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FIXED POINTS UNDER ψ-α-β CONDITIONS IN ORDERED PARTIAL METRIC SPACES

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Abstract. Recently, E. Karapinar and P. Salimi [Fixed point theorems via auxiliary functions, J. Appl. Math. 2012, Article ID 792174] have obtained fixed point results for increasing mappings in a partially ordered metric space using three auxiliary functions in the contractive condition. In this paper, these results are extended to 0-complete ordered partial metric spaces with a more general contractive condition. Examples are given showing that these extensions are proper.

1. Introduction

Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Subsequently, several authors (see, e.g., [2, 3, 4, 5, 6]) proved variuos more general fixed point results in partial metric spaces.

The notion of weakly contractive conditions in metric spaces was first used by Rhoades [7] who proved the following

Theorem 1.1. [7] *Let (X, d) be a metric space. If $T : X \to X$ satisfies the condition*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall\, x, y \in X,$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ iff $t = 0$, then T has a unique fixed point.

Subsequently, several authors (see, e.g., [8, 9]) proved various generalizations and refinements of this result.

Fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [10] who presented its applications to matrix equations. Further, a lot of results appeared, we mention just those contained in [11, 12, 13, 14, 15, 16]. These results use weaker contractive conditions (mostly just for comparable elements of the given space), but at the expense of some additional restrictions to the mappings involved.

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Consider the following classes of functions from $[0, +\infty)$ into itself:

$$\begin{aligned}\Psi &= \{\, \psi : \psi \text{ is nondecreasing and lower semicontinuous} \,\},\\ \Phi_1 &= \{\, \alpha : \alpha \text{ is upper semicontinuous} \,\},\\ \Phi_2 &= \{\, \beta : \beta \text{ is lower semicontinuous} \,\}.\end{aligned}$$

Very recently, Karapinar and Salimi [16] proved the following

Theorem 1.2. *Let $(X, \preceq, d)$ be a complete ordered metric space and let $T : X \to X$ be a nondecreasing selfmap. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $s, t \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0, \tag{1.1}$$

and

$$\psi(d(Tx, Ty)) \leq \alpha(d(x, y)) - \beta(d(x, y)) \tag{1.2}$$

for all comparable $x, y \in X$. Suppose that, either, T is continuous, or X is regular (see further Definition 2.5). If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

We shall prove in the present paper that this result can be extended to 0-complete ordered partial metric spaces with a contractive condition more general than (1.2). Examples will be given showing that this extension is proper.

Remark 1.3. It was shown very recently (see [17]) that in some cases fixed point results in partial metric spaces can be directly reduced to their standard metric counterparts. We note that the results of the present paper do not fall into this category. Moreover, using the method from [17], it is not possible to conclude that if x is a fixed point of the mapping under consideration, then $p(x, x) = 0$, which is an important conclusion for applications in Computer Science.

2. Preliminaries and auxiliary results

The following definitions and details can be seen in [1, 2, 3, 4, 18].

Definition 2.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) of a nonempty set X and a partial metric p on X. If, moreover, $\preceq$ is a partial order on X, then the triple $(X, \preceq, p)$ is called an ordered partial metric space.

It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2), $x = y$. But $p(x, x)$ may not be 0. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. This will be denoted as $x_n \to x$ $(n \to \infty)$ or $\lim_{n\to\infty} x_n = x$. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot, \cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ imply $p(x_n, y_n) \to p(x, y)$.

Example 2.2. (1) A paradigmatic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

(2) [1] Let $X = \{ [a, b] : a, b \in \mathbb{R}, a \leq b \}$ and let $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (X, p) is a partial metric space.

Definition 2.3. Let (X, p) be a partial metric space. Then:

(1) A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).
(2) The space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(3) [2] A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges (in τ_p) to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 2.4. *Let (X, p) be a partial metric space.*

(a) [3] *If $p(x_n, z) \to p(z, z) = 0$ as $n \to \infty$, then $p(x_n, y) \to p(z, y)$ as $n \to \infty$ for each $y \in X$.*

(b) [2] *If (X, p) is complete, then it is 0-complete.*

The converse assertion of (b) does not hold as an easy example in [2] shows.

Definition 2.5. Let $(X, \preceq, p)$ be an ordered partial metric space. We say that X is regular if the following holds: if $\{z_n\}$ is a nondecreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to \infty$ (in (X, p)), then $z_n \preceq z$ for all $n \in \mathbb{N}$ and if $\{z_n\}$ is a nonincreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to \infty$ (in (X, p)), then $z_n \succeq z$ for all $n \in \mathbb{N}$.

We will also need the following lemma which was proved in the metric case, e.g., in [19]. The proof is similar in the partial metric case, and so we omit it.

Lemma 2.6. *Let (X, p) be a partial metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{2.1}$$

If $\{x_n\}$ is not a 0-Cauchy sequence in (X, p), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$p(x_{m_k}, x_{n_k}), \quad p(x_{m_k}, x_{n_k+1}), \quad p(x_{m_k-1}, x_{n_k}), \quad p(x_{m_k-1}, x_{n_k+1}). \tag{2.2}$$

3. Main results

Our first result is the following

Theorem 3.1. *Let $(X, \preceq, p)$ be a 0-complete ordered partial metric space and let $T : X \to X$ be a nondecreasing selfmap. Assume that there exist functions $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $t, s \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0, \tag{3.1}$$

and

$$\psi(p(Tx, Ty)) \leq \alpha(M(x, y)) - \beta(M(x, y)) \tag{3.2}$$

for all comparable $x, y \in X$, where

$$M(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right\}.$$

Suppose that, either T is continuous, or X is regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point $z \in X$ satisfying that $p(z,z) = 0$.

Proof. Starting with the given x_0, construct the Picard sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$. Using the assumption $x_0 \preceq Tx_0$ and that T is nondecreasing, we conclude that

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}_0$, a fixed point of T is found. Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Apply assumption (3.2) for $x = x_n$ and $y = x_{n+1}$ to obtain

$$\begin{aligned} \psi(p(x_{n+1}, x_{n+2})) &= \psi(p(Tx_n, Tx_{n+1})) \\ &\le \alpha(M(x_n, x_{n+1})) - \beta(M(x_n, x_{n+1})), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \\ &\qquad \tfrac{1}{2}(p(x_n, x_{n+2}) + p(x_{2n+1}, x_{2n+1}))\} \\ &= \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \end{aligned}$$

(condition (p_4) of partial metric was used). Suppose that $p(x_{n+1}, x_{n+2}) > p(x_n, x_{n+1})$ for some $n \in \mathbb{N}_0$. Then (3.3) implies that

$$\psi(p(x_{n+1}, x_{n+2})) \le \alpha(p(x_{n+1}, x_{n+2})) - \beta(p(x_{n+1}, x_{n+2})).$$

By the assumption (3.1) it follows that $p(x_{n+1}, x_{n+2}) = 0$ and, hence, $x_{n+1} = x_{n+2}$, which is already excluded. Hence,

$$p(x_{n+1}, x_{n+2}) \le p(x_n, x_{n+1}),$$

and $M(x_n, x_{n+1}) = p(x_n, x_{n+1})$ for all $n \in \mathbb{N}_0$. Thus, the sequence $\{p(x_n, x_{n+1})\}$ is nonincreasing. Since it is bounded from below, there exists $r \ge 0$ such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = r$. It follows from (3.3) that

$$\psi(p(x_{n+1}, x_{n+2}) \le \alpha(p(x_n, x_{n+1})) - \beta(p(x_n, x_{n+1})),$$

and using the properties of functions ψ, α, β we get that

$$\begin{aligned} \psi(r) &\le \liminf \psi(p(x_{n+1}, x_{n+2})) \le \limsup \psi(p(x_{n+1}, x_{n+2})) \\ &\le \limsup[\alpha(p(x_n, x_{n+1})) - \beta(p(x_n, x_{n+1}))] \\ &= \limsup \alpha(p(x_n, x_{n+1})) - \liminf \beta(p(x_n, x_{n+1}))] \\ &\le \alpha(r) - \beta(r). \end{aligned}$$

Using again the condition (3.1), we get that it is only possible if $r = 0$.

Next, we claim that $\{x_n\}$ is a 0-Cauchy sequence in the partial metric space (X, p). Suppose that this is not the case. Then, using Lemma 2.6 we get that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and sequences (2.2) tend to ε when $k \to \infty$. Applying condition (3.2) to elements $x = x_{n_k-1}$ and $y = x_{m_k}$ we get that

$$\psi(p(x_{n_k}, x_{m_k+1})) \le \alpha(M(x_{n_k-1}, x_{m_k})) - \beta(M(x_{n_k-1}, x_{m_k})), \tag{3.4}$$

where

$$\begin{aligned}M(x_{n_k-1},x_{m_k}) = \max\{&p(x_{n_k-1},x_{m_k}),p(x_{n_k-1},x_{n_k}),p(x_{m_k},x_{m_k+1}),\\ &\tfrac{1}{2}[p(x_{n_k-1},x_{m_k+1})+p(x_{n_k},x_{m_k})]\}.\end{aligned}$$

Using Lemma 2.6 we get that $\lim_{k\to\infty} M(x_{n_k-1},x_{m_k}) = \varepsilon$. Passing to the upper limit in (3.4) and using properties of the functions ψ,α,β, we get that

$$\begin{aligned}\psi(\varepsilon) &\le \liminf \psi(p(x_{n_k},x_{m_k+1})) \le \limsup \psi(p(x_{n_k},x_{m_k+1}))\\ &\le \limsup[\alpha(M(x_{n_k-1},x_{m_k})) - \beta(M(x_{n_k-1},x_{m_k}))]\\ &= \limsup \alpha(M(x_{n_k-1},x_{m_k})) - \liminf \beta(M(x_{n_k-1},x_{m_k}))\\ &\le \alpha(\varepsilon)-\beta(\varepsilon).\end{aligned}$$

This is (because of $\varepsilon > 0$) a contradiction with (3.1). We conclude that $\{x_n\}$ is a 0-Cauchy sequence in (X,p).

Now, since (X,p) is a 0-complete partial metric space, it follows that there exists $x \in X$ such that $x_n \to x$ as $n\to\infty$, i.e.,

$$p(x_n,x)\to p(x,x)=0, \quad \text{as } n\to\infty.$$

Assume first that the mapping T is continuous. Then $\lim_{n\to\infty} p(Tx_n,Tx) = p(Tx,Tx)$. It follows that

$$\begin{aligned}p(Tx,x) &\le p(Tx,Tx_n)+p(x_{n+1},x)-p(x_{n+1},x_{n+1})\\ &\le p(Tx,Tx_n)+p(x_{n+1},x)\to p(Tx,Tx) \quad \text{as } n\to\infty.\end{aligned}$$

It follows that $p(Tx,x)=p(Tx,Tx)$. Since $x\preceq x$, we can apply (3.2) to obtain

$$\begin{aligned}\psi(p(Tx,Tx)) &\le \alpha(M(x,x))-\beta(M(x,x)) = \alpha(p(x,Tx))-\beta(p(x,Tx))\\ &= \alpha(p(Tx,Tx))-\beta(p(Tx,Tx)).\end{aligned}$$

By (3.1), this is possible only if $p(Tx,Tx)=0$, i.e., $p(x,Tx)=0$. Hence $Tx=x$.

Assume now that the space $(X,\preceq,p)$ is regular. Since the sequence $\{x_n\}$ is increasing and $x_n\to x$ as $n\to\infty$, we get that $x_n\preceq x$ for $n\in\mathbb{N}$. Hence, we can apply (3.2) to get

$$\psi(p(Tx_n,Tx)) \le \alpha(M(x_n,x))-\beta(M(x_n,x)), \tag{3.5}$$

where

$$\begin{aligned}M(x_n,x) &= \max\{p(x_n,x),p(x_n,x_{n+1}),p(x,Tx),\tfrac{1}{2}[p(x_n,Tx)+p(x_{n+1},x)]\}\\ &\to p(x,Tx) \quad \text{as } n\to\infty.\end{aligned}$$

On the other hand, since $p(x_n,x)\to p(x,x)=0$, Lemma 2.4.(1) implies that $p(Tx_n,Tx)=p(x_{n+1},Tx)\to p(x,Tx)$. Hence, passing to the upper limit in (3.5), similarly as in the previous case we get that it is only possible that $p(x,Tx)=0$, i.e., $Tx=x$.

In all possible cases we have obtained that x is a fixed point of the mapping T satisfying $p(x,x)=0$ and the theorem is proved. □

Remark 3.2. Taking p to be a standard metric in the contractive condition (3.2), and assuming $\psi=\alpha$ to be continuous, we obtain [13, Corollary 3.3].

In a similar way one can prove the following two assertions.

Theorem 3.3. *Let all the conditions of Theorem 3.1 be fulfilled, except that condition* (3.2) *is replaced by*

$$\psi(p(Tx,Ty)) \le \alpha(p(x,y)) - \beta(p(x,y)). \tag{3.6}$$

Then T has a fixed point in X.

Theorem 3.4. *Let all the conditions of Theorem 3.1 be fulfilled, except that condition* (3.2) *is replaced by*

$$\psi(p(Tx,Ty)) \le \alpha(M_1(x,y)) - \beta(M_1(x,y)), \tag{3.7}$$

where

$$M_1(x,y) = \max\left\{p(x,y), \frac{1}{2}[p(x,Tx)+p(y,Ty)], \frac{1}{2}[p(x,Ty)+p(y,Tx)]\right\}.$$

Then T has a fixed point in X.

The following simple example shows that conditions of Theorem 3.1 are not sufficient for the uniqueness of fixed points.

Example 3.5. Let $X = \{(1,0),(0,1)\}$, let $(a,b) \preceq (c,d)$ if and only if $a \le c$ and $b \le d$, and let p be the Euclidean metric. The function $T((a,b)) = (a,b)$ is continuous. The only comparable pairs of points in X are $x \preceq x$ for $x \in X$ and then $M(x,x) = 0$ and $p(Tx,Tx) = 0$, hence the condition $\psi(p(fx,fy)) \le \alpha(M(x,y)) - \beta(M(x,y))$ is fulfilled, e.g., for the functions $\psi \in \Psi$, $\alpha \in \Phi_1$, $\beta \in \Phi_2$ given as $\psi(t) = \alpha(t) = t$, $\beta(t) = kt$, $0 < k < 1$. However, T has two fixed points $(1,0)$ and $(0,1)$.

Using the same example we can show that there exist situations where conditions of Theorem 3.1, taken in the case without order may not be sufficient.

Example 3.6. If the previous example is considered without order, then one has also to take into account the case when $x \ne y$. But then $p(x,y) = \sqrt{2}$ and $M(Tx,Ty) = \sqrt{2}$, and so the condition (3.2) reduces to $\psi(\sqrt{2}) \le \alpha(\sqrt{2}) - \beta(\sqrt{2})$ and cannot be valid for any functions ψ, α, β satisfying (3.1).

Now we give a sufficient condition for the uniqueness of fixed point.

Theorem 3.7. *Let all the conditions of Theorem 3.1 be fulfilled and, moreover, the space $(X, \preceq, p)$ satisfy the following condition: For all $x, y \in X$ there exists $z \in X$, $z \preceq Tz$, satisfying both $x \preceq z$ and $y \preceq z$ or there exists $z \in X$, $z \preceq Tz$, satisfying both $x \succeq z$ and $y \succeq z$. Then the fixed point of T is unique.*

Proof. Let x and y be two fixed points of T, i.e., $Tx = x$ and $Ty = y$. Consider the following two possible cases.

1. x and y are comparable. Then we can apply condition (3.2) and obtain that

$$\psi(p(x,y)) = \psi(p(Tx,Ty)) \le \alpha(M(x,y)) - \beta(M(x,y)),$$

where

$$\begin{aligned} M(x,y) &= \max\{p(x,y), p(x,Tx), p(y,Ty), \tfrac{1}{2}[p(x,Ty)+p(y,Tx)]\} \\ &= p(x,y) \end{aligned}$$

and hence $\psi(p(x,y)) \le \alpha(p(x,y)) - \beta(p(x,y))$ which is possible only if $p(x,y) = 0$ and hence $x = y$.

2. Suppose now that x and y are not comparable. Choose an element $z \in X$, $z \preceq Tz$ comparable with both of them. Then also $x = T^n x$ is comparable with $T^n z$ for each n (since T is nondecreasing). Applying (3.1) one obtains that

$$\begin{aligned}\psi(p(x, T^n z)) &= \psi(p(TT^{n-1}x, TT^{n-1}z)) \\ &\leq \alpha(M(T^{n-1}x, T^{n-1}z)) - \beta(M(T^{n-1}x, T^{n-1}z)),\end{aligned}$$

where

$$\begin{aligned}M(T^{n-1}x, T^{n-1}z) &= \max\{p(T^{n-1}x, T^{n-1}z), p(T^{n-1}x, T^n x), p(T^{n-1}z, T^n z), \\ &\quad \tfrac{1}{2}[p(T^{n-1}x, T^n z) + p(T^n x, T^{n-1}z)]\} \\ &= \max\{p(x, T^{n-1}z), p(T^{n-1}z, T^n z), \tfrac{1}{2}[p(x, T^n z) + p(x, T^{n-1}z))\} \\ &\leq \max\{p(x, T^{n-1}z), p(x, T^n z)\},\end{aligned}$$

for n sufficiently large, because $p(T^{n-1}z, T^n z) \to 0$ when $n \to \infty$ (the last assertion can be proved, starting from the assumption $z \preceq Tz$, in the same way as a similar conclusion in the proof of Theorem 3.1).

Similarly as in the proof of Theorem 3.1, it can be shown that $p(x, T^n z) \leq M(x, T^{n-1}z) \leq p(x, T^{n-1}z)$. It follows that the sequence $p(x, T^n z)$ is nonincreasing and it has a limit $l \geq 0$. Assuming that $l > 0$ and passing to the limit in the relation

$$\psi(p(x, T^n z)) \leq \alpha(M(x, T^{n-1}z)) - \beta(M(x, T^{n-1}z))$$

one obtains that $l = 0$, a contradiction. In the same way it can be deduced that $p(y, T^n z) \to 0$ as $n \to \infty$. Now, passing to the limit in $p(x, y) \leq p(x, T^n z) + p(T^n z, y)$, it follows that $p(x, y) = 0$. Hence, $x = y$ and the uniqueness of the fixed point is proved. □

Remark 3.8. If two selfmaps $T, S : X \to X$ are given, then in a similar way a common fixed point result can be obtained, under an appropriate contractive condition and assuming, e.g., that T and S are weakly increasing (for the details see, e.g., [13]). The proof uses the standard method of Jungck sequences.

4. Examples

Our first example (inspired by [5]) shows that it may happen that the contractive condition (3.7) is not satisfied, hence the existence of a fixed point cannot be obtained using Theorem 3.3. However, the condition (3.2) is fulfilled and Theorem 3.1 can be used to obtain the conclusion.

Example 4.1. Consider the set $X = \{a, b, c\}$ and the function $p : X \times X \to \mathbb{R}$ given by $p(a, b) = p(b, c) = 1$, $p(a, c) = \frac{3}{2}$, $p(x, y) = p(y, x)$, $p(a, a) = p(c, c) = \frac{1}{2}$ and $p(b, b) = 0$. Obviously, p is a partial metric on X, not being a metric (since $p(x, x) \neq 0$ for $x = a$ and $x = c$). Define an order-relation $\preceq$ on X by $a \preceq b \preceq c$. Then, $(X, \preceq, p)$ is a 0-complete ordered partial metric space. Define a selfmap T on X by

$$T : \begin{pmatrix} a & b & c \\ b & b & a \end{pmatrix}.$$

Then T is not a (Banach)-contraction since

$$p(fc, fc) = p(a, a) = \frac{1}{2} = p(c, c)$$

and there is no $\lambda \in [0,1)$ such that $p(fc, fc) \leq \lambda p(c,c)$. Moreover, condition (3.7) (i.e., condition (1.2) of Theorem 1.2, with d replaced by p) cannot hold for functions $\psi \in \Psi$, $\alpha \in \Phi_1$, $\beta \in \Phi_2$ satisfying (1.1) because for otherwise $x = y = c$ would imply $\psi(\frac{1}{2}) - \alpha(\frac{1}{2}) + \beta(\frac{1}{2}) \leq 0$ which cannot hold.

We will check that T satisfies the condition (3.2) of Theorem 3.1 with functions ψ, α, β given as $\psi(t) = t$, $\alpha(t) = t$, $\beta(t) = \frac{1}{3}t$ (obviously belonging to the respective classes). Note that in this case $\alpha(t) - \beta(t) = \frac{2}{3}t$. If $x, y \in \{a, b\}$, then $p(Tx, Ty) = p(b,b) = 0$ and (3.2) trivially holds. Let, e.g., $y = c$; then we have the following three cases:

$$\begin{aligned}
\psi(p(Ta,Tc)) &= p(b,a) = 1 \leq \frac{2}{3}\cdot\frac{3}{2}\\
&= \frac{2}{3}\max\{p(a,c), p(a,Ta), p(c,Tc), \tfrac{1}{2}[p(a,Tc)+p(c,Ta)]\},\\
\psi(p(Tb,Tc)) &= p(b,a) = 1 \leq \frac{2}{3}\cdot\frac{3}{2}\\
&= \frac{2}{3}\max\{p(b,c), p(b,Tb), p(c,Tc), \tfrac{1}{2}[p(b,Tc)+p(c,Tb)]\},\\
\psi(p(Tc,Tc)) &= p(a,a) = \frac{1}{2} < \frac{2}{3}\cdot\frac{3}{2} = \frac{2}{3}\max\{p(c,c), p(c,Tc)\}.
\end{aligned}$$

Thus, conditions of Theorem 3.1 are satisfied and the existence of a fixed point of T follows.

Using an example of Romaguera [2], we present another example showing how Theorem 3.1 can be used. It also shows that there are situations when standard metric arguments cannot be used to obtain the existence of a fixed point.

Example 4.2. Let $X = [0,1] \cap \mathbb{Q}$ be equipped with the partial metric p defined by $p(x,y) = \max\{x,y\}$ for $x, y \in X$ and the standard order. Let $T : X \to X$ be given by

$$Tx = \frac{x^2}{1+x}.$$

It is easy to see that the space (X,p) is 0-complete. Take the functions ψ, α, β given by $\psi(t) = \alpha(t) = t$, $\beta(t) = \frac{1}{2}t$. The contractive condition (3.2) for (say) $x \geq y$ takes the form

$$\begin{aligned}
p(Tx,Ty) &= \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} = \frac{x^2}{1+x}\\
&\leq \frac{1}{2}\max\{p(x,y), p(x, \tfrac{x^2}{1+x}), p(y, \tfrac{y^2}{1+y}), \tfrac{1}{2}[p(x, \tfrac{y^2}{1+y}) + p(y, \tfrac{x^2}{1+x})]\}\\
&= \frac{1}{2}\max\{x, x, y, \tfrac{1}{2}[x + \max\{y, \tfrac{x^2}{1+x}\}]\} = \frac{1}{2}x,
\end{aligned}$$

and it is satisfied for all $x, y \in X$, since $0 \leq x \leq 1$. Hence, all the conditions of Theorem 3.1 and Theorem 3.7 are satisfied and T has a unique fixed point ($z = 0$).

However, if we consider the same ordered set $X = [0,1] \cap \mathbb{Q}$ equipped with the standard metric $d(x,y) = |x - y|$, we obtain a non-complete ordered metric space. Hence, the existence of a fixed point of T cannot be deduced using Theorem 1.2.

We present now an example (inspired by [16]) showing the usage of Theorem 3.1 with (at least some of) functions ψ, α, β not being continuous.

Example 4.3. Consider the set $X = [0, 1]$ equipped with the standard order and the partial metric given as $p(x, y) = \max\{x, y\}$. It is easy to show that $(X, \preceq, p)$ is a regular, 0-complete ordered partial metric space. Let $T : X \to X$ be given by $Tx = \frac{1}{2}x - \frac{1}{4}x^2$ and take the functions ψ, α, β defined by

$$\psi(t) = \begin{cases} t + \frac{3}{2}, & t > 0 \\ 1, & t = 0, \end{cases}, \qquad \alpha(t) = t + \frac{5}{2}, \qquad \beta(t) = \frac{1}{2}t + 1.$$

Note that ψ is lower semicontinuous and the condition (3.1) is satisfied.

Both conditions (3.2) and (3.7) are satisfied. We shall check, e.g., condition (3.2). Let $x, y \in X$ and, for example, $x \leq y$. Then

$$\psi(p(Tx, Ty)) = \psi(Ty) = \begin{cases} \frac{1}{2}y - \frac{1}{4}y^2 + \frac{3}{2}, & y > 0 \\ 1, & y = 0, \end{cases}$$

and

$$\begin{aligned} M(x, y) &= \max\{\max\{x, y\}, \max\{x, Tx\}, \max\{y, Ty\}, \tfrac{1}{2}[\max\{x, Ty\} + \max\{y, Tx\}]\} \\ &= \max\{y, x, y, \tfrac{1}{2}[\max\{x, Ty\} + y]\} = y, \end{aligned}$$

since $\max\{x, Ty\} \leq y$. The condition (3.2) reduces to $\frac{1}{2}y - \frac{1}{4}y^2 + \frac{3}{2} \leq \frac{1}{2}y + \frac{3}{2}$ if $y > 0$ and to $1 \leq \frac{3}{2}$ if $y = 0$ and it is fulfilled in both cases. By Theorems 3.1 and 3.7, T has a unique fixed point (which is $z = 0$).

Finally, the following example demonstrates the situation when contractive conditions using partial metric can guarantee the existence of a fixed point, while the respective conditions with the standard metric cannot.

Example 4.4. Consider the set $X = [0, +\infty)$ endowed with the partial metric $p(x, y) = \max\{x, y\}$ and the order given by

$$x \preceq y \implies x = y \vee (x, y \in [0, 1] \wedge x \leq y).$$

$(X, \preceq, p)$ is a regular, 0-complete ordered partial metric space. Let $T : X \to X$ be given by

$$Tx = \begin{cases} \dfrac{x^2}{1 + x}, & x \in [0, 1] \\ \dfrac{x}{2}, & x > 1 \end{cases}$$

and consider the functions $\psi \in \Psi$, $\alpha \in \Phi_1$, $\beta \in \Phi_2$ defined as

$$\psi(t) = \alpha(t) = t, \qquad \beta(t) = \begin{cases} \dfrac{t}{1 + t}, & t \in [0, 1] \\ \dfrac{t}{2}, & t > 1. \end{cases}$$

Clearly, ψ, α, β satisfy condition (3.1). We shall show that also condition (3.2) is satisfied. Let $x, y \in X$ and suppose that, e.g., $y \preceq x$. Then, there are two possibilities:

1. $x \in [0, 1]$ (and hence also $y \in [0, 1]$ and $y \leq x$). Then

$$p(Tx, Ty) = \max\left\{\frac{x^2}{1 + x}, \frac{y^2}{y + x}\right\} = \frac{x^2}{1 + x},$$

and

$$M(x,y)=\max\left\{x,x,y,\frac{1}{2}\left[x+\max\left\{y,\frac{x^2}{1+x}\right\}\right]\right\}=x,$$

since $\max\{y,\frac{x^2}{1+x}\}\le x$. Condition (3.2) reduces to

$$\frac{x^2}{1+x}\le x-\frac{x}{1+x}$$

and obviously holds true.

2. $x>1$ (and hence $y=x$). Then $p(Tx,Ty)=\frac{x}{2}$ and $M(x,y)=x$. Hence, (3.2) reduces to $\frac{x}{2}\le x-\frac{x}{2}$ and is also satisfied.

All the conditions of Theorems 3.1 and 3.7 are fulfilled and T has a unique fixed point (which is $z=0$).

Consider now the same problem but in the case that instead od the partial metric p, the standard metric $d(x,y)=|x-y|$ is used (note that this is the so-called associated metric of p, see, e.g., [1]). Take $x=1$ and $y=\frac{1}{2}$. Then

$$\psi(d(Tx,Ty))=\left|\frac{1}{2}-\frac{1}{6}\right|=\frac{1}{3},$$
$$M(x,y)=\max\left\{\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{2}\left(\frac{5}{6}+0\right)\right\}=\frac{1}{2},$$
$$\alpha(M(x,y))-\beta(M(x,y))=\frac{1}{2}-\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{6}<\frac{1}{3},$$

and the contractive condition (3.2) with $p=d$ is not satisfied (neither is condition (1.2) of Theorem 1.2).

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References

[1] S.G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728** (1994), 183–197.
[2] S. Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl. (2010) Article ID 493298, 6 pages.
[3] T. Abdeljawad, E. Karapinar, K. Taş, *Existence and uniqueness of a common fixed point on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1900–1904.
[4] E. Karapinar, I.M. Erhan, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1894–1899.
[5] C. Di Bari, Z. Kadelburg, H.K. Nashine, S. Radenović, *Common fixed points of g-quasi-contractions and related mappings in* 0*-complete partial metric spaces*, Fixed Point Theory Appl. **2012**:113 (2012), doi:10.1186/1687-1812-2012-113.
[6] N. Hussain, Z. Kadelburg, S. Radenović, F.R. Al-Solamy, *Comparison functions and fixed point results in partial metric spaces*, Abstract Appl. Anal. **2012**, Article ID 605781, 15 pages, doi:10.1155/2012/605781.
[7] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001), 2683–2693.
[8] P.N. Dutta, B.S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed Point Theory Appl. (2008), Article ID 406368, doi:10.1155/2008/406368.
[9] B.S. Choudhury, A. Kundu, (ψ,α,β)*-weak contractions in partially ordered metric spaces*, Appl. Math. Lett. **25** (2012), 6–10.

[10] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.

[11] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. **71** (2009), 3403–3410.

[12] J. Harjani, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal. **72** (3-4) (2010), 1188–1197.

[13] S. Radenović, Z. Kadelburg, *Generalized weak contractions in partially ordered metric spaces*, Comput. Math. Appl. **60** (2010), 1776–1783.

[14] Z. Golubović, Z. Kadelburg, S. Radenović, *Common fixed points of ordered g-quasicontractions and weak contractions in ordered metric spaces*, Fixed Point Theory Appl. **2012**:20 (2012), doi:10.1186/1687-1812-2012-20.

[15] H. Aydi, E. Karapinar, B. Samet, *Remarks on some recent fixed point theorems*, Fixed Point Theory Appl. **2012**:76 (2012), doi:10.1186/1687-1812-2012-76.

[16] E. Karapinar, P. Salimi, *Fixed point theorems via auxiliary functions*, J. Appl. Math. **2012**, Article ID 792174, 9 pages, doi:10.1155/2012/792174.

[17] R.H. Haghi, Sh. Rezapour, N. Shahzad, *Be careful on partial metric fixed point results*, Topology Appl. **160** (2013), 450–454.

[18] H.K. Nashine, Z. Kadelburg, S. Radenović, *Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces*, Math. Comput. Modelling **57** (2013), 2355–2365.

[19] S. Radenović, Z. Kadelburg, D. Jandrlić and A. Jandrlić, *Some results on weak contraction maps*, Bull. Iranian Math. Soc. **38**(3) (2012), 625–645.

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