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On Magnetic Curves According to Killing Vector Fields in Euclidean 3-Space

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Abstract. In the geometric theory of space curves, a magnetic field generates magnetic flow. The trajectories of magnetic flow are called magnetic curves. In the present paper, we obtain magnetic curves corresponding to killing magnetic fields in Euclidean 3-space $\mathbb{E}^3$. The magnetic curves of the spherical indicatrices of the tangent, principal normal and binormal for a regular space curve are said to be meant curves. Also, we investigate the magnetic curves of the tangent indicatrix and obtain the trajectories of the magnetic fields called $\mathrm{T_T}$-magnetic, $\mathrm{N_T}$-magnetic and $\mathrm{B_T}$-magnetic curves. Finally, some computational examples in support of our main results are given and plotted.

1. Introduction

The magnetic curves on three dimensional Riemannian manifold (M^3, g) are trajectories of charged particles moving on M^3 under the action of a magnetic field F. Each trajectory γ may be found by solving the Lorentz equation $\nabla_{\gamma'}\gamma' = \phi(\gamma')$, where ϕ is the Lorentz force corresponding to F and ∇ is the Levi Civita connection of g. In particular, the trajectories of (charged) particles moving without the action of a magnetic field are geodesics, which satisfy $\nabla_{\gamma'}\gamma' = 0$ (see for more details [1, 2]). In a three-dimensional space, when a charged particle moves along a regular curve, the tangent, normal and binormal vectors describe the kinematic and geometric properties of this particle. These vectors and the time dimension affect the trajectory of the charged particle during the motion in a magnetic field [3, 4]. Moreover, the study of magnetic curves was extended to other ambient spaces, such as complex space forms [5, 6], Sasakian 3-manifold [7, 8]. Recently, results of classification for the Killing

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magnetic trajectories on two special 3-dimensional manifolds, namely E^3 and $S^2 \times R$, were obtained in [9] and [10], respectively. Barros and Romero proved that if (M^3, g) has constant curvature, then the magnetic curves corresponding to a Killing magnetic field are center lines of Kirchhoff elastic rods [11]. The curves and their frames play an important role in differential geometry and in many branches of science such as mechanics and physics. So, we are interested here in studying some of these curves called magnetic curves, which have many applications in modern physics. In this work, we investigate the trajectories of the magnetic fields called as T_T-magnetic, N_T-magnetic and B_T-magnetic curves and obtain some solutions of the Lorentz force equation. We are looking forward to see that our results will be helpful to researchers who are specialized in mathematical modeling, mechanics and modern physics.

2. Basic concepts

In this section, we list some notions, formulae and conclusions for curves in three-dimensional Euclidean space which can be found in the text books on differential geometry (see for instance [1, 12, 13]). Let $\mathbb{E}^3$ denotes the real vector space with its usual vector structure. We denote by (x_1, x_2, x_3) the coordinates of a vector with respect to the canonical basis of $\mathbb{E}^3$. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, the metric g on $\mathbb{E}^3$ is defined by

$$g(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 + x_3y_3.$$

The norm of $\mathbf{x}$ is given by

$$\|\mathbf{x}\| = \sqrt{g(\mathbf{x}, \mathbf{x})},$$

and the vector product is denoted by

$$\mathbf{x} \times \mathbf{y} = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), (x_1y_2 - x_2y_1)).$$

The sphere of radius $r > 0$ with center at the origin is given by

$$S^2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{E}^3 : g(\mathbf{x}, \mathbf{x}) = r^2\}.$$

Let $\gamma = \gamma(s) : I \subset R \to \mathbb{E}^3$ be an arbitrary curve in $\mathbb{E}^3$, s be the arclength parameter of γ. It is well known that each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields T, N and B called the tangent, the principal normal and the binormal vector fields, respectively [14].

Let $\{T(s), N(s), B(s)\}$ be the moving frame along γ, where these vectors are mutually orthogonal vectors satisfying

$$\langle T(s), T(s)\rangle = \langle N(s), N(s)\rangle = \langle B(s), B(s)\rangle = 1.$$

The Frenet equations for γ are given by [15]

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are called the curvatures of γ.

For spherical images of a regular curve in Euclidean 3-space, we present the following definition:

Definition 2.1. *[16, 17] Let γ be a curve in Euclidean 3-space with Frenet vectors T, N and B. The unit tangent vectors along the curve $\gamma(s)$ generate a curve $\gamma_T = T$ on the sphere of radius 1 about the origin. The curve γ_T is called the spherical indicatrix of T or more commonly, γ_T is called tangent indicatrix of the curve γ. If $\gamma = \gamma(s)$ is a natural representations of the curve γ, then $\gamma_T = T(s)$ will be a representation of γ_T. Similarly, one can consider the principal normal indicatrix $\gamma_N = N(s)$ and binormal indicatrix $\gamma_B = B(s)$.*

Let γ be a curve in $\mathbb{E}^3$ and consider $\gamma_T = T(s)$ as the tangent indicatrix of γ with $\{T_T, N_T, B_T\}$ as its Frenet vectors. Then we have the Frenet formula as follows:

$$\begin{bmatrix} T'_T(s_T) \\ N'_T(s_T) \\ B'_T(s_T) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_T & 0 \\ -\kappa_T & 0 & \tau_T \\ 0 & -\tau_T & 0 \end{bmatrix} \begin{bmatrix} T_T(s_T) \\ N_T(s_T) \\ B_T(s_T) \end{bmatrix},$$

where

$$\begin{cases} T_T = N, \\ N_T = \frac{-1}{\sqrt{1+f^2}} T + \frac{f}{\sqrt{1+f^2}} B, \\ B_T = \frac{f}{\sqrt{1+f^2}} T + \frac{1}{\sqrt{1+f^2}} B, \end{cases}$$

and

$$s_T = \int \kappa(s) ds, \quad \kappa_T = \sqrt{1+f^2}, \quad \tau_T = \sigma \sqrt{1+f^2}, \quad f = \frac{\tau(s)}{\kappa(s)}, \tag{2.2}$$

taking into consideration that

$$\sigma = \frac{f'(s)}{\kappa(s) \left(1 + f^2(s)\right)^{3/2}},$$

is the geodesic curvature of the principal image of the principal normal indicatrix of the curve γ, s_T is a natural representation parameter of the tangent indicatrix of γ and also it is the total curvature of the curve γ and κ_T, τ_T are the curvature and torsion of γ_T. Therefore, we can see that $\frac{\tau_T}{\kappa_T} = \sigma$. Let us introduce the following notions [6, 11, 18].

Definition 2.2. *A magnetic field on a three-dimensional oriented Riemannian manifold (M^3, g) is defined as a closed 2-form F on M^3, related to a skew-symmetric $(1,1)-$tensor field ϕ called the Lorentz force of F, and we have*

$$g(\phi(X), Y) = F(X, Y), \quad \forall\, X, Y \in \chi(M).$$

The magnetic trajectories of F are curves γ on M^3 which satisfy the Lorentz equation:

$$\nabla_{\gamma'}\gamma' = \phi(\gamma').$$

Let V be a Killing vector field on M^3, then the Lorentz force can be written as

$$\phi(X) = V \times X, \tag{2.3}$$

in this case, the Lorentz force equation is given by

$$\nabla_{\gamma'}\gamma' = V \times \gamma'.$$

Note that, for a trivial magnetic field; $F = 0$, the Lorentz equation becomes $\nabla_{\gamma'}\gamma' = 0$ and then the solutions are geodesics.

Proposition 2.1. *Let $\gamma : I \subset R \to M^3$ be a curve in the three-dimensional oriented Riemannian Manifold (M^3, g) and V be a vector field along the curve γ. Then, one can take a variation of γ in the direction of V, say, a map $\Pi : I \times (-\epsilon, \epsilon) \to M^3$ which satisfies*

$$\Pi(s, 0) = \gamma(s), \left(\frac{\partial \Pi}{\partial s}(s, t)\right) = V(s).$$

In this setting, we have the following functions:

1. the speed function $v(s, t) = \left\|\frac{\partial \Pi}{\partial s}(s, t)\right\|$; t is the time dimension,

2. the curvature $\kappa(s, t)$ and the torsion $\tau(s, t)$ are functions of $\gamma(s)$. The variations of these functions at $t = 0$ are given as follows:

$$V(v) = \left(\frac{\partial v}{\partial t}(s, t)\right)\Big|_{t=0} = g(\nabla_T V, T),$$

$$V(\kappa) = \left(\frac{\partial \kappa}{\partial t}(s, t)\right)\Big|_{t=0} = g(\nabla_T^2 V, N) - 2\kappa\, g(\nabla_T V, T) + g(R(V, T)T, N),$$

$$V(\tau) = \left(\frac{\partial \tau}{\partial t}(s, t)\right)\Big|_{t=0} = \left[\frac{1}{\kappa} g(\nabla_T^2 V + R(V, T)T, B)\right]' + g(R(V, T)N, B) + \tau g(\nabla_T V, T) + 2\kappa\, g(\nabla_T V, B),$$

where R is the curvature tensor of M^3.

Corollary 2.1. *Let $V(s)$ be a restriction to $\gamma(s)$ of a Killing vector field V of M^3, then*

$$V(v) = V(\kappa) = V(\tau) = 0.$$

3. Magnetic curves of the tangent indicatrix

Definition 3.1. *[11, 18] Let $\gamma_T : I \to S^2 \subset \mathbb{E}^3$ be a tangent indicatrix of a regular curve γ in three-dimensional Euclidean space $\mathbb{E}^3$, and F be a magnetic field on M^3, then the curve γ_T is*

(i) T_T-magnetic curve if T_T satisfies the Lorentz force equation, $\nabla_{T_T} T_T = \phi(T_T) = V \times T_T$,

(ii) N_T-magnetic curve if N_T satisfies the Lorentz force equation, $\nabla_{T_T} N_T = \phi(N_T) = V \times N_T$,

(iii) B_T-magnetic curve if B_T satisfies the Lorentz force equation, $\nabla_{T_T} B_T = \phi(B_T) = V \times B_T$.

In the light of this definition, we can investigate the following.

3.1. T_T-**magnetic curve.**

Proposition 3.1. *Let γ_T be a T_T-magnetic curve in $\mathbb{E}^3$, with the Frenet apparatus $\{T_T, N_T, B_T, \kappa_T, \tau_T\}$. Then, we have the Frenet formula:*

$$\begin{bmatrix} T_T'(s_T) \\ N_T'(s_T) \\ B_T'(s_T) \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{1+f^2} & 0 \\ -\sqrt{1+f^2} & 0 & \sigma\sqrt{1+f^2} \\ 0 & -\sigma\sqrt{1+f^2} & 0 \end{bmatrix} \begin{bmatrix} T_T(s_T) \\ N_T(s_T) \\ B_T(s_T) \end{bmatrix},$$

and the Lorentz force in the Frenet frame can be written as

$$\begin{bmatrix} \phi(T_T) \\ \phi(N_T) \\ \phi(B_T) \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{1+f^2} & 0 \\ -\sqrt{1+f^2} & 0 & \Psi_1 \\ 0 & -\Psi_1 & 0 \end{bmatrix} \begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}. \tag{3.1}$$

where Ψ_1 is a certain function defined by $\Psi_1 = g(\phi(N_T), B_T)$.

Proof. From *Definition* 3.1, one can write

$$\phi(T_T) = \sqrt{1+f^2}\; N_T. \tag{3.2}$$

Since $\phi(N_T) \in span\{T_T, N_T, B_T\}$, we have

$$\phi(N_T) = \lambda_1 T_T + \lambda_2 N_T + \lambda_3 B_T.$$

Use the following equalities:

$$\begin{aligned} g(\phi(N_T), T_T) &= -g(\phi(T_T), N_T) = -\sqrt{1+f^2}, \\ g(\phi(N_T), N_T) &= 0, \quad g(\phi(N_T), B_T) = \Psi_1, \end{aligned}$$

to get

$$\lambda_1 = -\sqrt{1+f^2},\;\; \lambda_2 = 0,\;\; \lambda_3 = \Psi_1.$$

Hence,

$$\phi(N_T) = -\sqrt{1+f^2} T_T + \Psi_1 B_T. \tag{3.3}$$

Similarly, we can easily obtain

$$\phi(B_T) = -\Psi_1 N_T. \tag{3.4}$$

From Eqs. (3.2), (3.3) and (3.4), we get the required result. □

Proposition 3.2. *The curve γ_T is a T_T-magnetic trajectory of a magnetic field F if and only if the vector field V is given by*

$$V = \Psi_1 T_T + \sqrt{1+f^2}\; B_T. \tag{3.5}$$

Proof. Let γ_T be a T_T-magnetic trajectory of a magnetic field F. Then, by using Proposition 3.1 and Eq. (2.3), we can easily have

$$V = \Psi_1 T_T + \sqrt{1+f^2}\, B_T.$$

Conversely, we assume that Eq. (3.5) holds, then we get $V \times T_T = \phi(T_T)$ and so the curve γ_T is a T_T-magnetic curve. □

Theorem 3.1. *Let γ_T be a T_T-magnetic curve and V be a Killing vector field on a space form $(M^3(K), g)$. If γ_T is one of the T_T-magnetic trajectories of $(M^3(K), g, V)$, then its curvatures satisfying the following relations:*

$$\Psi_1 = const.,$$
$$(1+f^2)\left(\frac{\Psi_1}{2} - \sigma\sqrt{1+f^2}\right) = A_1,$$
$$\left(\sqrt{1+f^2}\right)'' + \sigma(1+f^2)\Psi_1 - \sigma^2(1+f^2)^{3/2} + K\sqrt{1+f^2} + \frac{1}{2}(1+f^2)^{3/2} = A_2\sqrt{1+f^2},$$

where K is the curvature of Riemannian space M^3 and A_1, A_2 are constants.

Proof. Let V be a vector field in Riemannian manifold M^3, then V satisfies Eq. (3.5). So, differentiating Eq. (3.5) with respect to s, we get

$$\nabla_T V = \Psi_1' T_T + \sqrt{1+f^2}(\Psi_1 - \sigma\sqrt{1+f^2})N_T + \left(\sqrt{1+f^2}\right)' B_T. \tag{3.6}$$

Since V is a Killing vector then from Corollary 2.1, $V(v) = 0$ and $\nabla_T V$ has no tangential component, i.e., $\Psi_1 = const$. Also, the differentiation of Eq. (3.6) and using Eq. (2.3) lead to

$$\begin{aligned} \nabla_T^2 V &= (1+f^2)(\sigma\sqrt{1+f^2} - \Psi_1)T_T + \left(\left(\sqrt{1+f^2}\right)'' + \sigma(1+f^2)\Psi_1 - \sigma^2(1+f^2)^{3/2}\right) B_T \\ &\quad + \left(\left(\sqrt{1+f^2}\right)'\left(\Psi_1 - 2\sigma\sqrt{1+f^2}\right) - \sqrt{1+f^2}\left(\sigma\sqrt{1+f^2}\right)'\right) N_T. \end{aligned} \tag{3.7}$$

Thus, from Eqs. (3.6) and (3.7) and Corollary 2.1, we have $(V(\sqrt{1+f^2}) = 0)$. So, we get

$$(1+f^2)\left(\frac{\Psi_1}{2} - \sigma\sqrt{1+f^2}\right) + A_1 = 0. \tag{3.8}$$

Similarly, according to Proposition 2.2, when Eqs. (3.6) and (3.7) are considered with the condition $V(\sigma\sqrt{1+f^2}) = 0$, we can easily obtain

$$\left[\frac{1}{\sqrt{1+f^2}}\left(\left(\sqrt{1+f^2}\right)'' + \sigma(1+f^2)\Psi_1 - \sigma^2(1+f^2)^{3/2} + g(R(V,T_T)T_T, B_T)\right)\right]'$$
$$+\sqrt{1+f^2}\left(\sqrt{1+f^2}\right)' = 0.$$

If M^3 has constant curvature K, then

$$g(R(V,T_T)T_T, B_T) = Kg(V, B_T) = K\sqrt{1+f^2},$$

therefore,

$$\left(\sqrt{1+f^2}\right)'' + \sigma(1+f^2)\Psi_1 - \sigma^2(1+f^2)^{3/2} + K\sqrt{1+f^2} + \frac{1}{2}(1+f^2)^{3/2} = A_2\sqrt{1+f^2}. \quad (3.9)$$

Hence, the proof is completed. □

3.2. N_T**-magnetic curve.**

Proposition 3.3. *Let γ_T be a N_T-magnetic curve in $\mathbb{E}^3$ with Frenet apparatus $\{T_T, N_T, B_T, \kappa_T, \tau_T\}$. Then, the Lorentz force in the Frenet frame can be written as*

$$\begin{bmatrix} \phi(T_T) \\ \phi(N_T) \\ \phi(B_T) \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{1+f^2} & \Psi_2 \\ -\sqrt{1+f^2} & 0 & \sigma\sqrt{1+f^2} \\ -\Psi_2 & -\sigma\sqrt{1+f^2} & 0 \end{bmatrix} \begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}, \quad (3.10)$$

where Ψ_2 is a function defined by $\Psi_2 = g(\phi(T_T), B_T)$.

Proof. From *Definition* 3.1, one can write

$$\phi(N_T) = -\sqrt{1+f^2}T_T + \sigma\sqrt{1+f^2}B_T. \quad (3.11)$$

Since $\phi(T_T) \in span\{T_T, N_T, B_T\}$, then we have

$$\phi(T_T) = \mu_1 T_T + \mu_2 N_T + \mu_3 B_T.$$

Using the following equalities:

$$\begin{aligned} g(\phi(T_T), T_T) &= 0, \\ g(\phi(T_T), B_T) &= \Psi_2, \\ g(\phi(T_T), N_T) &= -g(\phi(N_T), T_T) = \sqrt{1+f^2}, \end{aligned}$$

we get

$$\mu_1 = 0,\ \mu_2 = \sqrt{1+f^2},\ \mu_3 = \Psi_2,$$

and therefore,

$$\phi(T_T) = \sqrt{1+f^2}N_T + \Psi_2 B_T. \quad (3.12)$$

Similarly, we can easily obtain that

$$\phi(B_T) = -\Psi_2 T_T - \sigma\sqrt{1+f^2}N_T. \quad (3.13)$$

Hence, from Eqs. (3.11), (3.12) and (3.13), the proof is completed. □

Corollary 3.1. *Let γ_T be a curve in $\mathbb{E}^3$. Then, the curve γ_T is a N_T-magnetic trajectory of a magnetic field F if and only if the vector field V along γ is written as*

$$V = \sigma\sqrt{1+f^2}T_T - \Psi_2 N_T + \sqrt{1+f^2}B_T. \quad (3.14)$$

Proof. The proof is similar to that we have considered in Proposition 3.2. □

Theorem 3.2. *Let γ_T be a N_T-magnetic curve and V be a Killing vector field on a space form $(M^3(K), g)$. If the curve γ_T is one of the N_T-magnetic trajectories of $(M^3(K), g, V)$, then its curvatures satisfying the following relations:*

$$\Psi_2 = \frac{\left(\sigma\sqrt{1+f^2}\right)'}{\sqrt{1+f^2}},$$

$$\Psi_2\sigma^2(1+f^2) - \sigma\sqrt{1+f^2}\left(\sqrt{1+f^2}\right)' - \Psi_2'' = K\Psi_2,$$

$$\left(\sqrt{1+f^2}\right)'' - 2\Psi_2'\sigma\sqrt{1+f^2} - \Psi_2\left(\sigma\sqrt{1+f^2}\right)' + K\sqrt{1+f^2} + \frac{(1+f^2)^{3/2}(1+\sigma)}{2} = A_3\sqrt{1+f^2},$$

where A_3 is a constant.

Proof. Differentiating Eq. (3.14) with respect to s, we get

$$\nabla_T V = \left(\Psi_2\sqrt{1+f^2} + \left(\sigma\sqrt{1+f^2}\right)'\right) T_T - \Psi_2' N_T + \left(\left(\sqrt{1+f^2}\right)' - \Psi_2\sigma\sqrt{1+f^2}\right) B_T. \quad (3.15)$$

Since V is a Killing vector, then from *Proposition* 3.2 ($V(v) = 0$), we have

$$\Psi_2 = \frac{\left(\sigma\sqrt{1+f^2}\right)'}{\sqrt{1+f^2}}.$$

Also, differentiation of Eq. (3.15) together with Eq. (2.2), gives

$$\begin{aligned}\nabla_T^2 V &= \Psi_2'\sqrt{1+f^2}T_T + \left(\Psi_2\sigma^2(1+f^2) - \sigma\sqrt{1+f^2}\left(\sqrt{1+f^2}\right)' - \Psi_2''\right) N_T \\ &+ \left(\left(\sqrt{1+f^2}\right)'' - 2\Psi_2'\sigma\sqrt{1+f^2} - \Psi_2\left(\sigma\sqrt{1+f^2}\right)'\right) B_T. \end{aligned} \quad (3.16)$$

Thus, from Eqs. (3.15) and (3.16) together with Proposition 2.2 ($V(\sqrt{1+f^2}) = 0$), we get

$$\Psi_2\sigma^2(1+f^2) - \sigma\sqrt{1+f^2}\left(\sqrt{1+f^2}\right)' - \Psi_2'' + g(R(V, T_T)T_T, N_T) = 0.$$

If M^3 has a constant curvature K, then

$$g(R(V, T_T)T_T, N_T) = Kg(V, N_T) = -K\Psi_2,$$

and therefore

$$\Psi_2\sigma^2(1+f^2) - \sigma\sqrt{1+f^2}\left(\sqrt{1+f^2}\right)' - \Psi_2'' - K\Psi_2 = 0. \quad (3.17)$$

Using the condition $V(\sigma\sqrt{1+f^2}) = 0$ in Eqs. (3.15) and (3.16), we obtain

$$\begin{aligned}&\left[\frac{1}{\sqrt{1+f^2}}\left(\left(\sqrt{1+f^2}\right)'' - 2\Psi_2'\sigma\sqrt{1+f^2} - \Psi_2\left(\sigma\sqrt{1+f^2}\right)' + K\sqrt{1+f^2}\right)\right]' \\ &+\sqrt{1+f^2}\left(\sqrt{1+f^2}\right)' + \sigma\sqrt{1+f^2}\left(\sigma\sqrt{1+f^2}\right)' = 0. \end{aligned} \quad (3.18)$$

Integrating Eq. (3.18) leads to

$$\left(\sqrt{1+f^2}\right)'' - 2\Psi_2'\sigma\sqrt{1+f^2} - \Psi_2\left(\sigma\sqrt{1+f^2}\right)' + K\sqrt{1+f^2} + \frac{(1+f^2)^{3/2}(1+\sigma)}{2} = A_3\sqrt{1+f^2}. \quad (3.19)$$

Thus, this completes the proof. □

Corollary 3.2. *Let γ_T be a N_T-magnetic curve in Euclidean 3-space with Ψ_2 is zero, then γ_T is a circular helix. Moreover, the axis of the circular helix is the vector field.*

Proof. It is clear from *Theorem* 3.2. □

3.3. B_T**-magnetic curve.**

Proposition 3.4. *Let γ_T be a B_T-magnetic curve in $\mathbb{E}^3$ with Frenet apparatus $\{T_T, N_T, B_T, \kappa_T, \tau_T\}$. Then, the Lorentz force in the Frenet frame can be written as*

$$\begin{bmatrix} \phi(T_T) \\ \phi(N_T) \\ \phi(B_T) \end{bmatrix} = \begin{bmatrix} 0 & \Psi_3 & 0 \\ -\Psi_3 & 0 & \sigma\sqrt{1+f^2} \\ 0 & -\sigma\sqrt{1+f^2} & 0 \end{bmatrix} \begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}. \tag{3.20}$$

where Ψ_3 is given by $\Psi_3 = g(\phi(T_T), N_T)$.

Proof. As we mentioned the above, we can write

$$\begin{aligned} \phi(B_T) &= -\sigma\sqrt{1+f^2}N_T, \\ \phi(T_T) &= \upsilon_1 T_T + \upsilon_2 N_T + \upsilon_3 B_T. \end{aligned} \tag{3.21}$$

Using the following conditions:

$$\begin{aligned} g(\phi(T_T), T_T) &= 0, \\ g(\phi(T_T), N_T) &= \Psi_3, \\ g(\phi(T_T), B_T) &= -g(\phi(B_T), T_T) = 0, \end{aligned}$$

we can obtain

$$\mu_1 = 0, \ \mu_2 = \Psi_3, \ \mu_3 = 0.$$

From this, we get

$$\phi(T_T) = \Psi_3 N_T. \tag{3.22}$$

Also, we obtain

$$\phi(N_T) = -\Psi_3 T_T + \sigma\sqrt{1+f^2}B_T. \tag{3.23}$$

Therefore, the proof is completed. □

Corollary 3.3. *Let γ_T be a curve in $\mathbb{E}^3$. The curve γ_T is a B_T-magnetic trajectory of a magnetic field F if and only if the vector field V along γ is written as*

$$V = \sigma\sqrt{1+f^2}T_T + \Psi_3 B_T. \tag{3.24}$$

Theorem 3.3. *Let γ_T be a B_T-magnetic curve and V be a Killing vector field on a space form $(M^3(K), g)$. If the curve γ_T is one of the B_T-magnetic trajectories of $(M^3(K), g, V)$, then its curvatures satisfying the following relations:*

$$\sigma\sqrt{1+f^2} = const.,$$

$$\Psi_3' = \frac{1}{2}\left(\sqrt{1+f^2}\right)',$$

$$\Psi_3'' + \sigma^2(1+f^2)\left(\sqrt{1+f^2} - \Psi_3\right) + K\Psi_3 + \frac{(1+f^2)^{3/2}}{4} = A_4\sqrt{1+f^2}\ ;\ A_4 \text{ is constant.}$$

Proof. Since V is a vector field, differentiating Eq. (3.24) with respect to s, we get

$$\nabla_T V = \left(\sigma\sqrt{1+f^2}\right)' T_T + \sigma\sqrt{1+f^2}\left(\sqrt{1+f^2} - \Psi_3\right) N_T + \Psi_3' B_T. \tag{3.25}$$

Since V is a Killing vector, then we have

$$\sigma\sqrt{1+f^2} = const. \tag{3.26}$$

Again, differentiating Eq. (3.25) and using Eq. (2.2), we get

$$\begin{aligned} \nabla_T^2 V &= -\sigma(1+f^2)\left(\sqrt{1+f^2} - \Psi_3\right) T_T + \sigma\sqrt{1+f^2}\left(\left(\sqrt{1+f^2}\right)' - 2\Psi_3'\right) N_T \\ &\quad + \left(\Psi_3'' + \sigma^2(1+f^2)\left(\sqrt{1+f^2} - \Psi_3\right)\right) B_T, \end{aligned} \tag{3.27}$$

which leads to

$$\Psi_3' = \frac{1}{2}\left(\sqrt{1+f^2}\right)'. \tag{3.28}$$

Similarly, using the condition $V(\sigma\sqrt{1+f^2}) = 0$ in Eqs. (3.25) and (3.27), we obtain

$$\begin{aligned} &\left[\frac{1}{\sqrt{1+f^2}}\left(\Psi_3'' + \sigma^2(1+f^2)\left(\sqrt{1+f^2} - \Psi_3\right) + g(R(V, T_T)T_T, B_T)\right)\right]' \\ &\quad + \Psi_3'\sqrt{1+f^2} = 0. \end{aligned} \tag{3.29}$$

If $K = const.$, then we have

$$g(R(V, T_T)T_T, B_T) = Kg(V, B_T) = K\Psi_3',$$

and therefore

$$\Psi_3'' + \sigma^2(1+f^2)\left(\sqrt{1+f^2} - \Psi_3\right) + K\Psi_3' + \frac{(1+f^2)^{3/2}}{4} = A_4\sqrt{1+f^2}, \tag{3.30}$$

thus, this completes the proof. □

Corollary 3.4. *Let γ_T be a B_T-magnetic curve in Euclidean 3-space with Ψ_3 constant, then γ_T is a circular helix. Moreover, the axis of the circular helix is the vector field.*

Proof. It is obvious from Eq. (3.26) and Eq. (3.28). □

Using Eq. (3.30), we obtain the following second-order nonlinear ordinary differential equation

$$u''(s)+\sigma^2(1+f^2)u(s)+Ku'(s)+2u^3(s)-2A_4u(s)=0,\quad u(s)=\frac{1}{2}\sqrt{1+f^2};\ K\ \ and\ \ \sigma\sqrt{1+f^2}=const.$$

Now, we can consider the above differential equation in Euclidean 3- space $\mathbb{E}^3$, in 3- sphere $\mathbb{S}^3$ and in hyperbolic 3- space $\mathbb{H}^3$, respectively.

Case 3.1. Euclidean 3- space $\mathbb{E}^3$ $(K=0,\ \sigma\sqrt{1+f^2}=3)$:

$$u''(s)+2u^3(s)+7u(s)=0,$$

A sample of individual solutions for this equation is given in the following figures:

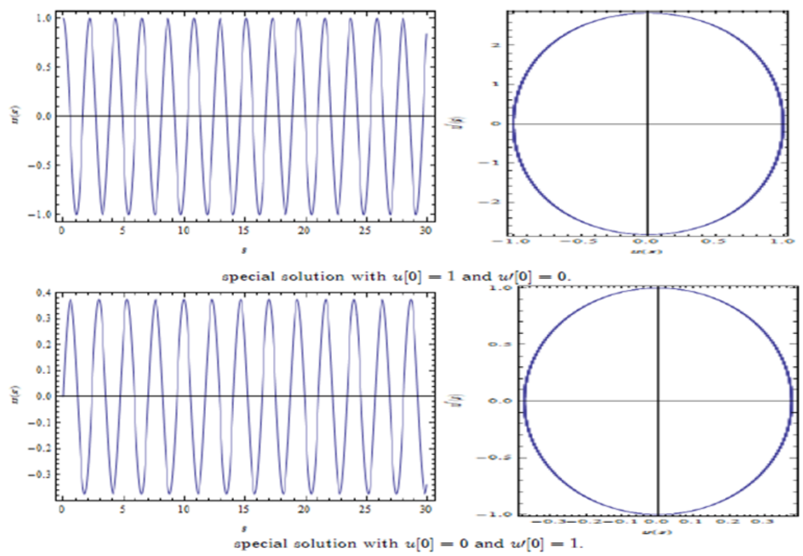


Figure 1

Sample solution family:

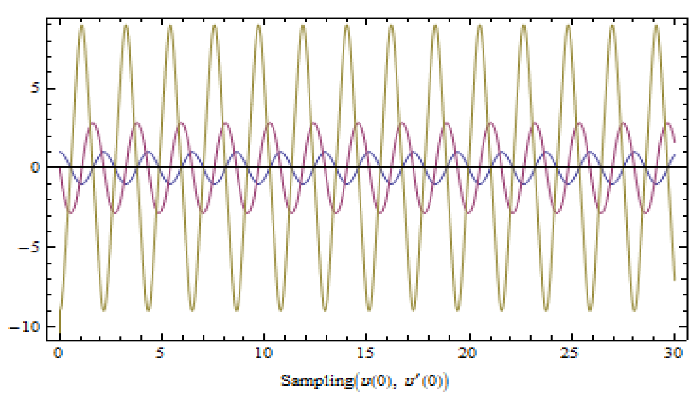


Figure 2. Trajectories of the curvature κ_T of B-magnetic curve in Euclidean 3-space.

Case 3.2. 3-sphere $\mathbb{S}^3$ ($K = 1,\ \sigma\sqrt{1+f^2} = 3$)*:*

$$u''(s) + u'(s) + 2u^3(s) + 7u(s) = 0,$$

A sample of individual solutions for this equation is given in the following figures:

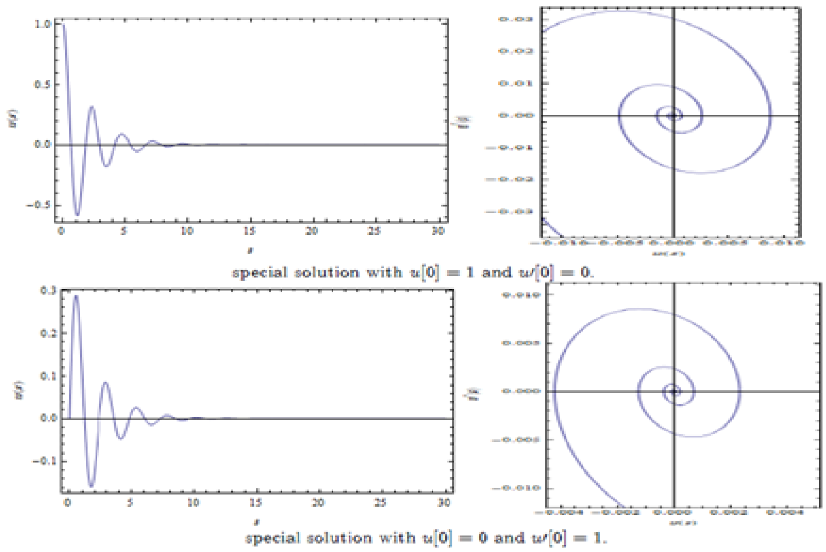


Figure 3.

Sample solution family:

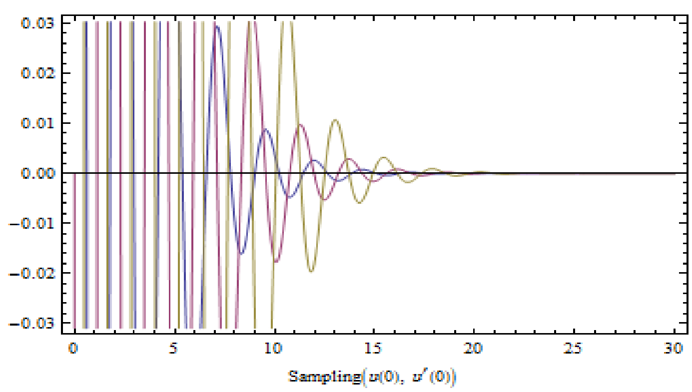


Figure 4. Trajectories of the curvature κ_T of B-magnetic curve in 3-sphere.

Case 3.3. 3- hyperbolic space $\mathbb{H}^3$($K = -1,\ \sigma\sqrt{1+f^2} = 3$)*:*

$$u''(s) - u'(s) + 2u^3(s) + 7u(s) = 0, \quad K = -1, \quad \sigma\sqrt{1+f^2} = 3,$$

A sample of individual solutions for this equation is given in the following figures:

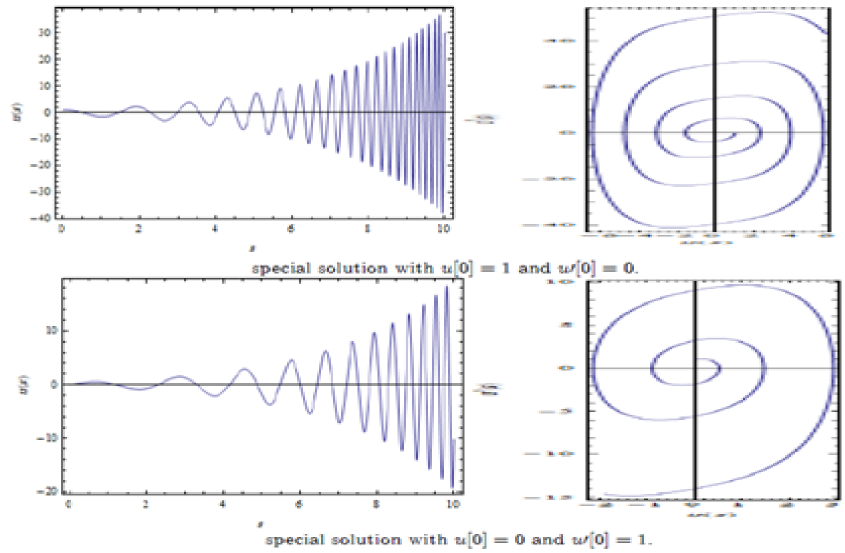


Figure 5.

Sample solution family:

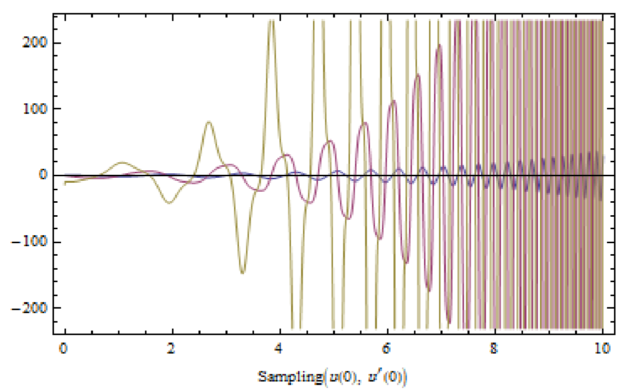


Figure 6. Trajectories of the curvature κ_T of B-magnetic curve in Hyperbolic 3-space.

Remark 3.1. *According to the study that we have considered in the case of magnetic curves of the tangent indicatrix of* γ*, we can do similar study for the other spherical indicatrices, the principal normal indicatrix and the binormal indicatrix.*

4. Applications

In what follows, we give two computational examples to illustrate our main results.

Example 4.1. *Let* $\alpha : I \to \mathbb{E}^3$ *be a regular curve in the three-dimensional Euclidean space* $\mathbb{E}^3$*, can be written as*

$$\alpha = \left(\frac{s}{2}\cos[\ln[\frac{s}{2}]], \frac{s}{2}\sin[\ln[\frac{s}{2}]], \frac{s}{\sqrt{2}}\right),$$

taking the first derivative of the curve α we get

$$T(s) = \left(\frac{1}{2}\left(\cos[\ln[\frac{s}{2}]] - \sin[\ln[\frac{s}{2}]]\right), \frac{1}{2}\left(\cos[\ln[\frac{s}{2}]] + \sin[\ln[\frac{s}{2}]]\right), \frac{1}{\sqrt{2}}\right).$$

Also, we can get the principal normal and binormal vectors of α respectively,

$$\begin{aligned} N(s) &= \left(-\frac{\cos[\ln[\frac{s}{2}]] + \sin[\ln[\frac{s}{2}]]}{\sqrt{2}}, \frac{\cos[\ln[\frac{s}{2}]] - \sin[\ln[\frac{s}{2}]]}{\sqrt{2}}, 0\right), \\ B(s) &= \left(\frac{1}{2}\left(\sin[\ln[\frac{s}{2}]] - \cos[\ln[\frac{s}{2}]]\right), \frac{1}{2}\left(-\sin[\ln[\frac{s}{2}]] - \cos[\ln[\frac{s}{2}]]\right), \frac{1}{\sqrt{2}}\right), \end{aligned}$$

the curvatures of α are

$$\kappa(s) = \tau(s) = \frac{1}{\sqrt{2}s}.$$

It is clear that α is a general helix. The tangent indicatrix of α is obtained as follows

$$\alpha_T = \left(\frac{1}{2}\left(\cos[\ln[\frac{s}{2}]] - \sin[\ln[\frac{s}{2}]]\right), \frac{1}{2}\left(\cos[\ln[\frac{s}{2}]] + \sin[\ln[\frac{s}{2}]]\right), \frac{1}{\sqrt{2}}\right),$$

From direct calculations, we can get the Frenet vectors of α_T

$$\begin{aligned} T_T(s_T) &= \left(-\frac{\cos[\ln[\frac{s}{2}]] + \sin[\ln[\frac{s}{2}]]}{\sqrt{2}}, \frac{\cos[\ln[\frac{s}{2}]] - \sin[\ln[\frac{s}{2}]]}{\sqrt{2}}, 0\right), \\ N_T(s_T) &= \left(\frac{1}{\sqrt{2}}\left(\sin[\ln[\frac{s}{2}]] - \cos[\ln[\frac{s}{2}]]\right), \frac{1}{\sqrt{2}}\left(-\sin[\ln[\frac{s}{2}]] - \cos[\ln[\frac{s}{2}]]\right), 0\right), \\ B_T(s_T) &= (0, 0, 1). \end{aligned}$$

The natural representation and the curvatures of α_T are respectively,

$$s_T = \frac{1}{\sqrt{2}}\ln[s], \quad f = 1, \quad \sigma = 0, \quad \kappa_T = \sqrt{2}, \quad \tau_T = 0,$$

In addition, the certain function of α_T is $\Psi_1 = const.$, it means that α_T is a T_T-magnetic curve.

Example 4.2. *We consider the circular helix γ in Euclidean $3-$ space defined by*

$$\gamma(s) = \left(\cos\left[\frac{s}{\sqrt{2}}\right], \sin\left[\frac{s}{\sqrt{2}}\right], \frac{s}{\sqrt{2}}\right).$$

Differentiating this equation, we get the tangent vector T as follows:

$$T(s) = \left(\frac{-1}{\sqrt{2}}\sin\left[\frac{s}{\sqrt{2}}\right], \frac{1}{\sqrt{2}}\cos\left[\frac{s}{\sqrt{2}}\right], \frac{1}{\sqrt{2}}\right).$$

It follows that, the principal normal and binormal vectors of γ respectively, are given by

$$\begin{aligned} N(s) &= \left(-\cos\left[\frac{s}{\sqrt{2}}\right], -\sin\left[\frac{s}{\sqrt{2}}\right], 0\right), \\ B(s) &= \left(\frac{1}{\sqrt{2}}\sin\left[\frac{s}{\sqrt{2}}\right], \frac{-1}{\sqrt{2}}\cos\left[\frac{s}{\sqrt{2}}\right], \frac{1}{\sqrt{2}}\right), \end{aligned}$$

and so, the curvatures of γ are obtained

$$\kappa(s) = \tau(s) = \frac{1}{2}.$$

From the above calculations, the tangent indicatrix of γ is given as follows

$$\gamma_T(s_T) = \left(\frac{-1}{\sqrt{2}}\sin\left[\frac{s}{\sqrt{2}}\right], \frac{1}{\sqrt{2}}\cos\left[\frac{s}{\sqrt{2}}\right], \frac{1}{\sqrt{2}}\right).$$

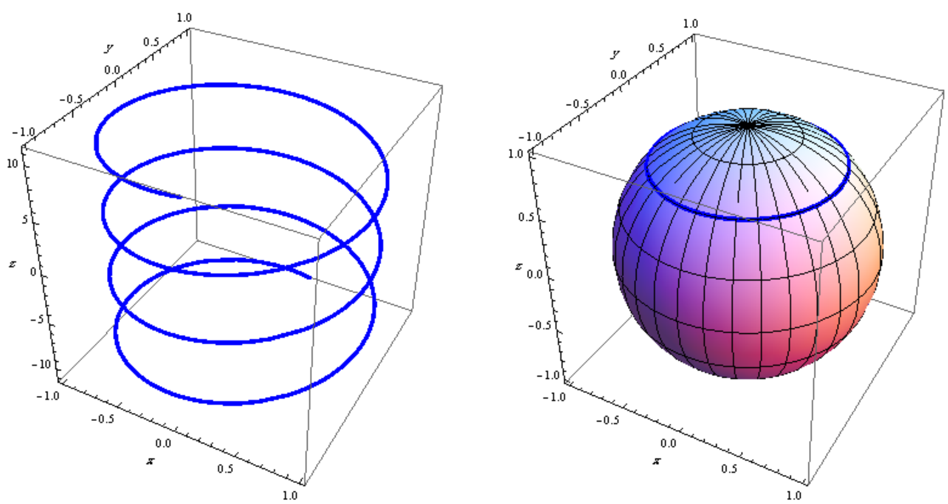
The Frenet vectors of γ_T are given as follows

$$\begin{aligned} T_T(s_T) &= \left(-\cos\left[\frac{s}{\sqrt{2}}\right], -\sin\left[\frac{s}{\sqrt{2}}\right], 0\right), \\ N_T(s_T) &= \left(\sin\left[\frac{s}{\sqrt{2}}\right], -\cos\left[\frac{s}{\sqrt{2}}\right], 0\right), \\ B_T(s_T) &= (0,0,1). \end{aligned}$$

Moreover, the natural representation and the curvature of γ_T are respectively,

$$s_T = \frac{1}{2}s, \quad f = 1, \quad \sigma = 0, \quad \kappa_T = \sqrt{2},$$

In addition, the torsion and the certain function of γ_T are respectively, $\tau_T = 0$ and $\Psi_2 = 0$, it means that γ_T is N_T-magnetic as well as B_T-magnetic curve.



(a) (b)

Figure 7. The circular helix γ and its spherical image γ_T.

5. Conclusion

The value of this paper is due to the important and prominent role of the theory of curves in differential geometry as well as magnetic fields that generate magnetic flow whose trajectories give so-called magnetic curves. In this sense, the idea of this work is devoted to examine some conditions to construct special magnetic curves of spherical images for a regular curve γ in Euclidean 3-space. Some characterizations of magnetic curves of the tangent indicatrix of γ are obtained. An application to confirm our main results is given and plotted.

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