International Journal of Analysis and Applications

Well-Posedness and Stability for a System of Klein-Gordon Equations

Naaima Latioui[1,*], Amar Guesmia[1], Amar Ouaoua[2]

[1] *Laboratory of Applied Mathematics and History and Didactics of Mathematics "LAMAHIS", Department of Mathematics, University 20 août 1955 Skikda, Algeria*

[2] *Laboratory of Applied Mathematics and History and Didactics of Mathematics "LAMAHIS", Department of Technologic, University 20 août 1955 Skikda, Algeria*

[*] *Corresponding author:* loubnalatioui@gmail.com, n.latioui@univ-Skikda.dz

Abstract. In this paper, we study the weak existence of solution for a non-linear hyperbolic coupled system of Klein-Gordon equations with memory and source terms using the Faedo-Galerkin method techniques and compactness results, we have demonstrated the uniqueness of the solution by using the classical technique. In addition, we show that the solution remains stable over time. The reaction of the proper Lyapunov function is the primary tool of the proof.

1. Introduction

In this paper, we consider a non-linear hyperbolic system of Klein-Gordon equations, defined as the following

$$\begin{cases} u_{tt} - \Delta u - \alpha u_t + k * \Delta u - div(|v|^2 \nabla u) + u|\nabla v|^2 = 0 \quad \text{in} \quad \Omega \times (0, T), \\ v_{tt} - \Delta v - \beta v_t + l * \Delta v - div(|u|^2 \nabla v) + v|\nabla u|^2 = 0 \quad \text{in} \quad \Omega \times (0, T), \end{cases} \tag{1.1}$$

with boundary conditions

$$u(x, t) = v(x, t) = 0 \qquad \text{on} \quad \Gamma \times (0, T), \tag{1.2}$$

$$u_t(x, t) = v_t(x, t) = 0 \qquad \text{on} \quad \Gamma \times (0, T), \tag{1.3}$$

Received: Dec. 3, 2021.

2010 *Mathematics Subject Classification.* 81Q05, 37L45, 93D20.

Key words and phrases. Klein-Gordon system; Faedo-Galerkin method; Lyapunov function.

https://doi.org/10.28924/2291-8639-20-2022-10

ISSN: 2291-8639

© 2022 the author(s).

and Initial conditions

$$u(x,0)=u_0(x), \quad v(x,0)=v_0(x) \qquad \text{on} \quad \Omega, \tag{1.4}$$

$$u_t(x,0)=u_1(x), \quad v_t(x,0)=v_1(x) \qquad \text{on} \quad \Omega. \tag{1.5}$$

Where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary Γ and let $T>0$, α and β are non-positive constants, and

$$(n * w)(t)=\int_0^t n(t-s)w(s)ds. \tag{1.6}$$

Several authors have studied the Klein-Gordon non-linear system among them
Medeiros & M. Miranda [8] considered the non-linear system

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-\Delta u+u-|v|^\rho|u|^\rho u=f_1, \\ \frac{\partial^2 v}{\partial t^2}-\Delta v+v-|u|^\rho|v|^\rho v=f_2, \end{cases} \tag{1.7}$$

they prove the existence and uniqueness of weak global solutions in $\Omega\times[0,T]$, where ρ is a real number meeting a specific condition and Ω is any domain of $\mathbb{R}^n$.

D. Andrade & A. Mognon [2], considered the non-linear system with memory term

$$\begin{cases} u_{tt}-\Delta u+f(u,v)+k*\Delta u=0, \\ v_{tt}-\Delta v+g(u,v)+l*\Delta v=0, \end{cases} \tag{1.8}$$

for $x\in\Omega$ and $t>0$ where

$$f(u,v)=|u|^{\rho-2}u|v|^\rho, \qquad \text{and} \qquad g(u,v)=|v|^{\rho-2}v|u|^\rho,$$

with

$$\rho>0 \quad \text{if} \quad n=1,2 \quad \text{and} \quad 1<\rho\leq\frac{n-1}{n-2} \quad \text{if} \quad n\geq 3,$$

they use the argument from Komornik and Zuazua [6] to prove the existence of weak and strong solutions in $\Omega\times(0,T)$ given initial and boundary conditions.

A. T. Louredo & M. M. Miranda [7], considered the non-linear system

$$\begin{cases} u''-\Delta u+\alpha v^2 u=0, \\ v''-\Delta v+\alpha u^2 v=0, \end{cases} \tag{1.9}$$

with the nonlinear boundary conditions,

$$\frac{\partial u}{\partial \nu}+h_1(.,u')=0 \quad \text{on} \quad \Gamma_1\times(0,\infty),$$
$$\frac{\partial v}{\partial \nu}+h_2(.,v')=0 \quad \text{on} \quad \Gamma_1\times(0,\infty),$$

and boundary conditions $u=v=0$ on $(\Gamma/\Gamma_1)\times(0,\infty)$, where Ω is a bounded open set of $\mathbb{R}^n$ $(n\leq 3)$, $\alpha>0$ a real number, Γ_1 is a subset of the border Γ of Ω and h_i a real function defined on $\Gamma_1\times(0,\infty)$. They use the Galerkin approach to demonstrate the existence of global solutions.

K. Zennir & A. Guesmia [10], considered the non-linear κth-order with non-linear sources and memory terms

$$\begin{cases} u_1'' + (-1)^\kappa \Delta^\kappa u_1 + m_1^2 u_1 + \alpha_1(t)\int_0^t g_1(t-s)\Delta^\kappa u_1(x,s)ds + |u_1'|^{r-2}|u_1'| \\ = |u_1|^{p-2}u_1|u_2|^p, \\ u_2'' + (-1)^\kappa \Delta^\kappa u_2 + m_2^2 u_2 + \alpha_2(t)\int_0^t g_2(t-s)\Delta^\kappa u_2(x,s)ds + |u_2'|^{r-2}|u_2'| \\ = |u_2|^{p-2}u_2|u_1|^p, \end{cases} \tag{1.10}$$

using the potential well method, they verify the existence of global solutions in the a bounded domain Ω of $\mathbb{R}^n$, where $m_i = 1, 2$ are non-negative constants, $r, p \geq 2, \kappa \geq 1$.
C. L. Frota & A. Vicente [5], studied the non-linear system of Klien-Gordon with acoustic boundary conditions

$$\begin{cases} u'' - \Delta u + |v|^{\rho+2}|u^\rho|u = f_1 \quad \text{in} \quad \Omega \times (0,T), \\ v'' - \Delta v + |u|^{\rho+2}|v^\rho|v = f_1 \quad \text{in} \quad \Omega \times (0,T), \end{cases} \tag{1.11}$$

they demonstrate the existence of both global and weak solutions, as well as their uniqueness.
Our objective is to prove that the problem (1.1)-(1.5) has a weak and unique solution such that the kernel terms k, l have some hypothesis as well as using some ideas from articles ([2]) and ([9]).

2. Preliminaries

Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary Γ let $T > 0$.
The inner product and norm in $L^2(\Omega)$ are denoted by

$$\langle u, v\rangle = \int_\Omega u(x)v(x)dx, \quad |u|_2 = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}. \tag{2.1}$$

The norm in $H_0^1(\Omega)$ is denoted by

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}. \tag{2.2}$$

We assume that k, l: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are non-increasing differentiable functions satisfying :

$$l_1 = \left(1 - \int_0^t l(s)ds\right) > 0 \quad \text{and} \quad k_1 = \left(1 - \int_0^t k(s)ds\right) > 0, \tag{2.3}$$

and

$$k'(t) \leq -k(t), \quad l'(t) \leq -l(t). \tag{2.4}$$

If $w = w(t,x)$ is a function in $L^2(0.T; H_0^1(\Omega))$ and k is continuous we put:

$$(k \circ w)(t) = \int_0^t k(t-s)|\nabla w(t) - \nabla w(s)|_2^2 ds.$$

Lemma 2.1. *[2] $w \in C^1((0,T); H_0^1(\Omega))$, $k \in C^1(0,\infty)$*

$$\int_0^t k(t-s)\left\langle \nabla w(s)\nabla w'(t)\right\rangle ds = -\frac{1}{2}\frac{d}{dt}(k\circ w)(t) + \frac{1}{2}\frac{d}{dt}\left(\int_0^t k(s)ds\right)|\nabla w(t)|_2^2 \quad (2.5)$$
$$+(k'\circ w)(t) - k(t)|\nabla w(t)|_2^2.$$

Lemma 2.2. *[9] (**Young's Inequality**)*
Let a, $b \geq 0$ and $\frac{1}{q}+\frac{1}{p}=1$ for $1<p,q<+\infty$, then one has the inequality $ab \leq \delta a^q + c(\delta)b^p$, where $\delta > 0$ is an arbitrary constant, and $c(\delta)$ is a positive constant depending on δ.

Lemma 2.3. *[1]* **(Sobolev-Poincaré inequality)**
Let s be a number with $2 \leq s < +\infty$ if $n \leq 2$ and $2 \leq s \leq \frac{2n}{n-2}$ if $n > 2$. Then there is a constant C depending on Ω and s such that

$$\|u\|_s \leq C\|\nabla u\|_2, \quad u \in H_0^1.$$

Theorem 2.1. *Let u_0, $v_0 \in L^2(\Omega)$ and u_1, $v_1 \in L^1(\Omega)$. Then, under assumptions on two functions k and l, the problem (1.1)-(1.5) has a local solution $(u(x,t), v(x,t))$ such that*

$$u, v \in L^\infty(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)), \quad (2.6)$$
$$u_t, v_t \in L^\infty(0,T; L^2(\Omega)). \quad (2.7)$$

Theorem 2.2. *Let $u, v :\rightarrow L^2(\Omega)$ be functions in the class (2.6) and (2.7) satisfying from (1.1) to (1.5), with $u, v \in H^2(\Omega)$. Then the solution (u,v) obtained in Theorem (2.1) is unique.*

3. Global Existence

Step 1: Approximate solution. Using the Faedo-Galerkin process, we will determine the existence of a local solution to the problem (1.1)-(1.5) in this section. Let $\{w_i\}$ be a basis for both $H^2(\Omega) \cap H_0^1(\Omega)$ and $L^2(\Omega)$ for each positive integer m we put

$$V = span\{w_1, w_2, \dots, w_m\}.$$

we look for an approximate solution in the form

$$u_m(t) = \sum_{i=1}^m u_{im} w_i \quad \text{and} \quad v_m(t) = \sum_{i=1}^m v_{im} w_i,$$

satisfying the approximate problem

$$\int_\Omega \{u_{tt}^m - \Delta u^m - \alpha u_t^m\} w_i dx - \int_0^t k(t-s)\langle \nabla u^m(s), \nabla w_i\rangle ds \tag{3.1}$$
$$+ \int_\Omega |v^m|^2 \nabla u^m \nabla w_i dx + \int_\Omega u^m |\nabla v^m| w_i dx = 0,$$
$$\int_\Omega \{v_{tt}^m - \Delta v^m - \beta v_t^m\} w_i dx - \int_0^t l(t-s)\langle \nabla v^m(s), \nabla w_i\rangle ds \tag{3.2}$$
$$+ \int_\Omega v^m |\nabla u^m| w_i dx + \int_\Omega |u^m|^2 \nabla v^m \nabla w_i = 0,$$

with initial conditions satisfying

$$\begin{cases} u^m(0) = u_0^m, \sum_{i=1}^m a_{im} w_i = u_0^m \to u_0, \quad v^m(0) = v_0^m, \sum_{i=1}^m b_{im} w_i = v_0^m \to v_0 \quad \text{in } L^2(\Omega), \\ u_t^m(0) = u_1^m, \sum_{i=1}^m a_{im}^1 w_i = u_1^m \to u_1, \quad v_t^m(0) = v_1^m, \sum_{i=1}^m b_{im}^1 w_i = v_1^m \to v_1 \quad \text{in } L^1(\Omega). \end{cases} \tag{3.3}$$

Since the vectors $\{w_i\}$ are linearly independent, this means $det(w_i, w_j) \neq 0$, the latter ensuring that the problem admits a local solution $(u^m(t), v^m(t))$ in the interval $[0, T_m]$.

Step 2: A priori estimate. Our system's energy functional $E(t)$ is given by

$$2E(t) = |u_t^m|_{L^2(\Omega)}^2 + |v_t^m|_{L^2(\Omega)}^2 + (k \circ u^m)(t) + (l \circ v^m)(t) \tag{3.4}$$
$$+ \left(1 - \int_0^t k(s)ds\right)|\nabla u^m|_2^2 + \left(1 - \int_0^t l(s)ds\right)|\nabla v^m|_2^2 + |v^m \nabla u^m|_2^2 + |u^m \nabla v^m|_2^2.$$

After that, we multiply (3.1) by u_t, (3.2) by v_t, and use identity (2.5) to get

$$\frac{d}{dt}E(t) = (k' \circ u^m)(t) + (l' \circ v^m)(t) - k(t)|\nabla u^m|_2^2 - l(t)|\nabla v^m|_2^2 + \alpha|u_t^m|_2^2 + \beta|v_t^m|_2^2 \leq 0. \tag{3.5}$$

We found that $\frac{d}{dt}E(t)$ is a non-positive function, this last indicates that $E(t)$ is a non-increasing function, meaning there exists a positive constant C_1, independent of t and m such that

$$|u_t^m|_2^2 + |v_t^m|_2^2 + |\nabla u^m|_2^2 + |\nabla v^m|_2^2 + |u^m \nabla v^m|_2^2 + |v^m \nabla u^m|_2^2 \leq C_1. \tag{3.6}$$

From this estimation, deduce that $T_m = T$. In addition, we get

$$\begin{cases} u^m, v^m & \text{is bounded in} \quad L^\infty(0, T; H_0^1(\Omega)), \\ u^m, v^m & \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)), \\ u_t^m, v_t^m & \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)). \end{cases} \tag{3.7}$$

By the Holder inequality, the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and (3.7), we obtain

$$|u^m|\nabla v^m|^2|_2^2 \leq \|u^m\|_{L^6(\Omega)}^2 \|\nabla v^m\|_{L^6(\Omega)}^4 \leq C_1, \tag{3.8}$$
$$||v^m|^2 \nabla u^m|_2^2 \leq \|v^m\|_{L^6(\Omega)}^4 \|\nabla u^m\|_{L^6(\Omega)}^2 \leq C_2 \quad \forall (u^m, v^m) \quad \text{in} \quad H^2(\Omega). \tag{3.9}$$

Therefore

$$(u^m|\nabla v^m|^2) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{3.10}$$

$$(|v^m|^2\nabla u^m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.11}$$

Analogously

$$(v^m|\nabla u^m|^2) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{3.12}$$

$$(|u^m|^2\nabla v^m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.13}$$

Step 3: passage to the limit. From (3.7), (3.10), (3.11), (3.12) and (3.13) there exists a subsequence of (u^m) and a subsequence of (v^m), denoted by same symbols, such that

$$\begin{cases} u^m \to u \quad \text{and} \quad v^m \to v \quad \text{weak star in} \quad L^\infty(0,T;H_0^1(\Omega)), \\ u^m \to u \quad \text{and} \quad v^m \to v \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)), \\ u_t^m \to u_t \quad \text{and} \quad v_t^m \to v_t \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)), \\ u^m|\nabla v^m|^2 \to \chi_1 \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)), \\ v^m|\nabla u^m|^2 \to \chi_2 \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)), \\ |v^m|^2\nabla u^m \to \chi_3 \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)), \\ |u^m|^2\nabla v^m \to \chi_4 \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{3.14}$$

From (3.14) and Aubin-Lions compactness Lemma in ([3]), we obtain

$$u^m \to u, \quad v^m \to v \quad \text{strongly in} \quad L^\infty(0,T;L^2(\Omega)), \tag{3.15}$$

since ∇u^m and ∇v^m are bounded, then we have

$$\begin{cases} u^m|\nabla v^m|^2 \to u|\nabla v|^2 \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \\ v^m|\nabla u^m|^2 \to v|\nabla u|^2 \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \\ |u^m|^2\nabla v^m \to |u|^2\nabla v \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \\ |v^m|^2\nabla u^m \to |v|^2\nabla u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \end{cases} \tag{3.16}$$

Then, there exists a subsequences of u^m and v^m, which we will denote by u^m, v^m respectively, such that

$$\begin{cases} u^m|\nabla v^m|^2 \to u|\nabla v|^2 \quad \text{almoust everywhere in} \quad (0,T)\times\Omega, \\ v^m|\nabla u^m|^2 \to v|\nabla u|^2 \quad \text{almoust everywhere in} \quad (0,T)\times\Omega, \\ |u^m|^2\nabla v^m \to |u|^2\nabla v \quad \text{almoust everywhere in} \quad (0,T)\times\Omega, \\ |v^m|^2\nabla u^m \to |v|^2\nabla u \quad \text{almoust everywhere in} \quad (0,T)\times\Omega. \end{cases} \tag{3.17}$$

From Lemma (3.15) in ([11]) and (3.17) we deduce

$$\begin{cases} u^m|\nabla v^m|^2 \to u|\nabla v|^2 & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)), \\ v^m|\nabla u^m|^2 \to v|\nabla u|^2 & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)), \\ |u^m|^2\nabla v^m \to |u|^2\nabla v & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)), \\ |v^m|^2\nabla u^m \to |v|^2\nabla u & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{3.18}$$

By the last formula (3.18) and (3.14) we get

$$\begin{aligned} \chi_1 &= u|\nabla v|^2, \\ \chi_2 &= v|\nabla u|^2, \\ \chi_3 &= |v|^2\nabla u, \\ \chi_4 &= |u|^2\nabla v. \end{aligned} \tag{3.19}$$

Taking $w_i = 1$ in (3.1) become

$$(u^m_{tt}, 1) - \alpha(u^m_t, 1) + (u^m|\nabla v^m|^2, 1) = 0 \tag{3.20}$$

$$|(u^m_{tt}, 1)| = \left|\alpha(u^m_t, 1) - (u|\nabla v^m|^2, 1)\right|.$$

Using the Cauchy Schwartz inequality, we have

$$\|u^m_{tt}\|_{L^1(\Omega)} \le |\alpha||u^m_t|_2 m^{\frac{1}{2}}(\Omega) + |u^m|\nabla v^m|^2|_2 m^{\frac{1}{2}}(\Omega),$$

such that, $m(\Omega)$ is a measure of Ω.

Since, the measure of Ω is finite, and (3.14), we obtain

$$\|u^m_{tt}\|_{L^1(\Omega)} \le C_1. \tag{3.21}$$

Analogously

$$\|v^m_{tt}\|_{L^1(\Omega)} \le C_2. \tag{3.22}$$

Then

$$\begin{cases} u^m_{tt} \quad \text{is bounded in} \quad L^\infty(0,T;L^1(\Omega)), \\ v^m_{tt} \quad \text{is bounded in} \quad L^\infty(0,T;L^1(\Omega)). \end{cases} \tag{3.23}$$

Similarly we have

$$\begin{cases} u^m_{tt} \to u_{tt} \quad \text{weakly star in} \quad L^\infty(0,T;L^1(\Omega)), \\ v^m_{tt} \to v_{tt} \quad \text{weakly star in} \quad L^\infty(0,T;L^1(\Omega)). \end{cases} \tag{3.24}$$

From (3.14), (3.24) and lemma (3.1.7) in ([12]) with $B = L^2(\Omega)$ and $B = L^1(\Omega)$ we get

$$\begin{cases} u^m_0 \to u(0), \quad v^m_0 \to v(0) \quad \text{weakly star in} \quad L^2(\Omega), \\ u^m_1 \to u_1(0), \quad v^m_1 \to v_1(0) \quad \text{weakly star in} \quad L^1(\Omega). \end{cases} \tag{3.25}$$

From (3.25) and (3.3) we get

$$u(0) = u_0, \quad v(0) = v_0, \tag{3.26}$$

$$u_1(0) = u_1, \quad v_1(0) = v_1. \tag{3.27}$$

Setting up $m \to \infty$ and passing to the limit in (3.1), (3.2), we obtained

$$\int_\Omega \{u_{tt} - \Delta u - \alpha u_t\} w_i dx - \int_0^t k(t-s) \langle \nabla u(s), \nabla w_i \rangle ds \tag{3.28}$$
$$+ \int_\Omega |v|^2 \nabla u \nabla w_i dx + \int_\Omega u|\nabla v| w_i dx = 0,$$

$$\int_\Omega \{v_{tt} - \Delta v - \beta v_t\} w_i dx - \int_0^t l(t-s) \langle \nabla v(s), \nabla w_i \rangle ds \tag{3.29}$$
$$+ \int_\Omega v|\nabla u| w_i dx + \int_\Omega |u|^2 \nabla v \nabla w_i = 0.$$

$i = 1..., m$. since $(wi)_{i=1}^\infty$ is a base of $H_0^1(\Omega)$, we deduce that (u, v) satisfies (1.1).
The proof is complete.

Lemma 3.1. *Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ be given. Assume that (2.3) and (2.4) are true. Then the problem's local solution (1.1)-(1.5) is global in time.*

proof. Since the map $t \to E(t)$ is a non-increasing function, i.e there exists a positive constant C_1, independent of t, such that
Delete m from these equations

$$C_1 \geq 2E(t) = |u_t|^2_{L^2(\Omega)} + |v_t|^2_{L^2(\Omega)} + (k \circ u)(t) + (l \circ v)(t) \tag{3.30}$$
$$+ \left(1 - \int_0^t k(s)ds\right) |\nabla u|_2^2 + \left(1 - \int_0^t l(s)ds\right) |\nabla v|_2^2 + |v \nabla u|_2^2 + |u \nabla v|_2^2 > 0,$$

which give

$$C_1 \geq 2E(t) \geq |u_t|^2_{L^2(\Omega)} + |v_t|^2_{L^2(\Omega)} + \left(1 - \int_0^t k(s)ds\right) |\nabla u|_2^2 + \left(1 - \int_0^t l(s)ds\right) |\nabla v|_2^2 \tag{3.31}$$
$$+ |v \nabla u|_2^2 + |u \nabla v|_2^2 > 0,$$

consequently, $\forall t \in [0, T]$, we have $|u_t|^2_{L^2(\Omega)} + |v_t|^2_{L^2(\Omega)} + |\nabla u|_2^2 + |\nabla v|_2^2 + |v \nabla u|_2^2 + |u \nabla v|_2^2 \leq C_1$.
This deduces that the solution.

4. Uniqueness

Let (u, v) and (u_1, v_1) two solutions of (1.1), we assume that $U = u - u_1$ and $V = v - v_1$ satisfy

$$U_{tt} - \Delta U - \alpha U_t + k * \Delta U - div(|v|^2\nabla u - |v_1|^2\nabla u_1) + (u|\nabla v|^2 - u_1|\nabla v_1|^2) = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{4.1}$$

$$V_{tt} - \Delta V - \beta V_t + k * \Delta V - div(|u|^2\nabla v - |u_1|^2\nabla v_1) + (v|\nabla u|^2 - v_1|\nabla u_1|^2) = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{4.2}$$

with

$$U(0) = V(0) = 0 \quad U_t(0) = V_t(0) = 0. \tag{4.3}$$

Let as put

$$\begin{aligned} 2E_2(t) &= |U_t|_2^2 + |V_t|_2^2 + (k \circ U)(t) + (l \circ V)(t) \\ &+ \left(1 - \int_0^t k(s)ds\right)|\nabla U|_2^2 + \left(1 - \int_0^t l(s)ds\right)|\nabla V|_2^2. \end{aligned} \tag{4.4}$$

Multiplying (4.1) by $U_t(t)$ and (4.2) by $(V_t(t))$ and summing up the product result we have

$$\begin{aligned} \frac{d}{dt}E_2(t) &\le \int_\Omega div(|v|^2\nabla u - |v_1|^2\nabla u_1)U_t - (u|\nabla v|^2 - u_1|\nabla v_1|^2)U_t dx \\ &+ \int_\Omega div(|u|^2\nabla v - |u_1|^2\nabla v_1)V_t - (v|\nabla u|^2 - v_1|\nabla u_1|^2)V_t dx, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \frac{d}{dt}E_2(t) &\le \int_\Omega \big||v|^2\nabla u - |v_1|^2\nabla u_1\big||\nabla U_t| + \big|u|\nabla v|^2 - u_1|\nabla v_1|^2\big||U_t| dx \\ &+ \int_\Omega \big||u|^2\nabla v - |u_1|^2\nabla v_1\big||\nabla V_t| + \big|v|\nabla u|^2 - v_1|\nabla u_1|^2\big||V_t| dx, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\int_\Omega (|v|^2\nabla u - |v_1|^2\nabla u_1)\nabla U_t + (|u|\nabla v|^2 - u_1|\nabla v_1|^2)U_t dx \\ &= \int_\Omega |v|^2\nabla U\nabla U_t + \nabla u_1\left[|v|^2 - |v_1|^2\right]\nabla U_t + U_t|\nabla v|^2 U + u_1\left[|\nabla v|^2 - |\nabla v_1|^2\right] dx. \end{aligned} \tag{4.7}$$

From the mean value theorem, it follows that

$$\begin{aligned} &\int_\Omega \big||v|^2\nabla u - |v_1|^2\nabla u_1\big||\nabla U_t| + \big|u|\nabla v|^2 - u_1|\nabla v_1|^2\big||U_t| dx \\ &\le \int_\Omega |v|^2|\nabla U||\nabla U_t| + 2|\nabla u_1|^2\left[|v| + |v_1|\right]|\nabla U_t||V| + |\nabla v|^2|U_t||U|. \end{aligned}$$

Working in the same way as in argument of Lemma (2.2) in ([2]) there exists $C > 0$ such that

$$\int_\Omega \big||v|^2\nabla u - |v_1|^2\nabla u_1\big||\nabla U_t| + \big||u|\nabla v|^2 - u_1|\nabla v_1|^2\big||U_t| dx \le C\left\{|\nabla U|_2^2 + |\nabla V|_2^2 + |U_t|_2^2\right\}.$$

Analogously we have

$$\int_\Omega \big||u|^2\nabla v - |u_1|^2\nabla v_1\big||\nabla V_t| + \big||v|\nabla u|^2 - v_1|\nabla u_1|^2\big||V_t| dx \le C\left\{|\nabla V|_2^2 + |\nabla U|_2^2 + |V_t|_2^2\right\},$$

and from (4.6) we have

$$\frac{d}{dt}E_2(t) \leq C\left\{|\nabla U|_2^2 + |\nabla V|_2^2 + |U_t|_2^2 + |V_t|_2^2\right\} \tag{4.8}$$

$$\frac{d}{dt}E_2(t) \leq CE_2(t). \tag{4.9}$$

Then, by using Gronwall's lemma (1.3) in ([4]) we get

$$|\nabla U|_2^2 = |\nabla V|_2^2 = |U_t|_2^2 = |V_t|_2^2 = 0. \tag{4.10}$$

This proves the uniqueness of the solution.

5. Stability

Theorem 5.1. *Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ be given. Assume that (2.3) and (2.4) hold. Then there exists two positive constants μ_1 and μ_2 independent of t such that $0 < E(t) \leq \mu_1 e^{-\mu_2 t}, \forall t \geq 0$.*

Proof. We define the function of Laypunov, for $\epsilon > 0$ as follows

$$L(t) = E(t) + \epsilon \int_\Omega u_t u + v_t v \, dx. \tag{5.1}$$

We prove that $L(t)$ and $E(t)$ are equivalent, meaning that there exist two positive constants N and M depending on ϵ such that for $t \geq 0$

$$NE(t) \leq L(t) \leq ME(t). \tag{5.2}$$

From the Lemma (2.2), we have

$$L(t) \leq E(t) + \epsilon\left[\frac{1}{2\delta}|u_t|_2^2 + \delta|u|_2^2\right] + \epsilon\left[\frac{1}{2\delta}|v_t|_2^2 + \delta|v|_2^2\right].$$

By using the Poincaré inequality, we get

$$L(t) \leq E(t) + \epsilon\left[\frac{1}{2\delta}|u_t|_2^2 + \delta C_1|\nabla u|_2^2\right] + \epsilon\left[\frac{1}{2\delta}|v_t|_2^2 + \delta C_2|\nabla v|_2^2\right].$$

From (3.31) we have

$$L(t) \leq E(t) + \epsilon\left[\frac{1}{\delta}E(t) + 2\delta\frac{C_1}{k_1}E(t)\right] + \epsilon\left[\frac{1}{\delta}E(t) + 2\delta\frac{C_2}{l_1}E(t)\right]$$

$$L(t) \leq E(t) + 2\epsilon\frac{1}{\delta}E(t) + 2\epsilon\delta\frac{C_1}{k_1}E(t) + 2\epsilon\delta\frac{C_2}{l_1}E(t)$$

$$L(t) \leq ME(t) \quad \text{such that} \quad M = 1 + 2\epsilon\frac{1}{\delta} + 2\epsilon\delta\frac{C_1}{k_1} + 2\epsilon\delta\frac{C_2}{l_1}.$$

On the other hand, we have

$$\begin{aligned}L(t) &\geq E(t) - \epsilon\left[\frac{1}{2\delta}|u_t|_2^2 + \delta|u|_2^2\right] - \epsilon\left[\frac{1}{2\delta}|v_t|_2^2 + \delta|v|_2^2\right]\\ &\geq E(t) - \epsilon\left[\frac{1}{2\delta}|u_t|_2^2 + \delta C_1|\nabla u|_2^2\right] - \epsilon\left[\frac{1}{2\delta}|v_t|_2^2 + \delta C_2|\nabla v|_2^2\right]\\ &\geq E(t) - \epsilon\left[\frac{1}{\delta}E(t) + 2\delta\frac{C_1}{k_1}E(t)\right] - \epsilon\left[\frac{1}{\delta}E(t) + 2\delta\frac{C_2}{l_1}E(t)\right],\end{aligned}$$

$$L(t) \geq NE(t) \quad \text{such that} \quad N = 1 - 2\epsilon\frac{1}{\delta} - 2\epsilon\delta\frac{C_1}{k_1} - 2\epsilon\delta\frac{C_2}{l_1}.$$

Now we have

$$\frac{d}{dt}L(t) = \frac{d}{dt}E(t) + \epsilon\int_\Omega\left[u_t^2 + u_{tt}u + v_t^2 + v_{tt}v\right]dx \tag{5.3}$$

$$\begin{aligned}&\epsilon\int_\Omega u_{tt}udx = \epsilon\int_\Omega\left[u.\Delta u + \alpha uu_t - u.k*\Delta u + u.div(|v|^2\nabla u) - u|\nabla v|^2.u\right]dx\\ &\leq \epsilon\left[-|\nabla u|_2^2 + \alpha\frac{1}{2\delta}|u_t|_2^2 + \alpha\delta|u|_2^2 - |v\nabla u|_2^2 - |u\nabla v|_2^2 + \int_\Omega\nabla u\int_0^t k(t-s)\nabla u(s)dsdx\right] \quad (5.4)\\ &\leq \epsilon\left[-|\nabla u|_2^2 + \alpha\frac{1}{2\delta}|u_t|_2^2 + \alpha C_1\delta|\nabla u|_2^2 - |v\nabla u|_2^2 - |u\nabla v|_2^2 + \int_\Omega\nabla u\int_0^t k(t-s)\nabla u(s)dsdx\right].\end{aligned}$$

Analogous

$$\begin{aligned}\epsilon\int_\Omega v_{tt}vdx &= \epsilon\int_\Omega\left[v.\Delta v + \beta vv_t - v.k*\Delta v + v.div(|u|^2\nabla v) - v|\nabla u|^2.v\right]dx \quad (5.5)\\ &\leq \epsilon\left[-|\nabla v|_2^2 + \beta\frac{1}{2\delta}|v_t|_2^2 + \beta C_1\delta|\nabla v|_2^2 - |u\nabla v|_2^2 - |v\nabla u|_2^2 + \int_\Omega\nabla v\int_0^t l(t-s)\nabla v(s)dsdx\right].\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}L(t) \leq{}& \frac{d}{dt}E(t) + \epsilon|u_t|_2^2 + \epsilon|v_t|_2^2\\ &- \epsilon|\nabla u|_2^2 + \epsilon\alpha\frac{1}{2\delta}|u_t|_2^2 + \epsilon\alpha C_1\delta|\nabla u|_2^2 - \epsilon|v\nabla u|_2^2 - \epsilon|u\nabla v|_2^2 + \epsilon\int_\Omega\nabla u\int_0^t k(t-s)\nabla u(s)dsdx\\ &- \epsilon|\nabla v|_2^2 + \epsilon\beta\frac{1}{2\delta}|v_t|_2^2 + \epsilon\beta C_1\delta|\nabla v|_2^2 - \epsilon|u\nabla v|_2^2 - \epsilon|v\nabla u|_2^2 + \epsilon\int_\Omega\nabla v\int_0^t l(t-s)\nabla v(s)ds.dx.\end{aligned} \tag{5.6}$$

The last term of relation (5.6) can be estimated as follow.

$$\begin{aligned}&\left|\int_\Omega\nabla u\int_0^t k(t-s)\nabla u(s)dsdx\right|\\ &\leq \int_\Omega\left(\int_0^t k(t-s)|\nabla u(s) - \nabla u(t)|ds\right)dx + \int_0^t k(s)ds|\nabla u|_2^2 \quad (5.7)\\ &\leq (1+\eta)(1-k_1)|\nabla u|_2^2 + \frac{1}{4\eta}(k\circ\nabla u)(t) \quad \text{for} \quad \eta > 0.\end{aligned}$$

Analogously

$$\begin{aligned}&\left|\int_\Omega \nabla v \int_0^t l(t-s)\nabla v(s)dsdx\right|\\&\leq (1+\eta)(1-l_1)|\nabla v|_2^2+\frac{1}{4\eta}(l\circ\nabla v)(t) \quad \text{for} \quad \eta>0.\end{aligned}$$

So

$$\begin{aligned}\frac{d}{dt}L(t)\leq{}& (k'\circ u)(t)+(l'\circ v)(t)-k(t)|\nabla u|_2^2-l(t)|\nabla v|_2^2+\alpha|u_t|_2^2+\beta|v_t|_2^2+\epsilon|u_t|_2^2+\epsilon|v_t|_2^2\\&-\epsilon|\nabla u|_2^2+\epsilon\alpha\frac{1}{2\delta}|u_t|_2^2+\epsilon\alpha C_1\delta|\nabla u|_2^2-\epsilon|v\nabla u|_2^2-\epsilon|u\nabla v|_2^2+\epsilon(1+\eta)(1-k_1)|\nabla u|_2^2\\&+\epsilon\frac{1}{4\eta}(k\circ\nabla u)(t)-\epsilon|\nabla v|_2^2+\epsilon\beta\frac{1}{2\delta}|v_t|_2^2+\epsilon\beta C_1\delta|\nabla v|_2^2-\epsilon|u\nabla v|_2^2-\epsilon|u\nabla u|_2^2\\&+\epsilon(1+\eta)(1-l_1)|\nabla v|_2^2+\epsilon\frac{1}{4\eta}(l\circ\nabla v)(t),\end{aligned}\tag{5.8}$$

so

$$\begin{aligned}\frac{d}{dt}L(t)\leq{}&\left(\alpha+\epsilon+\epsilon\alpha\frac{1}{2\delta}\right)|u_t|_2^2+\left(\beta+\epsilon+\epsilon\beta\frac{1}{2\delta}\right)|v_t|_2^2\\&+\left(-k(t)-\epsilon+\epsilon\alpha C_1\delta+\epsilon(1+\eta)(1-k_1)\right)|\nabla u|_2^2\\&+(-l(t)-\epsilon+\epsilon\beta C_1\delta+\epsilon(1+\eta)(1-l_1))|\nabla v|_2^2+(-2\epsilon-1)|v\nabla u|_2^2+(-2\epsilon-1)|v\nabla u|_2^2\\&-(k\circ u)(t)-(l\circ v)(t)+|u\nabla v|_2^2+|v\nabla u|_2^2+\epsilon\frac{1}{4\eta}(k\circ\nabla u)(t)+\epsilon\frac{1}{4\eta}(l\circ\nabla v)(t),\end{aligned}\tag{5.9}$$

so

$$\frac{d}{dt}L(t)\leq\gamma E(t)+\lambda,\tag{5.10}$$

We choosing ϵ small enough, such that

$$\begin{aligned}\gamma={}&Min(\alpha+\epsilon+\epsilon\alpha\frac{1}{2\delta};\beta+\epsilon+\epsilon\beta\frac{1}{2\delta};-k(t)-\epsilon+\epsilon\alpha C_1\delta+\epsilon(1+\eta)(1-k_1)\\&-l(t)-\epsilon+\epsilon\beta C_1\delta+\epsilon(1+\eta)(1-l_1);(-2\epsilon-1);-1)<0,\end{aligned}\tag{5.11}$$

and

$$\lambda=|u\nabla v|_2^2+|v\nabla u|_2^2+\epsilon\frac{1}{4\eta}(k\circ\nabla u)(t)+\epsilon\frac{1}{4\eta}(l\circ\nabla v)(t).\tag{5.12}$$

From (5.2), we have

$$\frac{d}{dt}L(t)\leq\frac{\gamma}{M}L(t)+\lambda,\tag{5.13}$$

by integrating the previous differential inequality (5.13) between 0 and t, we obtain the following estimate for the function L

$$L(t)\leq ce^{\frac{\gamma}{M}t}-\frac{\lambda M}{\gamma},\quad\forall t\geq 0,\tag{5.14}$$

by using (5.2), we conclude

$$E(t)\leq c_1e^{\frac{\gamma}{M}t}-\frac{\lambda M}{\gamma N},\quad\forall t\geq 0.\tag{5.15}$$

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R.A. Adams, J.J.F. Fournier, Sobolev spaces, Academic Press, New York (2003).

[2] D. Andrade, A Mognon, Global solutions for a system of Klein-Gordon equations with memory, Bol. Soc. Paran. Mat. Ser. 3, 21 (2003), 127–136.

[3] S. Brrimi, S.A. Messaoudi, Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping, Electron. J. Differ. Equ. 2004 (2004), No. 88, pp. 1–10.

[4] Y. Boukhatem and B. Benabdrrahmane, A. Rahmoune, Mèthode de faedo-Galerkin pour un problème aux limites non linéaire. Anale Universităţii Oradea, Fasc. Math. Tom XVI (2009), 167-181.

[5] C.L. Frota, A. Vicente, A hyperbolic system of Klein-Gordon type with acoustic boundary conditions, Int. J. Pure Appl. Math. 47 (2008), 185-198.

[6] V. Komornik, E. A Zuazua, A direct method for boundary stabilization of the wave equation, J. Math. Pure Appl. 69 (1990), 33-54.

[7] A.T. Louredo, M.M. Miranda, Nonlinear boundary dissipation for a coupled system of Klein-Gordon equations, Electron. J. Differ. Equ. 2010 (2010), No. 120, pp. 1-19.

[8] M.M. Miranda, L.A. Medeiros, On existence of global solutions of a coupled nonlinear Klein-Gordon equation, Funkcialaj Ekvacioj 30 (1987), 147-161.

[9] A. Ouaoua, M. Maouni, A. Khaldi, Exponential decay of solutions with L^p norm for class to semilinear wave equation with damping and source terms, Open J. Math Anal. 4 (2020), 123-131. https://doi.org/10.30538/psrp-oma2020.0071.

[10] K. Zennir, A. Guesmia, Existence of solutions to nonlinear κth-order coupled Klein-Gordon equations with nonlinear sources and memory terms, Appl. Math. E-Notes, 15 (2015), 121-136.

[11] S. Zheng, Nonlinear evolution equations, Chapman & Hall/ CRC Monographs and Surveys in Pure and Applied Mathematics 133, Chapman & Hall/CRC, Boca Raton, (2004).

[12] Y. Zhijian, Initial boundary value problem for a class of non-linear strongly damped wave equations, Math. Meth. Appl. Sci. 26 (2003), 1047–1066. https://doi.org/10.1002/mma.412.