

## On the Solutions of the Second Kind Nonlinear Volterra-Fredholm Integral Equations via Homotopy Analysis Method

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**Abstract.** In this paper, we discuss the existence and uniqueness of the solution of the second kind nonlinear Volterra-Fredholm integral equations (NV-FIEs) which appear in mathematical modeling of many phenomena, using Picard's method. In addition, we use Banach fixed point theorem to show the solvability of the first kind NV-FIEs. Moreover, we utilize the homotopy analysis method (HAM) to approximate the solution and the convergence of the method is investigated. Finally, some examples are presented and the numerical results are discussed to show the validity of the theoretical results.

### 1. Introduction

In the application of physical mathematics and engineering, the second kind of NV-FIEs are often arisen [1–8]. Therefore, there exist great efforts to approximate the solution of this kind of NV-FIEs. Yousefi and Razzaghi [9] presented a numerical method based upon Legendre wavelet approximations for solving the NV-FIEs. Cui and Du [10] obtained the representation of the exact solution for the NV-FIEs in the reproducing kernel space and the exact solution was given by the form of series. The approximate solutions of the NV-FIEs were presented using modified decomposition method by Bildik and Inc [11]. Ghasemi et al. [12] presented homotopy perturbation method for solving NV-FIEs. In addition, rationalized Haar functions are developed to approximate the solution of the NV-F-Hammerstein IEs by Ordokhani and Razzaghi [13]. He's variational iteration method was used by Yousefi [14] to approximate the solution of a type of NV-FIEs. Hashemizadeh et al. [15] introduced an approximation

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method based on hybrid Legendre and Block-Pulse functions for solving the NV-FIEs. A computational technique based on the composite collocation method was presented by Marzban et al. [16] for the solution of the NV-F-Hammerstein IEs. Moreover, Maleknejad et al. [17] utilized a method to solve NV-F-Hammerstein IEs in terms of Bernstein polynomials. Parand and Rad [18] proposed the collocation method based on radial basis functions to approximate the solution of NV-F-Hammerstein IEs. A numerical method based on hybrid of block-pulse functions and Taylor series is proposed by Mirzaee and Hoseini [19] to approximate the solution of NV-FIEs. Chen and Jiang [20] developed a simple and effective method for solving NV-FIEs based on Lagrange interpolation functions. The approximate solution of the NV-F-Hammerstein IEs is obtained by Gouyandeh et al. [21] using the Tau-Collocation method.

The present paper shall utilize HAM for solving the NFVIEs of the first and second kind. Foremost, in Section 2, we discuss the solvability of the second kind NF-VIEs using Picard's method. Moreover, in Section 3, Banach fixed point theorem is used to discuss the existence and uniqueness of the solution of the first kind NF-VIEs. In addition, the basic idea of HAM and how to utilize HAM for the NF-VIEs of the second and first kind are presented in Section 4. Finally, we present the numerical results in Section 5.

## 2. Existence and uniqueness of the second kind NV-FIEs

Consider the following second kind NV-FIEs of the form

$$\mu u(t) = f(t) + \lambda \int_a^b K_1(t, s) \mathcal{N}_1(u(s)) ds + \lambda \int_0^t K_2(t, s) \mathcal{N}_2(u(s)) ds. \quad (2.1)$$

Now, we shall discuss the solvability of Eq.(2.1) under the following assumptions

- (1) The function  $f(t)$  is continuous in the space  $C[0, T]$ , such that
 
$$\|f(t)\|_{C[0, T]} = \max_{t \in [0, T]} |f(t)| \leq P_1 \text{ and } \mu \in \mathbb{R} - \{0\}.$$
- (2) The Kernels  $K_1(t, s)$  and  $K_2(t, s)$  are continuous in  $C[0, T]$  and satisfy  $|K_1(t, s)| \leq P_2$  and  $|K_2(t, \tau)| \leq P_3, \forall t, \tau \in [0, T]$ , and  $0 \leq \tau \leq t \leq T < 1$ .
- (3) The nonlinear functions  $\mathcal{N}_i(u(t)), i = 1, 2$  satisfy
  - i. the Lipschitz condition
 
$$|\mathcal{N}_i(u_2(t)) - \mathcal{N}_i(u_1(t))| \leq \mathcal{L}_i |u_2(t) - u_1(t)|,$$
  - ii. the following inequality
 
$$\|\mathcal{N}_i(u(t))\| \leq \sigma_i \|u(t)\|,$$

where  $P_1 : P_3, \mathcal{L}_1, \mathcal{L}_2, \sigma_1$  and  $\sigma_2$  are positive constants.

**Theorem 2.1.** *If assumptions (1), (2) and (3.i) are satisfied and*

$$|\lambda| < \frac{|\mu|}{P_2 \mathcal{L}_1 + P_3 \mathcal{L}_2 T}, \quad (2.2)$$

*then Eq.(2.1) has a unique solution  $u(t)$  in the space  $C[0, T]$ .*

*Proof.* Foremost, using Picard's method, the solution of Eq.(2.1) can be expressed as a sequence of functions  $\{u_n(t)\}$  as  $n \rightarrow \infty$  based on

$$u_m(t) = f(t) + \lambda \int_a^b K_1(t, s)\mathcal{N}_1(u_{m-1}(s))ds + \lambda \int_0^t K_2(t, s)\mathcal{N}_2(u_{m-1}(s))ds, \quad (2.3)$$

with  $u_0(t) = f(t)$ . Let

$$v_m(t) = u_m(t) - u_{m-1}(t), \text{ and } v_0(t) = f(t). \quad (2.4)$$

with

$$u_n(t) = \sum_{i=0}^n v_i(t), \quad n = 1, 2, 3, \dots, \quad (2.5)$$

where  $v_m(t), m = 1, 2, \dots$ , are continuous functions. Now we shall prove that the series  $\sum_{i=0}^{\infty} v_i(t)$  is uniformly convergent. Using Eqs.(2.3), (2.4) and norm properties yield

$$\begin{aligned} |\mu| \|u_m(t) - u_{m-1}(t)\| \leq & |\lambda| \left\| \int_a^b K_1(t, s) [\mathcal{N}_1(u_{m-1}(s)) - \mathcal{N}_1(u_{m-2}(s))] ds \right\| \\ & + |\lambda| \left\| \int_0^t K_2(t, s) [\mathcal{N}_2(u_{m-1}(s)) - \mathcal{N}_2(u_{m-2}(s))] ds \right\|, \end{aligned} \quad (2.6)$$

at  $n = 1$ , from (2.6) and using the given assumptions, we get

$$|\mu| \|v_1(t)\| \leq |\lambda| \left\| \int_a^b |K_1(t, s)| \mathcal{L}_1 |v_0| ds \right\| + |\lambda| \left\| \int_0^t |K_2(t, s)| \mathcal{L}_2 |v_0| ds \right\|. \quad (2.7)$$

Hence, we obtain

$$\|v_1(t)\| \leq \frac{|\lambda|}{|\mu|} (P_2 \mathcal{L}_1 + P_3 \mathcal{L}_2 T) P_1, \quad (2.8)$$

where  $\max_{t \in [0, T]} |t| = T$ . In addition, at  $n = 2$ , we have

$$|\mu| \|v_2(t)\| \leq |\lambda| \left\| \int_a^b |K_1(t, s)| \mathcal{L}_1 |v_1| ds \right\| + |\lambda| \left\| \int_0^t |K_2(t, s)| \mathcal{L}_2 |v_1| ds \right\|, \quad (2.9)$$

which leads to

$$\|v_2(t)\| \leq \left( \frac{|\lambda|}{|\mu|} (P_2 \mathcal{L}_1 + P_3 \mathcal{L}_2 T) \right)^2 P_1. \quad (2.10)$$

Subsequently, the mathematical induction is applied to obtain

$$\|v_m(t)\| \leq \gamma_1^m P_1, \quad \gamma_1 = \frac{(P_2 \mathcal{L}_1 + P_3 \mathcal{L}_2 T) |\lambda|}{|\mu|}. \quad (2.11)$$

Note that, the series  $\sum_{i=0}^{\infty} v_i(t)$  is uniformly convergent if and only if the series  $\sum_{m=0}^{\infty} \gamma_1^m P_3$  is convergent.

Therefore, since  $|\lambda| < \frac{|\mu|}{P_2 \mathcal{L}_1 + P_3 \mathcal{L}_2 T}$ , we get  $\gamma_1 < 1$  and this implies that the series  $\sum_{m=0}^{\infty} \gamma_1^m P_3$  is convergent. Thus, for  $n \rightarrow \infty$ , we get

$$u(t) = \sum_{i=0}^{\infty} v_i(t) \quad (2.12)$$

represents a solution of Eq (2.1).

Now, to show the solution is unique, we assume that there exists another continuous solution  $\tilde{u}(t)$  of Eq.(2.1). So, we get

$$\begin{aligned} \|u(t) - \tilde{u}(t)\| \leq & \left\| \lambda \int_a^b K_1(t, s) (\mathcal{N}_1(u(s)) - \mathcal{N}_1(\tilde{u}(s))) ds \right\| \\ & + \left\| \lambda \int_a^t K_2(t, s) (\mathcal{N}_2(u(s)) - \mathcal{N}_2(\tilde{u}(s))) ds \right\|. \end{aligned} \quad (2.13)$$

Note that under the given conditions, inequality (2.13) yields

$$\|u(t) - \tilde{u}(t)\| \leq \gamma_1 \|u(t) - \tilde{u}(t)\|. \quad (2.14)$$

If  $\|u(t) - \tilde{u}(t)\| \neq 0$ , then (2.14) yields  $\gamma_1 \geq 1$  which is a contradiction. Therefore,  $\|u(t) - \tilde{u}(t)\| = 0$  and it is implied that  $u(t) = \tilde{u}(t)$  which means the solution is unique.  $\square$

### 3. Existence and uniqueness of the first kind NV-FIEs

If we have  $\mu = 0$  in Eq (2.1), we get the first kind NV-FIEs

$$f(t) + \lambda \int_a^b K_1(t, s) \mathcal{N}_1(u(s)) ds + \lambda \int_0^t K_2(t, s) \mathcal{N}_2(u(s)) ds = 0. \quad (3.1)$$

Now, we shall use Banach fixed point theorem which is used in case of failure of Picard's method at  $\mu = 0$ . So, Eq.(3.1) will be first expressed in its integral operator form

$$\bar{\mathbb{U}}u = f + \mathbb{U}u, \quad (3.2)$$

where

$$\mathbb{U}u = \mathbb{U}_1u + \mathbb{U}_2u,$$

$$\mathbb{U}_1u = \lambda \int_a^b K_1(t, s) \mathcal{N}_1(u(s)) ds, \text{ and } \mathbb{U}_2u = \lambda \int_0^t K_2(t, s) \mathcal{N}_2(u(s)) ds.$$

**For the normality of the integral operator**, we use (3.2) with the help of the given assumptions and norm properties to obtain

$$\begin{aligned} \|\mathbb{U}u\| & \leq \left\| \lambda \int_a^b K_1(t, s) \mathcal{N}_1(u(s)) ds \right\| + \left\| \lambda \int_0^t K_2(t, s) \mathcal{N}_2(u(s)) ds \right\| \\ & \leq |\lambda| P_2 \sigma_1 \|u(s)\| + |\lambda| P_3 \sigma_2 T \|u(s)\| \\ & \leq \gamma_2 \|u(s)\|, \quad \gamma_2 = |\lambda| (P_2 \sigma_1 + P_3 \sigma_2 T). \end{aligned} \quad (3.3)$$

If  $|\lambda| < \frac{1}{P_2 \sigma_1 + P_3 \sigma_2 T}$ , we get  $\gamma_2 < 1$  which means  $\mathbb{U}$  is a contraction operator and this implies that the integral operator  $\bar{\mathbb{U}}$  has a normality which leads directly after using the condition (1) to the normality of the operator  $\bar{\mathbb{U}}$ .

**For the continuity of the integral operator**, if we assume that the two functions  $u_1(t)$  and  $u_2(t) \in C[0, T]$  with the help of the norm properties under the given conditions, then we get

$$\begin{aligned} \|\bar{\mathcal{U}}u_1 - \bar{\mathcal{U}}u_2\| &= \|\mathcal{U}u_1 - \mathcal{U}u_2\| \\ &\leq \left\| \lambda \int_a^b K_1(t, s)(\mathcal{N}_1(u_1(s)) - \mathcal{N}_1(u_2(s)))ds \right\| \\ &\quad + \left\| \lambda \int_0^t K_2(t, s)(\mathcal{N}_2(u_1(s)) - \mathcal{N}_2(u_2(s)))ds \right\| \\ &\leq |\lambda|(P_2\mathcal{L}_1 + P_3\mathcal{L}_2T)\|u_1(s) - u_2(s)\| \\ &\leq \gamma_3\|u_1(s) - u_2(s)\|, \quad \gamma_3 = (P_2\mathcal{L}_1 + P_3\mathcal{L}_2T)|\lambda|. \end{aligned} \tag{3.4}$$

If  $|\lambda| < \frac{1}{P_2\mathcal{L}_1 + P_3\mathcal{L}_2T}$ , we get  $\gamma_2 < 1$  which means  $\bar{\mathcal{U}}$  is a contraction operator and leads to the continuity of the integral operator  $\bar{\mathcal{U}}$  in the space  $C[0, T]$ . Using Banach fixed point theorem,  $\bar{\mathcal{U}}$  has a unique fixed point that means the NV-FIEs (3.1) of the first kind has a unique solution.

#### 4. Homotopy analysis method for NV-FIEs

We shall introduce the basic idea of HAM [22, 23] for solving the operator equation

$$\mathcal{N}(u(t)) = 0, \quad t \in [0, T] \tag{4.1}$$

where  $\mathcal{N}$  denotes the nonlinear operator, and  $u(t)$  is an unknown function. Foremost, we define the homotopy operator  $\mathcal{H}$ ,

$$\mathcal{H}(\Phi, p) = (1 - p)(\Phi(t; p) - u_0(t)) - ph\mathcal{N}(\Phi(t; p)), \tag{4.2}$$

where  $p \in [0, 1]$  is the embedding parameter,  $h \neq 0$  denotes the convergence control parameter,  $u_0(t)$  describes the initial approximation of the solution of (4.1). Considering  $\mathcal{H}(\Phi, p) = 0$ , we get the so-called zero-order deformation equation

$$(1 - p)(\Phi(t; p) - u_0(t)) = ph\mathcal{N}(\Phi(t; p)). \tag{4.3}$$

For  $p = 0$ , we have  $\Phi(t; 0) - u_0(t) = 0$  which implies that  $\Phi(t; 0) = u_0(t)$ , whereas for  $p = 1$ , we have  $\mathcal{N}(\Phi(t; 1)) = 0$  that means  $\Phi(t; 1) = u(t)$ , where  $u(t)$  is the solution of (4.1). In this way, the variation of parameter  $p : 0 \rightarrow 1$  corresponds with the change of problem from the trivial problem to the original one (and with the change of solution from  $u_0(t) \rightarrow u(t)$ ). Expanding  $\Phi(x; p)$  into the Maclaurin series with respect to  $p$ , we get

$$\Phi(t; p) = \Phi(t; 0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(t; p)}{\partial p^m} \Big|_{p=0} p^m, \tag{4.4}$$

By distinguishing

$$v_m(t) = \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(t; p)}{\partial p^m} \Big|_{p=0}, \quad m = 1, 2, 3, \dots \tag{4.5}$$

Eq.(4.4) becomes

$$\Phi(t; p) = v_0(t) + \sum_{m=1}^{\infty} v_m(t)p^m. \quad (4.6)$$

If the above series is convergent at  $p = 1$ , we obtain

$$u(t) = \sum_{m=0}^{\infty} v_m(t). \quad (4.7)$$

To determine function  $v_m(t)$ , we differentiate the both sides of Eq.(4.3)  $m$  times with respect to  $p$ , next we divide the received result by  $m!$  and we substitute  $p = 0$ . Herein, we get the so-called  $m$  th-order deformation equation ( $m > 0$ )

$$v_m(t) - \chi_m v_{m-1}(t) = hR_m(\bar{v}_{m-1}, t) \quad (4.8)$$

where  $\bar{v}_{m-1} = \{v_0(t), v_1(t), \dots, v_{m-1}(t)\}$ ,

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} \quad (4.9)$$

and

$$R_m(\bar{v}_{m-1}, t) = \frac{1}{(m-1)!} \left( \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N} \left( \sum_{i=0}^{\infty} v_i(t)p^i \right) \right) \Big|_{p=0}. \quad (4.10)$$

Since, we can not determine the sum of series in (4.7), we shall accept the partial sum of this series

$$u(t) \approx u_n(t) = \sum_{m=0}^n v_m(t) \quad (4.11)$$

as the approximate solution of considered equation.

Secondly, we introduce HAM for NF-VIE (2.1) and operator  $\mathcal{N}$  can be defined as

$$\begin{aligned} \mathcal{N}(v(t)) = & \mu v(t) - f(t) - \lambda \int_a^b K_1(t, s) \mathcal{N}_1(v(s)) ds \\ & - \lambda \int_0^t K_2(t, s) \mathcal{N}_2(v(s)) ds. \end{aligned} \quad (4.12)$$

Applying the HAM, we get the following formula for functions  $v_m(t)$

$$v_m(t) = \chi_m v_{m-1}(t) + hR_m(\bar{v}_{m-1}, t), \quad (4.13)$$

where  $\chi_m$  and  $R_m$  are defined by (4.9) and (4.10), respectively. Using definitions of the respective operators, we obtain

$$\begin{aligned} v_1(t) = & hR_1(\bar{v}_0, t) \\ = & h \left( \mu v_0(t) - f(t) - \lambda \int_a^b K_1(t, s) \mathcal{N}_1(v_0(s)) ds \right. \\ & \left. - \lambda \int_0^t K_2(t, s) \mathcal{N}_2(v_0(s)) ds \right) \end{aligned} \quad (4.14)$$

and for  $m \geq 2$ , we get

$$\begin{aligned}
 v_m(t) = & (1 + h\mu)v_{m-1}(t) \\
 & - \frac{\lambda h}{(m-1)!} \int_a^b K_1(t, s) \left[ \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N}_2 \left( \sum_{i=0}^{\infty} v_i(s) p^i \right) \right]_{p=0} ds \\
 & - \frac{\lambda h}{(m-1)!} \int_a^t K_2(t, s) \left[ \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N}_2 \left( \sum_{i=0}^{\infty} v_i(s) p^i \right) \right]_{p=0} ds.
 \end{aligned} \tag{4.15}$$

In case of  $\mu = 0$ , Eq.(4.14) becomes

$$\begin{aligned}
 v_1(t) = & hR_1(\bar{v}_0, t) \\
 = & h \left( -f(t) - \lambda \int_a^b K_1(t, s) \mathcal{N}_1(v_0(s)) ds - \lambda \int_0^t K_2(t, s) \mathcal{N}_2(v_0(s)) ds \right)
 \end{aligned} \tag{4.16}$$

and Eq.(4.15) gives

$$\begin{aligned}
 v_m(t) = & v_{m-1}(t) \\
 & - \frac{\lambda h}{(m-1)!} \int_a^b K_1(t, s) \left[ \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N}_2 \left( \sum_{i=0}^{\infty} v_i(s) p^i \right) \right]_{p=0} ds \\
 & - \frac{\lambda h}{(m-1)!} \int_a^t K_2(t, s) \left[ \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N}_2 \left( \sum_{i=0}^{\infty} v_i(s) p^i \right) \right]_{p=0} ds.
 \end{aligned} \tag{4.17}$$

**Theorem 4.1.** *Suppose that the nonlinear operators  $\mathcal{N}_1$  and  $\mathcal{N}_2$  satisfies Lipschitz condition (3.i). If the series  $\sum_{m=0}^{+\infty} v_m(t)$  converges to  $u(t)$ , where  $v_m(t)$  is governed by Eq.(4.8) under the definitions (4.9) and (4.10), then  $u(t)$  will be the exact solution of the NF-VIE (2.1).*

*Proof.* Firstly, we define

$$\mathcal{H}_i(t) = \frac{1}{m!} \left( \frac{\partial^m}{\partial p^m} \mathcal{N}_i \left( \sum_{j=0}^{\infty} v_j(t) p^j \right) \right) \Big|_{p=0}, \quad i = 1, 2. \tag{4.18}$$

From (4.9), we have

$$\begin{aligned}
 \sum_{m=1}^n [v_m(t) - \chi_m v_{m-1}(t)] = & v_1(t) + [v_2(t) - v_1(t)] + [v_3(t) - v_2(t)] \\
 & + \dots + [v_n(t) - v_{n-1}(t)] = v_n(t).
 \end{aligned} \tag{4.19}$$

From the convergence of  $\sum_{m=0}^{+\infty} v_m(t)$ ,

$$\lim_{m \rightarrow \infty} v_m(t) = 0, \quad t \in [0, T]. \tag{4.20}$$

Using Eq.(4.20), Eq.(4.19) becomes

$$\sum_{m=1}^{\infty} [v_m(t) - \chi_m v_{m-1}(t)] = \lim_{n \rightarrow \infty} v_n(t) = 0. \tag{4.21}$$

Eq.(4.21) and Eq.(4.13) yield

$$h \sum_{m=1}^{\infty} R_m(\bar{v}_{m-1}, t) = \sum_{m=1}^{\infty} v_m(t) - \chi_m v_{m-1}(t) = 0. \quad (4.22)$$

Since  $h \neq 0$ , Eq.(4.22) gives

$$\sum_{m=1}^{\infty} R_m(\bar{v}_{m-1}, t) = 0. \quad (4.23)$$

Now, Eq.(4.12) and definitions (4.5) and (4.10) give

$$\begin{aligned} 0 &= \sum_{m=1}^{+\infty} [R_m(\bar{v}_{m-1}, t)] \\ &= \sum_{m=1}^{+\infty} \left[ \mu v_{m-1}(t) - (1 - \chi_m) f(t) - \lambda \int_a^b K_1(t, s) \frac{\partial^{m-1}}{(m-1)! \partial \rho^{m-1}} \mathcal{N}_1 \left[ \sum_{k=0}^{k=+\infty} v_k(s) \rho^k \right] \Big|_{\rho=0} ds \right. \\ &\quad \left. - \lambda \int_a^t K_2(t, s) \frac{\partial^{m-1}}{(m-1)! \partial \rho^{m-1}} \mathcal{N}_2 \left[ \sum_{k=0}^{k=+\infty} v_k(s) \rho^k \right] \Big|_{\rho=0} ds \right] \\ &= \sum_{m=1}^{+\infty} \left[ \mu v_{m-1}(t) - (1 - \chi_m) f(t) - \lambda \int_a^b K_1(t, s) \mathcal{H}_{1m-1}(s) ds - \lambda \int_a^t K_2(t, s) \mathcal{H}_{2m-1}(s) ds \right] \\ &= \sum_{m=1}^{+\infty} \left[ \mu v_{m-1}(t) - f(t) - \lambda \int_a^b K_1(t, s) \sum_{m=1}^{+\infty} \mathcal{H}_{1m-1}(s) ds - \lambda \int_a^t K_2(t, s) \sum_{m=1}^{+\infty} \mathcal{H}_{2m-1}(s) ds \right] \end{aligned} \quad (4.24)$$

Since the nonlinear operators  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are contraction; therefore, if the series  $\sum_{m=0}^{+\infty} v_m(t)$  converges to  $u(t)$ , then the series  $\sum_{m=0}^{+\infty} \mathcal{H}_{1m-1}(t)$  and  $\sum_{m=0}^{+\infty} \mathcal{H}_{2m-1}(t)$  will converge to  $\mathcal{N}_1(u(t))$  and  $\mathcal{N}_2(u(t))$ , respectively [24]. So, Eq.(4.24) becomes

$$\mu u(t) = f(t) + \lambda \int_a^b K_1(t, s) \mathcal{N}_1(u(s)) ds + \lambda \int_a^t K_2(t, s) \mathcal{N}_2(u(s)) ds. \quad (4.25)$$

Hence,  $u(t)$  is the exact solution of NF-VIE (2.1). In case of  $\mu = 0$ , Eq.(4.24) gives

$$f(t) + \lambda \int_a^b K_1(t, s) \mathcal{N}_1(u(s)) ds + \lambda \int_0^t K_2(t, s) \mathcal{N}_2(u(s)) ds = 0. \quad (4.26)$$

which indicates that  $u(t)$  is the exact solution of NF-VIE (3.1).  $\square$

#### Theorem 4.2. (Convergence Theorem)

Assume that  $h$  is properly chosen for which there exists  $0 < \alpha < 1$  so that  $\|v_{k+1}\| \leq \alpha \|v_k\|$ ,  $\forall k \geq k_0$ , for some  $k_0 \in \mathbb{N}$ , then  $u_n(t, h)$  in (4.11) converges as  $n \rightarrow +\infty$ .

*Proof.* Define the sequence  $U_m$  as

$$\begin{aligned} U_0 &= v_0 \\ U_1 &= v_0 + v_1 \\ &\vdots \\ U_n &= v_0 + v_1 + \cdots + v_m \end{aligned} \quad (4.27)$$



Now, we show that  $U_m$  is a Cauchy sequence in the space  $C[0, T]$ . Consider

$$\|U_{m+1} - U_m\| = \|v_{m+1}\| \leq \alpha \|v_m\| \leq \alpha^2 \|v_{m-1}\| \leq \dots \leq \alpha^{m-k_0+1} \|v_{k_0}\| \quad (4.28)$$

For every  $l, m \in \mathbb{N}, m \geq l > k_0$ , we have

$$\begin{aligned} \|U_m - U_l\| &= \|(U_m - U_{m-1}) + (U_{m-1} - U_{m-2}) + \dots + (U_{l+1} - U_l)\| \\ &\leq \|(U_m - U_{m-1})\| + \|(U_{m-1} - U_{m-2})\| + \dots + \|(U_{l+1} - U_l)\| \\ &\leq \alpha^{m-k_0} \|v_{k_0}\| + \alpha^{m-k_0-1} \|v_{k_0}\| + \dots + \alpha^{m-k_0+1} \|v_{k_0}\| \\ &= \frac{1 - \alpha^{m-l}}{1 - \alpha} \alpha^{l-k_0+1} \|v_{k_0}\|. \end{aligned} \quad (4.29)$$

Since  $0 < \alpha < 1$ , we get

$$\lim_{m, l \rightarrow \infty} \|U_m - U_l\| = 0. \quad (4.30)$$

Therefore,  $U_m$  is a Cauchy sequence in the space  $C[0, T]$  and  $U_n = u_n(t, h)$  converges as  $n \rightarrow \infty$  and the proof is complete.  $\square$

**Theorem 4.3.** *If assumptions of Theorem 4.2 are satisfied,  $n \in \mathbb{N}$  and  $n \geq k_0$ , then we obtain the estimation of error of the approximate solution defined by*

$$\|u(t) - u_n(t)\| \leq \frac{\alpha^{n+1-k_0}}{1 - \alpha} \|v_{k_0}\|. \quad (4.31)$$

*Proof.* . Let  $n \in \mathbb{N}$  and  $n \geq k_0$ , we get

$$\begin{aligned} \|u(t) - u_n(t)\| &= \sup_{t \in [0, T]} \left| u(t) - \sum_{m=0}^n v_m(x, t) \right| \\ &\leq \sup_{t \in [0, T]} \left( \sum_{m=n+1}^{\infty} |v_m(t)| \right) \\ &\leq \sum_{m=n+1}^{\infty} \sup_{t \in [0, T]} (|v_m(x, t)|) \\ &\leq \sum_{m=n+1}^{\infty} \alpha^{m-k_0} \|v_{k_0}\| \\ &= \frac{\alpha^{n+1-k_0}}{1 - \alpha} \|v_{k_0}\|. \end{aligned} \quad (4.32)$$

$\square$

## 5. Illustrating examples

In this section, some examples will be given to investigate the efficiency and accuracy of the proposed method

**Example 5.1.** Consider the following strongly NV-FIE

$$u(t) = f(t) + \int_0^1 (t^2 - s)u^2(s)ds + \int_0^t ts^2u^3(s)ds, \quad (5.1)$$

where  $f(t) = e^{-t}t^2 - \frac{6(e^2-7)t^2+109}{8e^2} + \frac{e^{-3t}t}{2187}(3t(3t(t(3t(3t(t(3t(3t+8)+56)+112)+560)+2240)+2240)+4480)+4480) - \frac{4480t}{2187} + \frac{15}{8}$  and (the exact solution  $u(t) = t^2e^{-t}$ ). In this example, Fig.(1) shows the behaviour of error using HAM at  $n = 10$  and  $h = -1.9$ ,  $t \in [0, 0.8]$ . Also, Fig.(2) presents the valid region of  $h$  at  $n = 10$ .

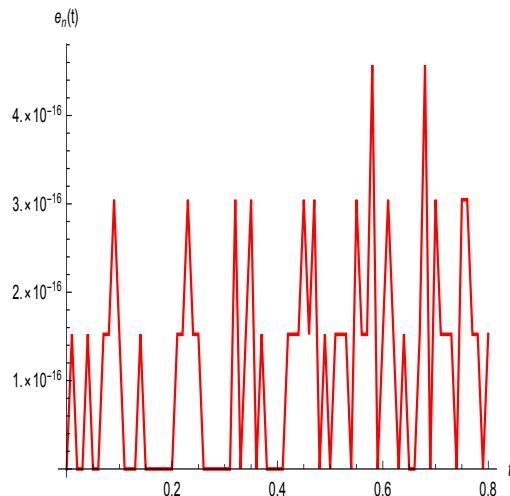


Figure 1. The error behaviour at  $n = 10$ ,  $h = -0.1$  using HAM

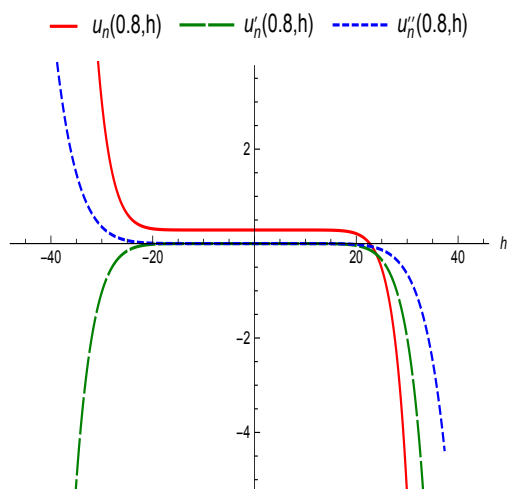


Figure 2. The h-curves at  $n = 10$

**Example 5.2.** Consider the following strongly NV-FIE [20]

$$u(t) = f(t) + \int_0^1 (t-s)u^5(s)ds + \int_0^t (t+s)u^4(s)ds, \tag{5.2}$$

where  $f(t) = \frac{77107623-70t(792t^5+324787)}{151200} + \frac{5t-6}{25e^5} + \frac{1}{16}e^{-4t}(8t+1) + \frac{5(10t-13)}{32e^4} + \frac{10(17t-26)}{27e^3} + \frac{5(19t-42)}{4e^2} + \frac{5(65t-326)}{e} + \frac{4}{27}e^{-3t}(9t(2t+1)+2) + \frac{1}{4}e^{-2t}(6t(t(4t+5)+4)+9) + e^{-t}(4t(t(t(2t+7)+18)+30)+97)$  and (the exact solution  $u(t) = t + e^t$ ). In this example, Fig.(3) displays the behaviour of error using HAM at  $n = 10$  and  $h = -1.9$ . In addition, Fig.(4) explains the valid region of  $h$  at  $n = 10$ . Moreover, Table 1 presents the maximum error  $e_{max} = \max_i |u(t_i) - u_n(t_i)| \quad \forall t_i \in [0, 0.9]$  for every  $n = 2, 5, 8, 12, 16, 20$  and a comparison with results in Ref. [20].

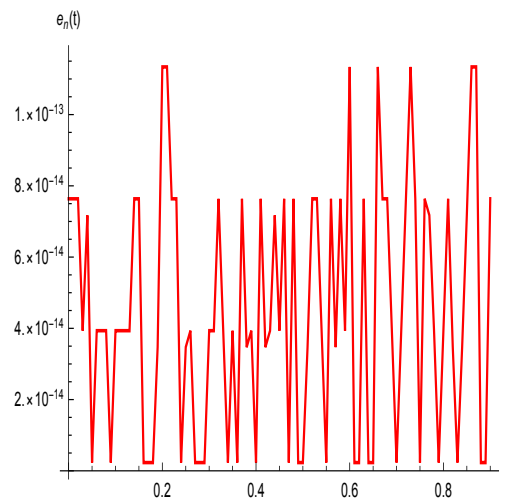


Figure 3. The error behaviour at  $n = 10, h = -1.9$  using HAM

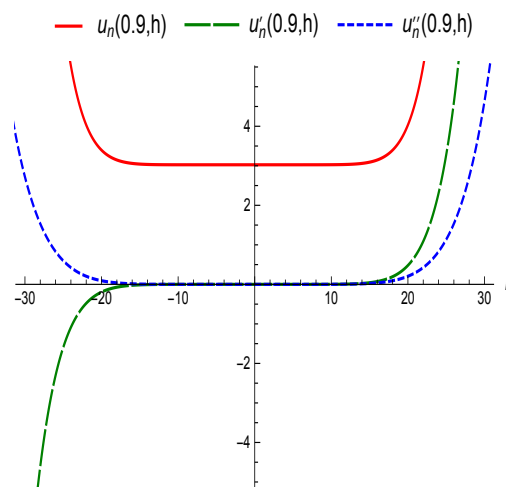


Figure 4. The h-curves at  $n = 10$

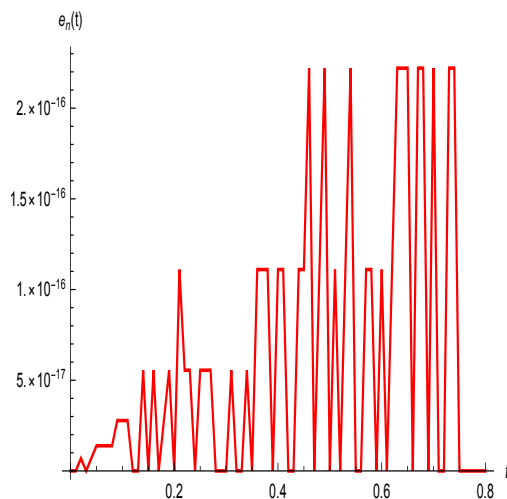
Table 1. The maximum error  $e_{max}$  for different values of  $n$  with corresponding  $h$  for example (5.2)

$n$	$e_{max}$ in [20]	$e_{max}$ of HAM at $h = -0.1$
2	$2.94 \times 10^{-2}$	$4.3876 \times 10^{-12}$
5	$5.95 \times 10^{-4}$	$9.45667 \times 10^{-13}$
8	$1.27E \times 10^{-4}$	$1.3152 \times 10^{-13}$
12	$3.93 \times 10^{-5}$	$1.65706 \times 10^{-13}$
16	$1.60 \times 10^{-5}$	$1.96266 \times 10^{-13}$
20	$7.66 \times 10^{-6}$	$2.02851 \times 10^{-13}$

**Example 5.3.** Consider the following strongly NV-FIE

$$f(t) = \int_0^1 s^2 t u^2(s) ds + \int_0^t s t^2 u^3(s) ds, \quad (5.3)$$

where  $f(t) = -\frac{1}{4}3t^3 \cosh(t) + \frac{1}{12}t^3 \cosh(3t) + \frac{3}{4}t^2 \sinh(t) - \frac{1}{36}t^2 \sinh(3t) - \frac{t}{6} + \frac{3}{8}t \sinh(2) - \frac{1}{4}t \cosh(2)$  and (the exact solution  $u(t) = \sinh(t)$ ). In this example, Fig.(5) shows the behaviour of error using HAM at  $n = 20$ ,  $h = 0.1$  and  $t \in [0, 0.8]$ . In addition, Fig. (6) investigates the valid region of  $h$  at  $n = 20$ .

Figure 5. The error behaviour at  $n = 20$ ,  $h = 0.1$  using HAM

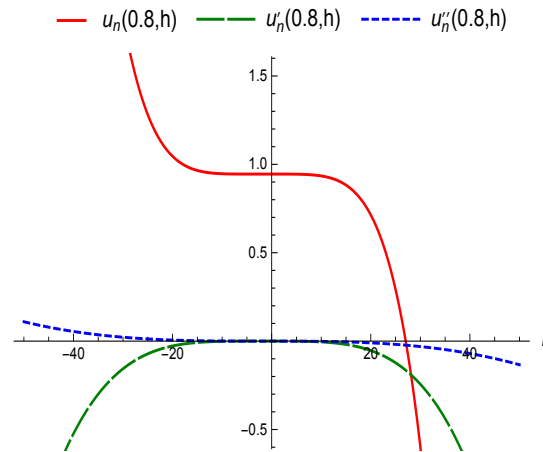


Figure 6. The h-curves at  $n = 20$

## 6. Conclusion

In this paper, we used Picard's method to prove the existence and uniqueness of the solution of the second kind NV-FIEs which has many application in mathematical physics. Moreover, we utilized Banach fixed point theorem to discuss the solvability of the first kind NV-FIEs. In addition, we applied the HAM to approximate the solution and discussed the convergence analysis. Furthermore, we investigated illustrative examples to indicate the validity and accurately of the presented method showing the error behaviour. Based on the results, we observed that that HAM is an effective method for solving the first and second kind NF-VIEs.

**Conflicts of Interest:** The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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