

(ii) **Cauchy** if for each $\epsilon > 0$, $t > 0$, ∃ N(ϵ) ∈ N implies $F_{x_n,x_m,a}$ (ϵ) > 1 – t,

$\forall\, a \in X\ and\ n, m \geq N(\epsilon)$.

(iii) If the cauchy sequence converges in X then it is referred as a **complete** 2-Menger space.

Definition 2.4 [8] Self-mappings P, S in 2-Menger space (X, F, t) are called

(i) **Compatible** If $F_{PSx_n, SPx_n,\ a}(\delta) \to 1,\ \ \forall\, a \in X$ and $\delta > 0$ whenever a sequence $\langle x_n \rangle$ in X $\ni Px_n, Sx_n \to z$ where z is an element of X as n $\to \infty$.

(ii) **Weakly compatible** if the mappings commute at their coincidence points.

(iii) **Occasionally weakly compatible** (OWC) if $\exists$ x in X $\ni$ Px = Sx $\Rightarrow$ PSx = SPx.

Remark 2.5 Two weakly compatible mappings are obviously OWC mappings, but the converse does not have to be the case.

Example 2.6. By treating $X = [0,\ 1]$ and d be the usual metric on X and for all $t_1 \in [0, 1]$, define

$$F_{u,\ v,\ a}(t_1) = \begin{cases} \frac{t_1}{t_1 + |\ \alpha - \beta\ |}, & t_1 > 0 \\ 0 & , t_1 = 0 \end{cases} \qquad \forall\ \alpha, \beta \text{ in X and fixed a}, t_1 > 0.$$

Define mappings $P, S : X \to X\ as\ P(x) = \frac{x^2}{2}$, x $\in [0,\ 1]$ and $S(x) = \frac{x}{3},\ \ x \in [0,\ 1]$.

We notice that the pair (P, S) has two coincidence points 0, $\frac{2}{3}$.

If x = $\frac{2}{3}$ then P $(\frac{2}{3})$ = S $(\frac{2}{3}) = \frac{2}{9}$ (2.6.1)

$PS\left(\frac{2}{3}\right) = P(\frac{2}{9}) = \frac{2}{81}$, (2.6.2)

$SP(\frac{2}{3})$ = S$(\frac{2}{9}) = \frac{2}{27}$. (2.6.3)

From (2.6.2) and (2.6.3) PS$(\frac{2}{3}) \neq$ SP$(\frac{2}{3})$.

At x = 0, P (0) = S (0) and $PS(0) = SP(0)$.

This shows the mappings P, S are OWC but not weakly compatible.

Definition 2.7 [9] "Self maps P and S of a 2-Menger space (X, F, t) are said to satisfy CLR'$_S$ – property (common limit range property) if there exists a sequence $\langle x_n \rangle \in X \ni Px_n, Sx_n \to Sz$, for some element z $\in X$ as n $\to \infty$.

This example shows that mappings P, S satisfy CLR'$_S$- property but they do not have closed ranges.

Example 2.8. Take X = (0, 1] and $t \in [0, 1]$, define

$F_{u,v}(\mathrm{t}) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & , t = 0 \end{cases}$ $\forall\ \alpha, \beta$ in X and $t\ >\ 0$.

$$Define\ P, S: X \rightarrow X\ as\ P(x) = \begin{cases} 1 - x, & x \in (0,\ \frac{\mathbf{2}}{\mathbf{3}}) \\ x, & x \in [\frac{\mathbf{2}}{\mathbf{3}},\ 1] \end{cases} \qquad (2.8.1)$$

and

$$S(x) = \begin{cases} 2x, & x \in (0,\ \frac{2}{3}] \\ 1, & x \in \left(\frac{2}{3},\ 1\right]. \end{cases} \qquad (2.8.2)$$

Consider a sequence $x_n = \frac{1}{3} - \frac{1}{3n}$ for n = 1, 2, 3… then (2.8.3)

$\mathrm{P}x_n = 1 - \left(\frac{\mathbf{1}}{\mathbf{3}} - \frac{1}{3n}\right) = \frac{\mathbf{2}}{\mathbf{3}} + \frac{1}{3n} \rightarrow \frac{\mathbf{2}}{\mathbf{3}}$ (2.8.4)

$\mathrm{S}x_n = 2(\frac{\mathbf{1}}{\mathbf{3}} - \frac{1}{3n}) = \frac{\mathbf{2}}{\mathbf{3}} - \frac{2}{9n} \rightarrow \frac{\mathbf{2}}{\mathbf{3}}$ as n $\rightarrow \infty$. (2.8.5)

Thus $Px_n, Sx_n \rightarrow S(\frac{\mathbf{1}}{\mathbf{3}}) = \frac{\mathbf{2}}{\mathbf{3}}$ as n $\rightarrow \infty$. (2.8.6)

Where P(X) = $(\frac{1}{3}, 1]$, S(X) = $(0,\ \frac{4}{3}]$ U{1} this shows that P, S are satisfy CLR'$_S$- property but they do not have closed ranges.

Now we give the statement of Theorem (A). It is proved by V. K. Gupta et al.

Theorem (A) [8] " Let A, B, S and T be self -mappings on a complete probabilistic 2-metric space ($\tilde{\mathrm{X}}$, F, t) satisfying:

(A_1) A($\tilde{\mathrm{X}}$) ⊆ T($\tilde{\mathrm{X}}$), B($\tilde{\mathrm{X}}$) ⊆ S($\tilde{\mathrm{X}}$)

(A_2) one of A($\tilde{\mathrm{X}}$), B($\tilde{\mathrm{X}}$), T($\tilde{\mathrm{X}}$) or S($\tilde{\mathrm{X}}$) is complete,

(A_3) pairs (A, S) and (B, T) are weakly compatible,

(A_4) $F_{Ax.\ By,\ a}$ (t) ≥ r $F_{Sx.\ Ty,\ a}$ (t) for all x, y and t > 0,

where r: [0, 1] → [0, 1] is some continuous function such that r(t) > t for each o < t < 1,

then A,B,S and T have unique common fixed point in $\tilde{\mathrm{X}}$ ''.

We now generalize Theorem(A) as under.

3. MAIN RESULT

Theorem 3.1 Let A, B, S and T be self -mappings on a complete probabilistic 2-metric space

($\tilde{\mathrm{X}}$, F, t^*) satisfying :

(3.1.1) A($\tilde{\mathrm{X}}$) ⊆ T($\tilde{\mathrm{X}}$) , B($\tilde{\mathrm{X}}$) ⊆ S($\tilde{\mathrm{X}}$),

(3.1.2) the pairs (A, S), (B, T) share the CLR'$_S$ property with OWC, (3.1.3) $F_{Ax.\ By,\ a}(t^*) \geq$ r $F_{Sx.\ Ty,\ a}(t^*)$ for all x, y and $t^* > 0$,

where r: [0, 1] → [0, 1] is some continuous function such that r(t^*) > t^* for each o < t^* < 1

then A, B, S and T have unique common fixed point in $\widetilde{X}$.

Proof:

Iteratively the sequences $\langle y_n \rangle$ and $\langle x_n \rangle$ can be constructed as

$x_0 \in \widetilde{X} \Rightarrow Ax_0 \in A(\widetilde{X}) \subseteq T(\widetilde{X})$, $\exists\, x_1 \in \widetilde{X}$ in such a way that $Ax_0 = Tx_1$,

$Bx_1 \in B(\widetilde{X}) \subseteq S(\widetilde{X})$ then we have $x_2 \in \widetilde{X}$ with $Bx_1 = Sx_2$

$\langle y_{2n} \rangle = Ax_{2n} = Tx_{2n+1}$ and $\langle y_{2n+1} \rangle = Bx_{2n+1} = Sx_{2n+2}$. (3.1.4)

Now our claim is to show $\langle y_n \rangle$ is cauchy sequence.

For this take $x = x_{2n}$, $y = x_{2n+1}$ in (3.1.3) we get

$F_{Ax_{2n},\ B x_{2n+1},\ a}(t^*) \geq rF_{S x_{2n}\,.\ T x_{2n+1}}(t^*)$, (3.1.5)

$\Rightarrow F_{y_{2n},\ y_{2n+1},\ a}(t^*) \geq rF_{y_{2n-1}\,.\ y_{2n},\ a}(t^*) > F_{y_{2n-1}\,.\ y_{2n},\ a}(t^*)$. (3.1.6)

Similarly

$F_{y_{2n+1},\ y_{2n+2},\ a}(t^*) > F_{y_{2n}\,.\ y_{2n+1},\ a}(t^*)$. (3.1.7)

In general we have $F_{y_{n+1}.\ y_n,\ a}(t^*) > F_{y_n, y_{n-1},\ a}(t^*)$ for all values of n.

Then < $F_{y_{n+1}.\ y_n,\ a}(t^*)$ > is an increasing sequence bounded above by 1 therefore it must converge to L, where L ≤ 1.

If L < 1 then $F_{y_{n+1}.\ y_n,\ a}(t^*)$ = L > r(1) > 1 as a result of the contradiction, L= 1.

Hence $F_{y_{n+1}.\ y_n,\ a}$ (t) = 1 for all n and p.

As a result, because Cauchy sequence exists in complete space X, it has a limit z in $\widetilde{X}$ and consequently each sub sequence has the same limit z.

That is Ax_{2n}, $Sx_{2n} \to z$ and Bx_{2n+1}, $Tx_{2n+1} \to z$ as $n \to \infty$. (3.1.8)

On using CLR_S-Property of (A, S) , (B, T) implies there are

sequences (a_n) as well as (b_n) in order for

Aa_n, Sa_n, $B\,b_n$, $T\,b_n \to S\mu$ as $n \to \infty$ for some μ in $\widetilde{X}$. (3.1.9)

To prove z = Sμ put $x = a_{2n}$, $y = x_{2n+5}$ in (3.1.3) we get

$F_{Aa_{2n}.\ Bx_{2n+5},\ a}(t^*) \geq$ r $(F_{Sa_{2n}.\ Tx_{2n+5},\ a}(t^*))$ as $n \to \infty$ (3.1.10)

$\Rightarrow F_{S\mu.\ z,\ a}(t^*) \geq$ r $(F_{S\mu.\ z,\ a}(t^*))$.> $F_{S\mu.\ z,\ a}(t^*)$. (3.1.11)

Resulting $F_{S\mu.\ z,\ a}(t^*)$. > $F_{S\mu.\ z,\ a}(t^*)$ (3.1.12)

which is a contradiction. Hence Sμ =z. (3.1.13)

Claim A μ = S μ.

Put x = μ, y = x_{2n+3} in (3.1.3) we get

$F_{A\mu.\ Bx_{2n+3},\ a}(t^*) \geq$ r ($F_{S\mu.\ Tx_{2n+3},\ a}(t^*)$) as n → ∞ (3.1.14)

⇒ $F_{A\mu.\ z,\ a}(t^*) \geq$ r ($F_{S\mu.\ z,\ a}(t^*)$) using (3.1.13) (3.1.15)

⇒ $F_{A\mu.\ z,\ a}(t^*) \geq$ r ($F_{z.\ z,\ a}(t^*)$) = r(1) = 1. (3.1.16)

This results A μ = S μ = z. (3.1.17)

Since the pair (A, S) obeys OWC resulting

A μ = S μ ⇒ SA μ = AS μ. That is Az = Sz. (3.1.18)

Claim Az = z.

Substitute y = x_{2n+3}, x = z in (3.1.3) we have

$F_{Az.\ Bx_{2n+3},\ a}(t^*) \geq$ r $F_{Sz.\ Tx_{2n+3},\ a}(t^*)$ letting n → ∞. (3.1.19)

⇒ $F_{Az,\ z,\ a}(t^*) \geq rF_{Sz\,.\ z,\ a}(t^*)$ using (3.1.18)

⇒ $F_{Az,\ z,\ a}(t^*) \geq rF_{Az\,.\ z,\ a}(t^*) > F_{Az\,.\ z,\ a}(t^*)$, (3.1.20)

⇒ $F_{Az,\ z,\ a}(t^*) > F_{Az\,.\ z,\ a}(t^*)$. (3.1.21)

This is a contradiction. Thus z = Az.

Resulting Az = Sz = z. (3.1.22)

Since Az ∈ A($\widetilde{X}$) ⊆ T($\widetilde{X}$) then ∃ ρ ∈ $\widetilde{X}$ such that Az = T ρ. (3.1.23)

Claim z = Bρ.

By employing x = x_{4n}, y = ρ of (3.1.3) we obtain

$F_{Ax_{2n},\ B\rho}(t^*) \geq rF_{Sx_{2n}\,.\ T\rho,\ a}(t^*)$ as n → ∞ . (3.1.24)

From (3.1.22) & (3.1.23)

⇒ $F_{z,\ B\rho}(t^*) \geq rF_{z\,.\ T\rho,\ a}(t^*)$ = r(1) = 1. (3.1.25)

Thus z = Bρ = Tρ.

Since the pair of mappings (B, T) obeys OWC, this results

Bρ = Tρ ⇒ BTρ = TBρ. That is Bz = Tz. (3.1.26)

Claim z = Bz.

By substituting y = z, x = z in (3.1.3) results

$F_{Az,\ Bz,\ a}(t^*) \geq rF_{Sz\,.\ Tz,\ a}(t^*)$ using (3.1.22) &(3.1.26) (3.1.27)

$F_{z,\ Bz,\ a}(t^*) \geq rF_{z\,.\ Bz,\ a}(t^*) > F_{z\,.\ Bz,\ a}(t^*)$. (3.1.28)

Resulting $F_{z,\ Bz,\ a}(t^*) > F_{z\,.\ Bz,\ a}(t^*)$. It is impossible. Therefore Bz =z.

Combining all we get Az = Bz = z = Sz = Tz.

Thus z is the required common fixed point for these mappings A, B, S and T.

Uniqueness:

Assume z_1 is second common fixed point.

Now assume $z \neq z_1$.

By considering $y = z_1$, $x = z$ in (3.1.4) we obtain

$F_{Az,\ Bz_1,\ a}\ (t^*) \geq rF_{Sz\ .\ Tz_1,\ a}\ (t^*)$

$F_{z,\ z_1,\ a}\ (t^*) \geq rF_{z\ .\ z_1,\ a}\ (t^*) > F_{z\ .\ z_1,\ a}\ (t^*)$

$F_{z,\ z_1,\ a}\ (t^*) > F_{z\ .\ z_1,\ a}\ (t^*)$ which is absurd. Hence $z = z_1$.

As a result, four self- mappings A, B, S, and T have the only one common fixed point.

Now we justify our theorem as under.

3.2 Example

Let us take X = $[0,\ \pi]$ and each $t\ \in\ [0,\ 1]$, define

$$F_{u,v}(\mathrm{t}) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & ,\ t = 0 \end{cases} \quad \mathit{for\ all}\ \alpha, \beta\ \mathit{in}\ X\ ,\ t\ >\ 0.$$

Define mappings $P, S, T\ \&\ Q\ :\ X\ \rightarrow\ X\ as$

$$A\ (x) = B(x) = \begin{cases} 2e^{-\pi x}, & x\ \in [0,\ \frac{\pi}{2}) \\ \pi - x, & x\ \in [\frac{\pi}{2},\ \pi] \end{cases} \tag{3.2.1}$$

$$and\ \ S(x) = T(x) = \begin{cases} 2e^{-\pi x^2}, & x\ \in [0,\ \frac{\pi}{2}) \\ x, & x\ \in [\frac{\pi}{2},\ \pi] \end{cases} \tag{3.2.2}$$

Now A(X) = B(X) = [0, 2] and S(X) = T(X) = $[0, \pi]$

implies $A(X\ ) \subseteq T(X)\ and\ B(X\ )\ \subseteq\ S(X)$.

Clearly $\frac{\pi}{2}$ and 1 are coincidence points for the mappings B, T.

At x = $\frac{\pi}{2}$, B($\frac{\pi}{2}$) = T ($\frac{\pi}{2}$) and $BT\left(\frac{\pi}{2}\right) = B\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$,

$TB\left(\frac{\pi}{2}\right)$ = T($\frac{\pi}{2}$) = $\frac{\pi}{2}$ implies $BT\left(\frac{\pi}{2}\right) = TB\left(\frac{\pi}{2}\right)$.

At x = 1, B(1) = T(1) and BT(1) $\neq$ TB(1).

Thus the pairs (A, S), (B, T) satisfy OWC but are not weakly compatible.

If x_n = $\frac{\pi}{2} - \frac{1}{n}$ for all n $\geq$ 1. Then (3.2.3)

Sx_n= Tx_n= S ($\frac{\pi}{2} - \frac{1}{n}$) = $\frac{\pi}{2} - \frac{1}{n} \rightarrow \frac{\pi}{2}$. (3.2.4)

Ax_n= Bx_n= A ($\frac{\pi}{2} - \frac{1}{n}$) = π – ($\frac{\pi}{2} - \frac{1}{n}$) = $\frac{\pi}{2} + \frac{1}{n}$ $\rightarrow \frac{\pi}{2}$ as n $\rightarrow\ \infty$. (3.2.5)

$\Rightarrow$ Ax_n, Sx_n, Tx_n, Bx_n $\rightarrow$ S($\frac{\pi}{2}$) as n $\rightarrow$ ∞. (3.2.6)

This gives the pairs of maps (A, S), (B, T) sharing the CLR'$_S$ property with OWC.

Thus A, B, S and T satisfy all the norms of Theorem and having the unique commonly fixed point at $\frac{\pi}{2}$ as A ($\frac{\pi}{2}$)= S($\frac{\pi}{2}$) = B($\frac{\pi}{2}$) = T($\frac{\pi}{2}$) = $\frac{\pi}{2}$.

Now we present another generalization of Theorem (A) as under.

Theorem 3.3 Let A, B, S and T be self -mappings on a complete probabilistic 2-metric space ($\widetilde{X}$, F, t^*) satisfying :

(3.3.1) A($\widetilde{X}$) ⊆ T($\widetilde{X}$), B($\widetilde{X}$) ⊆ S($\widetilde{X}$)

(3.3.2) the pair (A, S) satisfies CLR'$_S$ property with OWC and (B, T) satisfies OWC.

(3.3.3) Further $F_{Ax.\ By,\ a}\ (t^*) \geq$ r $F_{Sx.\ Ty,\ a}\ (t^*)$ for all elements x, y in $\widetilde{X}$ and t^*> 0

r is continuous self-map on [0, 1] such that r(t^*) > t^* for each o < t^*< 1.

Then A, B, S and T have unique common fixed point in $\widetilde{X}$.

Proof:

Take the constructed sequences < x_n> , < y_n> in Theorem (3.1) as

$\langle y_{2n} \rangle$ = Ax_{2n}= T$x_{2n+1}$and $\langle y_{2n+1} \rangle$ = Bx_{2n+1} = Sx_{2n+2} . (3.3.4)

It is already shown that $\langle y_n \rangle$ as cauchy sequence.

As a result each sub sequence has the same limit point z in complete space $\widetilde{X}$.

That is Ax_{2n} , Sx_{2n} → z and Bx_{2n+1}, Tx_{2n+1} → z.

The pair (A, S) obeys CLR$_S$-property this implies there is a sequence $\langle z_n \rangle$ such that $Az_n, Sz_n \rightarrow Sv$ for some v in $\widetilde{X}$.

Claim z = Sv.

By putting y = x_{2n+1} , x = z_n, in (3.3.3), that results

$F_{Az_n,\ Bx_{2n+1},\ a}\ (t^*) \geq rF_{Sz_n\ .\ Tx_{2n+1}\ ,\ a}\ (t^*)$ as n → ∞ (3.3.5)

$\Rightarrow F_{Sv,\ z,\ a}\ (t^*) \geq rF_{Sv\ .\ z,\ a}\ (t^*) > F_{Sv\ .\ z,\ a}\ (t^*)$, (3.3.6)

$\Rightarrow F_{Sv,\ z,\ a}\ (t^*) > F_{Sv\ .\ z,\ a}\ (t^*)$. (3.3.7)

This is absurd. As a result Sv =z. (3.3.8)

Claim Av = Sv. (3.3.9)

By inserting x = v, y = x_{2n+3} in (3.3.3), that results

$F_{Av,\ Bx_{2n+3},\ a}\ (t^*) \geq rF_{Sv\ .\ Tx_{2n+3}\ ,\ a}(t^*)$ letting as n → ∞ (3.3.10)

$\Rightarrow F_{Av,\ z,\ a}\ (t^*) \geq rF_{Sv\ .\ z,\ a}\ (t^*)$ using (3.3.8) (3.3.11)

$\Rightarrow F_{Av,\ Sv,\ a}\ (t^*) \geq rF_{Sv\ .\ Sv,\ a}\ (t^*)$ = r(1) = 1. (3.3.12)

$\Rightarrow$ Av = Sv = z. (3.3.13)

Since the pair (A, S) satisfies OWC property, that results

Av = Sv $\Rightarrow$ SAv = ASv. This gives Az = Sz. (3.3.14)

Claim Az = z.

By replacing y = $x_{2n+1,}$ x = z in (3.3.3), as a result

$F_{Az,\ Bx_{2n+1},\ a}\ (t^*) \geq rF_{Sz\ .\ Tx_{2n+1}\ ,\ a}\ (t^*)$. As n $\rightarrow \infty$ (3.3.15)

$\Rightarrow F_{Az,\ z,\ a}\ (t^*) \geq rF_{Sz\ .\ z,\ a}\ (t^*)$, using (3.3.14) (3.3.16)

$\Rightarrow F_{Az,\ z,\ a}\ (t^*) \geq rF_{Az\ .\ z,\ a}\ (t^*) > F_{Az\ .\ z,\ a}\ (t^*)$, (3.3.17)

$\Rightarrow F_{Az,\ z,\ a}\ (t^*) > F_{Az\ .\ z,\ a}\ (t^*)$.

This is a contradiction. Consequently Az = z. (3.3.18)

By combining (3.3.14) and (3.3.18) gives z = Sz = Az. (3.3.19)

Since Az $\in$ A($\widetilde{X}$) $\subseteq$ T($\widetilde{X}$) then $\exists\ \omega \in\ \widetilde{X}$ such that Az = T ω. (3.3.20)

Claim z = B ω.

By using x = x_{4n}, y = ω of (3.3.4), we obtain

$F_{Ax_{\ 2n,},\ B\omega}\ (t^*) \geq rF_{Sx_{\ 2n,}\ .\ T\omega\ ,\ a}\ (t^*)$. (3.3.21)

Taking limit as n $\rightarrow \infty$ and from (3.3.19) and (3.3.20) we get

$\Rightarrow F_{z,\ B\omega}\ (t^*) \geq rF_{z\ .\ z,\ a}\ (t^*)\ =\ r(1) = 1$. (3.3.22)

Thus z = B ω = T ω. (3.3.23)

Since the pair (B, T) obeys OWC property gives

B ω = T ω $\Rightarrow$ BT ω = TB ω implying Bz = Tz. (3.3.24)

Claim z = Bz.

Applying x = y = z in (3.3.3), this resulting

$F_{Az,\ Bz,\ a}\ (t^*) \geq rF_{Sz\ .\ Tz,\ a}\ (t^*)$, (3.3.25)

using (3.3.19) and (3.3.24)

$\Rightarrow F_{z,\ Bz,\ a}\ (t^*) \geq rF_{z\ .\ Bz,\ a}\ (t^*) > F_{z\ .\ Bz,\ a}\ (t^*)$, (3.3.26)

$\Rightarrow F_{z,\ Bz,\ a}\ (t^*)\ > F_{z\ .\ Bz,\ a}\ (t^*)$. (3.3.27)

Contradicting the fact implies Bz =z.

As a result Az = Bz = Sz = Tz = z.

As a consequence four self-mappings A, B, S, and T, there is a fixed point commonly.

Uniqueness can be easily proved as in the Theorem (3.1).

Now the Theorem (3.3) can be supported by discussing with suitable example.

3.4 Example

We choose X = [0, 1], d be usual metric on X and each $t \in [0, 1]$,
define

$$F_{u,v}(t) = \begin{cases} \frac{t}{t+|\alpha-\beta|}, & t > 0 \\ 0 & , t = 0 \end{cases} \quad for\ all\ \alpha, \beta\ in\ X,\ t > 0.$$

Choose mappings $P, S, T \,\&\, Q : X \to X\ as$

$$P(x) = Q(x) = \begin{cases} 1-2x, & x \in [0, 0.2] \\ x^2, & x \in (0.2, 1] \end{cases} \tag{3.4.1}$$

$$and\ \ S(x) = T(x) = \begin{cases} 3x, & x \in [0, 0.2] \\ x^3, & x \in (0.2, 1] \end{cases}. \tag{3.4.2}$$

Now P(X) = Q(X) = (0.04, 1] and S(X) = T(X) = [0, 1]

so that $P(X) \subseteq T(X)\ and\ Q(X) \subseteq S(X)$.

Clearly 0.2 and 1 are coincedence points of the graphs Q, T.

At x = 0.2, Q (0.2) = T(0.2) = 0.6 but QT(0.2) = Q(0.6) = 0.36, TQ(0.2) = T(0.6) = 0.216

At x = 1, Q (1) = T(1) and $QT(1) = Q(1) = 1 = T(1) = TQ(1)$.

This demonstrates that the pairs (P, S), (Q, T) are OWC mappings, although they are not weakly compatible.

If we choose $x_n = 1 - \frac{4}{3n}$ for all n ≥ 1. Then (3.4.3)

$Px_n = P\left(1 - \frac{4}{3n}\right) = \left(1 - \frac{4}{3n}\right)^2 \to 1$ (3.4.4)

$Sx_n = S\left(1 - \frac{4}{3n}\right) = \left(1 - \frac{4}{3n}\right)^3 \to 1$ as $n \to \infty$. (3.4.5)

This implies $Px_n, Sx_n \to S(1)$ as $n \to \infty$. (3.4.6)

This gives the pair (P, S) satisfies CLR'$_S$-property with OWC and the pair (Q, T) is OWC.

Thus the mappings P, Q, S and T satisfy all the norms of the Theorem (3.3), containing unique common fixed point at 1 as 1=P(1) = Q(1) = S(1) = T(1).

4. CONCLUSION

In this paper Theorem (A) is generalized in two ways.

(a) Theorem (3.1) is formulated by employing CLR'$_s$-property and applying OWC for both the pairs instead of assuming weakly compatible mappings.

(b) Theorem (3.2) is formulated by employing CLR'$_s$-property and OWC for one pair and OWC for the other pair instead of assuming weakly compatible mappings.

Further these two results are justified with suitable examples.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

[1] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. USA. 28(12) (1942) ,535-537.

[2] C. Alsina, B. Schweizer, and A. Sklar, On the definition of a probabilistic normed space, Aequationes Math. 46(1-2) (1993), 91-98.

[3] S. N. Mishra, N. Sharma and S. L. Singh, Common fixed points of maps on fuzzy Metric spaces, Int. J. Math. Math. Sci. 17(1994), 253-258.

[4] Altun, Ishak, and Duran Turkoglu, Some fixed point theorems for weakly compatible mappings satisfying an implicit relation, Taiwan. J. Math. 13 (2009), 1291-1304.

[5] X. Zhang, H. He, Y. Xu, A fuzzy logic system based on Schweizer-Sklar t-norm, Sci. China Ser. F: Inform. Sci. 49(2) (2006), 175-188.

[6] V. M. Sehgal, A. T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, Math. Syst. Theory 6(1-2) (1972), 97-102.

[7] M. A. Al-Thagafi, N. Shahzad, A note on occasionally weakly compatible Maps, Int. J. Math. Anal 3.2 (2009), 55-58.

[8] V.K. Gupta, A. Jain and R. Kumar, Common fixed point theorem in probabilistic 2-Metric space by weak compatibility, Int. J. Theor. Appl. Sci. 11(1) (2019), 09-12.

[9] S. Chauhan, W. Sintunavarat, and P. Kumam, Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces using (JCLR) property, Appl. Math. 3 (2012), 976-982.

[10] K. Satyanna, V. Srinvas, Fixed point theorem using semi compatible and sub sequentially continuous mappings in Menger space, J. Math. Comput. Sci. 10(6) (2020), 2503-2515.