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Some Hermite-Hadamard Inequalities via Generalized Fractional Integral on the Interval-Valued Coordinates

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Abstract. In this paper, we established the Hermite-Hadamard inequalities via generalized fractional. Meanwhile, interval analysis is a particular case of set-interval analysis. We established the fractional inequalities and these results are an extension of a previous research.

1. Introduction

The Hermite-Hadamard inequality, which is the first basic result of convex mappings with a nature geometric interpretation and extensive use, has attracted attention with great interest in elementrary mathematics. The Hermite-Hadamard type inequality, which is defined by:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2},$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the closed bounded interval I of $\mathbb{R}$,and $a, b \in I$ with $a < b$.

In the past decade, fractional calculus has been regarded as one of best tools to describe long-memory processes. Many researchers are interested in such a model. The subject of fractional calculus has gained considerable popularity and importance due mainly to its demonstrated applications in numerous seemly diverse and widespread fields of science and engineering. The most important of these models are described by differential equations with fractional derivatives.

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2. Fractional Integrals

In [28], Sarikaya et al. obtained the following Hermite-Hadamard's inequalities in fractional integral form:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a^{+}}^{\alpha} f(b)+I_{b^{-}}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2},$$

where $g:[a,b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be a positive convex function on $[a,b]$, $g \in L^{1}[a,b]$ with $a<b$, and $I_{a^{+}}^{\alpha}+I_{b^{-}}^{\alpha}$ are the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha>0$, these are respectively defined as []:

$$I_{a^{+}}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)} \int_{a}^{x}(x-t)^{\alpha-1} f(t)\, dt, \quad x>a,$$

and

$$I_{b^{-}}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)} \int_{x}^{b}(t-x)^{\alpha-1} f(t)\, dt, \quad x<b.$$

In [16],Katugampola introduced a new fractional which generalizes the Riemann-Liouville and the Hadamard fractional integrals into a single form as follow.

Definition 2.1

Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integralsod order $\alpha>0$ of $f \in X_{c}^{p}(a,b)$ are defined by

$${}^{\rho} I_{a^{+}}^{\alpha} f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{x} \frac{t^{\rho-1}}{\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}} f(t)\, dt$$

and

$${}^{\rho} I_{b^{-}}^{\alpha} f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{b} \frac{t^{\rho-1}}{\left(t^{\rho}-x^{\rho}\right)^{1-\alpha}} f(t)\, dt$$

where $a<x<b$ and $\rho>0$, if the integral exists.

Theorem 2.2

Let $\alpha>0$ and $\rho>0$. Then for $x>a$,

1. $\lim\limits_{\rho \rightarrow 1} {}^{\rho} I_{a^{+}}^{\alpha} f(x)=J_{a^{+}}^{\alpha} f(x)$,
2. $\lim\limits_{\rho \rightarrow 0^{+}} {}^{\rho} I_{a^{+}}^{\alpha} f(x)=H_{a^{+}}^{\alpha} f(x)$.

Similar results also hold for right-sided operators.

Theorem 2.3

Let $\alpha>0$ and $\rho>0$. Let $f:[a^{\rho}, b^{\rho}] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a \leq b$ and $f \in X_{c}^{p}(a,b)$. If f is also a convex function on $[a,b]$, then the following inequalities hold:

$$f\left(\frac{a^{\rho}+b^{\rho}}{2}, \frac{c^{\rho}+d^{\rho}}{2}\right) \leq \frac{\rho^{\alpha} \Gamma(\alpha+1)}{2\left(b^{\rho}-a^{\rho}\right)^{\alpha}}\left[{}^{\rho} I_{a^{+}}^{\alpha} f\left(b^{\rho}\right)+{}^{\rho} I_{b^{-}}^{\alpha} f\left(a^{\rho}\right)\right] \leq \frac{f\left(a^{\rho}\right)+f\left(b^{\rho}\right)}{2}$$

where the fractional integral are considered for the function $f(x^\rho)$ and evaluated at a and b, respectively.

In [31], Sarikaya and Ertuğral gave the definition of generalized fractional integrals (GFIs) as following:

Definition 2.4

The left-sided and right-sided GFIs are denoted by ${}_{a^+}I_\varphi$ and ${}_{b^-}I_\varphi$ as followings:

$$ {}_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} dt, \quad x > a, $$

and

$$ {}_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} dt, \quad x < b, $$

where a function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the condition $\int_0^1 \frac{\varphi(t)}{t} dt < \infty$.

In [31], Sarikaya et al. obtained the following Hermite-Hadamard's inequalities for GFIs under the condition of convexity as follows:

Theorem 2.5

For a convex function $f : [a,b] \to \mathbb{R}$ on $[a,b]$ with $a < b$, then the following inequalities hold:

$$ f\left(\frac{a+b}{2}\right) \leq \frac{1}{\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) +{}_{b^-} I_\varphi f(a)\right] \leq \frac{f(a)+f(b)}{2}, $$

where $\Lambda(x) = \int_0^x \frac{\varphi((b-a)t)}{t} dt < \infty$.

The most important feature of generalized fractional integrals is that they generalize some type of fractional integrals such as the Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional, and Hadamard feractional integrals. These important special cases of integral operator are mentioned below.

(1) If we choose $\varphi(x) = x$, the operators ${}_{a^+}I_\varphi f(x)$ and ${}_{b^-}I_\varphi f(x)$ are reduce to the Riemann integral.

(2) Considering $\varphi(x) = \frac{x^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$, the operators ${}_{a^+}I_\varphi f(x)$ and ${}_{b^-}I_\varphi f(x)$ are reduce to the Riemann-Liouville fractioal integrals $I_{a^+}^\alpha f(x)$ and $I_{b^-}^\alpha f(x)$, respectively. Here, Γ is a gamma function.

(3) For $\varphi(x) = \frac{1}{k\Gamma(\alpha)} x^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators ${}_{a^+}I_\varphi f(x)$ and ${}_{b^-}I_\varphi f(x)$ are reduce to the k-Riemann-Liouville fractional integrals $I_{a^+,k}^\alpha f(x)$ and $I_{b^-,k}^\alpha f(x)$, respectively. Here, Γ_k is a k-gamma function.

On the other hand, interval analysis is a particular case of set-valued analysis which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. An old example of interval enclosure is Archimede's method which is related to the computation of the circumference of circle. In 1966, the first book related to interval analysis was given by Moore who is known as the first user of intervals in computational mathematics.

After this book, several scientists started to investigate theory and application of interval arithmetic. Nowadays, because of its application, interval analysis is a useful tool in various areas related to uncertain data. We can see applications in computer graphics, experimental and computational physics, error analysis, robotics and many others.

3. Interval Calculus

A real valued interval X is bounded, closed subset of $\mathbb{R}$ and is defined by

$$X = \left[\underline{X}, \overline{X}\right] = \left\{t \in \mathbb{R} : \underline{X} \leq t \leq \overline{X}\right\}$$

where $\underline{X}, \overline{X} \in \mathbb{R}$ and $\underline{X} \leq \overline{X}$. The number $\underline{X}$ and $\overline{X}$ are called the left and right endpoints of interval X, respectively. When $\underline{X} = \overline{X} = a$, the interval X is said to be degenerate and we use the form $X = a = [a, a]$. Also we call X positive if $\underline{X} > 0$ or negative if $\overline{X} < 0$. The set of all closed intervals of $\mathbb{R}$, the sets of all closed positive intervals of $\mathbb{R}$ and closed negative intervals of $\mathbb{R}$ is denoted by $\mathbb{R}_I, \mathbb{R}_I^+$ and $\mathbb{R}_I^-$, respectively. The Pompeiu-Hausdorff distance between the intervals X and Y is defined by

$$d\left(X, Y\right) = d\left(\left[\underline{X}, \overline{X}\right], \left[\underline{Y}, \overline{Y}\right]\right) = \max\left\{\left|\underline{X} - \underline{Y}\right|, \left|\overline{X} - \overline{Y}\right|\right\}.$$

It is known that $(\mathbb{R}_I, d)$ is a complete metric space.

Now, we give the definitions of basic interval arithmetic operations for the intervals X and Y as follows:

$$X + Y = \left[\underline{X} + \underline{Y}, \overline{X} + \overline{Y}\right],$$

$$X - Y = \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right],$$

$$X \cdot Y = [\min S, \max S] \text{ where } S = \left\{\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}\right\},$$

$$X/Y = [\min T, \max T] \text{ where } T = \left\{\underline{X}/\underline{Y}, \underline{X}/\overline{Y}, \overline{X}/\underline{Y}, \overline{X}/\overline{Y}\right\} \text{ and } 0 \notin Y.$$

Scalar multiplication of the interval X is defined by

$$\lambda X = \lambda\left[\underline{X}, \overline{X}\right] = \begin{cases} \left[\lambda\underline{X}, \lambda\overline{X}\right], & \lambda > 0, \\ 0, & \lambda = 0, \\ \left[\lambda\overline{X}, \lambda\underline{X}\right], & \lambda < 0, \end{cases}$$

where $\lambda \in \mathbb{R}$.

The opposits of the interval X is

$$-X := (-1)\,X = \left[-\overline{X}, -\underline{X}\right],$$

where $\lambda = -1$.

The subtraction is given by

$$X - Y = X + (-Y) = \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right].$$

In general, $-X$ is not additive inverse for X, i.e. $X - X \neq 0$.

Use of monotonic functions

$$F\,(X) = \left[F\left(\underline{X}\right), F\left(\overline{X}\right)\right].$$

The definitons of operations lead to a number of algebraic properties which allows $\mathbb{R}_I$ to be quasi-linear space. They can be listed as follows

(1)(Associativity of addition) $(X + Y) + Z = X + (Y + Z)$ for all $X, Y, Z \in \mathbb{R}_I$,

(2)(Additivity elemant) $X + 0 = 0 + X = X$ for all $X \in \mathbb{R}_I$,

(3)(Commutativity of addition) $X + Y = Y + X$ for all $X, Y \in \mathbb{R}_I$,

(4)(Cancellation law) $X + Z = Y + Z \Longrightarrow X = Y$ for all $X, Y, Z \in \mathbb{R}_I$,

(5)(Associativity of multiplication) $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$ for all $X, Y, Z \in \mathbb{R}_I$,

(6)(Commutativity of multiplication) $X \cdot Y = Y \cdot X$ for all $X, Y \in \mathbb{R}_I$,

(7)(Unity element) $X \cdot 1 = 1 \cdot X$ for all $X \in \mathbb{R}_I$,

(8)(Associativity law) $\lambda\,(\mu X) = (\lambda\mu)\,X$ for all $X \in \mathbb{R}_I$, and for all $\lambda, \mu \in \mathbb{R}$,

(9)(First distributiviyu law) $\lambda\,(X + Y) = \lambda X + \lambda Y$ for all $X, Y \in \mathbb{R}_I$, and for all $\lambda \in \mathbb{R}$,

(10)(Second distributiviyu law) $(\lambda + \mu)\,X = \lambda X + \mu X$ for all $X \in \mathbb{R}_I$, and for all $\lambda, \mu \in \mathbb{R}$.

But, this law holds in certain cases. If $Y \cdot Z > 0$, then

$$X \cdot Y + Z = X \cdot Y + X \cdot Z.$$

What's more, one of the set property is the inclusion $\subseteq$ that is given by

$$X \subseteq Y \Longleftrightarrow \underline{Y} \leq \underline{X} \text{ and } \overline{X} \leq \overline{Y}.$$

Considering together with arthmetic operations and inclusion, one has the following property which is called inclusion isotone of interval operations:

Let $\odot$ be the addition, multiplication, subtraction or division. If X, Y, Z and T areintervals such that

$$X \subseteq Y \text{ and } Z \subseteq T,$$

then the following relation is valid

$$X \odot Z \subseteq Y \odot T.$$

4. Intgral of Interval-Valued Functions

In this section, the notion of integral is mentioned for interval-valued functions. Before the definition of integral, the necessary concepts will be given as the following:

A function F is said to be an interval-valued function of t on $[a, b]$, if it assigns a nonempty interval to each $t \in [a, b]$,

$$F(t) = \left[\underline{F}(t), \overline{F}(t)\right].$$

A partition of $[a, b]$ is any finite ordered subset P having the form:

$$P : a = t_0 < t_1 < ... < t_n = b.$$

The mesh of a partition P defined by

$$mesh(P) = \max\{t_i - t_{i-1} : i = 1, 2, ..., n\}.$$

We denoted by $P([a, b])$ the set of all partition of $[a, b]$. Let $P(\delta, [a, b])$ be the set of all $P \in P([a, b])$ such that $mesh(P) < \delta$. Choose an arbitrary point ξ_i in interval $[t_{i-1}, t_i]$, $(i = 1, 2, ..., n)$ and let us define the sum

$$S(F, P, \delta) = \sum_{i=1}^{n} F(\xi_i)[t_i - t_{i-1}],$$

where $F : [a, b] \to \mathbb{R}_I$. We call $S(F, P, \delta)$ a Riemann sum of F corresponding to $P \in P(\delta, [a, b])$.

Definition 4.1

A function $F : [a, b] \to \mathbb{R}_I$ is called interval Riemann intrgrable ((IR)-integrable) on $[a, b]$, if there exists $A \in \mathbb{R}_I$ such that, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(S(F, P, \delta), A) < \epsilon$$

for every Riemann sum S of F corresponding to each $P \in P(\delta, [a, b])$ and independent from choice of $\xi_i \in [t_{i-1}, t_i]$ for all $1 \leq i \leq n$. In this case, A is called the (IR)-integral of F on $[a, b]$ and is denoted by

$$A = (IR)\int_a^b F(t)\,dt.$$

The collection of all functions that are (IR)-integrable on $[a, b]$ will be denoted by $IR_{([a,b])}$.

The following theorem gives relation between $(IR)-$integrable and Riemann integrable (R)-integrable.

Theorem 4.2

Let $F:[a,b]\to\mathbb{R}_I$ be an interval-valued function such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$. $F\in IR_{([a,b])}$ if and only if $\underline{F}(t),\overline{F}(t)\in R_{([a,b])}$ and

$$(IR)\int_a^b F(t)\,dt=\left[(R)\int_a^b \underline{F}(t)\,dt,(R)\int_a^b \overline{F}(t)\,dt\right],$$

where $R_{([a,b])}$ denoted the all R-integrable functions.

It is seen easily that, if $F(t)\subseteq G(t)$ for all $t\in[a,b]$, then

$$(IR)\int_a^b F(t)\,dt\subseteq(IR)\int_a^b G(t)\,dt.$$

Furthermore, if $\{t_{i-1},t_i\}_{i=1}^m$ is a δ-fine P_1 of $[a,b]$ and if $\left\{s_{j-1},s_j\right\}_{j=1}^n$ is a δ-fine P_2 of $[c,d]$, then retangles

$$\triangle_{i,j}=[t_{i-1},t_i]\times\left[s_{j-1},s_j\right]$$

are the partition of retangle $\triangle=[a,b]\times[c,d]$ and the point $\left(\xi_i,\eta_j\right)$ are inside the retangles $[t_{i-1},t_i]\times\left[s_{j-1},s_j\right]$. And we denote the set of all δ-fine partition P of $\triangle$ with $P_1\times P_2$, where $P_1\in P(\delta,[a,b])$ and $P_2\in P(\delta,[c,d])$. Let $\triangle A_{i,j}$ be the area retangle $\triangle_{i,j}$, where $1\le i\le m, 1\le j\le n$, choose arbitrary $\left(\xi_i,\eta_j\right)$ and get

$$S(F,P,\delta,\triangle)=\sum_{i=1}^m\sum_{j=1}^n F\left(\xi_i,\eta_j\right)\triangle A_{i,j}.$$

Definition 4.3

A function $F:\triangle\to\mathbb{R}_I$ is called interval double Riemann integrable ((ID)-integrable) on $\triangle=$ $[a,b]\times[c,d]$ with the ID-integral $I=(ID)\iint_\triangle F(t,s)\,dA$, if there exists $I\in\mathbb{R}_I$ such that, for each $\epsilon>0$, there exists $\delta>0$ such that

$$d\left(S(F,P,\delta,\triangle),I\right)<\epsilon$$

for each $P\in P(\delta,\triangle)$. We denote by $IR_{(\triangle)}$ the set of all ID-integrable function on $\triangle$, and by $R_{([a,b])}$, $IR_{([a,b])}$, the set of all R-integrable and IR-integrable functions on $[a,b]$,respectively.

Theorem 4.4

Let $\triangle=[a,b]\times[c,d]$. If $F:\triangle\to\mathbb{R}_I$ is ID-integrable on $\triangle$, then we have

$$(ID)\iint_\triangle F(t,s)\,dA=(IR)\int_a^b(IR)\int_c^d F(s,t)\,dsdt.$$

In [25], Sadowska obtained the following Hermite-Hadamard inequality for interval-valued functions:

Theorem 4.5

Let $F : [a, b] \to \mathbb{R}_I^+$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $F \in IR_{([a,b])}$. Then

$$\frac{F(a) + F(b)}{2} \subseteq \frac{1}{b-a}(IR)\int_a^b F(t)\,dt \subseteq F\left(\frac{a+b}{2}\right).$$

5. Hermite-Hadamard Inequalities for Generalized Fractional on the Interval-Value Coordinates

Throughout this study, we hope to generalize the Hermite-Hadamard inequalities for generalized fractional on the interval-value coordinates. For bievity, we define

$$_{a^+,c^+}I_\varphi f(x,y) = \int_c^y \int_a^x \frac{\varphi(x-t)\,\varphi(y-s)}{(x-t)(y-s)}dtds,$$

$$_{a^+,d^-}I_\varphi f(x,y) = \int_y^d \int_a^x \frac{\varphi(x-t)\,\varphi(s-y)}{(x-t)(s-y)}dtds,$$

$$_{b^-,c^+}I_\varphi f(x,y) = \int_c^y \int_x^b \frac{\varphi(t-x)\,\varphi(y-s)}{(t-x)(y-s)}dtds,$$

and

$$_{b^-,d^-}I_\varphi f(x,y) = \int_y^d \int_x^b \frac{\varphi(t-x)\,\varphi(s-y)}{(t-x)(s-y)}dtds.$$

Theorem 5.1

Let $f : I \times I \to \mathbb{R}$ be an interval-valued convex function such that $f(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ and $a, b, c, d \in I$ with $a < b$ and $c < d$. If $f \in ID_{([a,b]\times[c,d])}$, then the following inequalities for generalized fractional integral hold:

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \supseteq\ & \frac{1}{4\Omega(1,1)}\left[_{a^+,c^+}I_\varphi f(b,d) +_{a^+,d^-} I_\varphi f(b,c) +_{b^-,c^+} I_\varphi f(a,d) +_{b^-,d^-} I_\varphi f(a,c)\right] \\ \supseteq\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}, \end{aligned}$$

where

$\Omega(x,y) = \int_0^x \int_0^y \frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}dtds.$

proof :

For $t \in [0,1]$, let $x = ta + (1-t)b, y = (1-t)a + tb, z = sc + (1-s)d, w = (1-s)c + sd$. Due to the convexity of f,

$$f\left(\frac{x+y}{2},\frac{z+w}{2}\right)\supseteq\frac{f(x,z)+f(x,w)+f(y,z)+f(y,w)}{4}$$

we get

$$\begin{aligned}
&4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\supseteq\ & f(ta+(1-t)b,sc+(1-s)d)+f((1-t)a+tb,sc+(1-s)d)\\
&+f(ta+(1-t)b,(1-s)c+sd)+f((1-t)a+tb,(1-s)c+sd).
\end{aligned}$$

Multiplying both side by $\frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}$, then integrating the resulting inequality with reapect to t over $(0,1]$, we obtain

$$\begin{aligned}
&4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)(ID)\int_0^1\int_0^1\frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}dtds\\
\supseteq\ &(ID)\int_0^1\int_0^1\frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}f(ta+(1-t)b,sc+(1-s)d)\,dtds\\
&+(ID)\int_0^1\int_0^1\frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}f((1-t)a+tb,sc+(1-s)d)\,dtds\\
&+(ID)\int_0^1\int_0^1\frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}f(ta+(1-t)b,(1-s)c+sd)\,dtds\\
&+(ID)\int_0^1\int_0^1\frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}f((1-t)a+tb,(1-s)c+sd)\,dtds\\
=\ &{}_{a^+,c^+}I_\varphi f(b,d)+{}_{a^+,d^-}I_\varphi f(b,c)+{}_{b^-,c^+}I_\varphi f(a,d)+{}_{b^-,d^-}I_\varphi f(a,c).
\end{aligned}$$

And then, since f is convex, for every $t\in[0,1]$, we have

$$\begin{aligned}
&f(ta+(1-t)b,sc+(1-s)d)+f((1-t)a+tb,sc+(1-s)d)\\
&+f(ta+(1-t)b,(1-s)c+sd)+f((1-t)a+tb,(1-s)c+sd)\\
\supseteq\ &tsf(a,c)+t(1-s)f(a,d)+(1-t)(1-s)f(b,c)+(1-t)sf(b,d)\\
&+t(1-s)f(a,c)+tsf(a,d)+(1-t)sf(b,c)+(1-t)(1-s)f(b,d)\\
&+(1-t)sf(a,c)+(1-t)(1-s)f(a,d)+tsf(b,c)+t(1-s)f(b,d)\\
&+(1-t)(1-s)f(a,c)+(1-t)sf(a,d)+t(1-s)f(b,c)+tsf(b,d)\\
=\ &f(a,c)+f(a,d)+f(b,c)+f(b,d).
\end{aligned}$$

Multiplying both side by $\frac{\varphi((b-a)t)\varphi((d-c)s)}{ts}$, then integrating the resulting inequality with reapect to t over $(0,1]$, we obtain

$$\begin{aligned}&{}_{a^+,c^+}I_\varphi f(b,d)+{}_{a^+,d^-}I_\varphi f(b,c)+{}_{b^-,c^+}I_\varphi f(a,d)+{}_{b^-,d^-}I_\varphi f(a,c)\\ \supseteq\ & [f(a,c)+f(a,d)+f(b,c)+f(b,d)]\,(ID)\int_0^1\int_0^1\frac{\varphi((b-a)t)\,\varphi((d-c)s)}{ts}dtds.\end{aligned}$$

Lemma 5.2

Let $f: I\times I\to\mathbb{R}$ be an interval-valued convex function such that $f(t)=\left[\underline{f}(t),\overline{f}(t)\right]$ and $a,b,c,d\in I$ with $a<b$ and $c<d$.If f and $\frac{\partial^2}{\partial t\partial s}f\in ID_{([a,b]\times[c,d])}$ and $\left|\frac{\partial^2}{\partial t\partial s}f\right|$ is convex, then the following equalities for generalized fractional integrals hold:

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\\ &+\frac{1}{4\Omega(1,1)}\left[{}_{a^+,c^+}I_\varphi f(b,d)+{}_{a^+,d^-}I_\varphi f(b,c)+{}_{b^-,c^+}I_\varphi f(a,d)+{}_{b^-,d^-}I_\varphi f(a,c)\right]\\ =\ &\frac{(b-a)(d-c)}{4\Omega(1,1)}\\ &\times(ID)\int_0^1\int_0^1\Omega(t,s)\left[\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,sc+(1-s)d)\right.\\ &\qquad-\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,(1-s)c+sd)\\ &\qquad-\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,sc+(1-s)d)\\ &\qquad\left.+\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,(1-s)c+sd)\right]dsdt\\ &+\frac{\Delta(1)}{4\Omega(1,1)}\left[{}_{a^+}I_\varphi f(b,d)+{}_{a^+}I_\varphi f(b,c)+{}_{b^-}I_\varphi f(a,d)+{}_{b^-}I_\varphi f(a,c)\right]\\ &+\frac{\Lambda(1)}{4\Omega(1,1)}\left[{}_{c^+}I_\varphi f(b,d)+{}_{d^-}I_\varphi f(b,c)+{}_{c^+}I_\varphi f(a,d)+{}_{d^-}I_\varphi f(a,c)\right]\end{aligned}$$

where $\Lambda(t)=(IR)\int_0^t\frac{\varphi((b-a)u)}{u}du$ and $\Delta(s)=(IR)\int_0^s\frac{\varphi((d-c)\lambda)}{\lambda}d\lambda$.

proof :

Here, we apply integration by parts, then we have

$$S_1=\frac{(b-a)(d-c)}{4\Omega(1,1)}(ID)\int_0^1\int_0^1\Lambda(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,sc+(1-s)d)\,dsdt$$

$$S_2=\frac{(b-a)(d-c)}{4\Omega(1,1)}(ID)\int_0^1\int_0^1-\Lambda(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,(1-s)c+sd)\,dsdt$$

$$S_3=\frac{(b-a)(d-c)}{4\Omega(1,1)}(ID)\int_0^1\int_0^1\Lambda(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,sc+(1-s)d)\,dsdt$$

and

$$S_4 = \frac{(b-a)(d-c)}{4\Omega(1,1)}(ID)\int_0^1\int_0^1 -\Lambda(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,(1-s)c+sd)\,dsdt.$$

If we add from S_1 to S_4 and multiply by $(b-a)(d-c)$, we obtain the proof.

Theorem 5.3

Let $f : I \times I \to \mathbb{R}$ be an interval-valued convex function such that $f(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ and $a, b, c, d \in I$ with $a < b$ and $c < d$. If f and $\frac{\partial^2}{\partial t\partial s}f \in ID_{([a,b]\times[c,d])}$ and $\left|\frac{\partial^2}{\partial t\partial s}f\right|$ is convex, then the following inequalities for generalized fractional integral hold:

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&-\frac{1}{4\Omega(1,1)}\left[{}_{a^+,c^+}I_\varphi f(b,d) + {}_{a^+,d^-}I_\varphi f(b,c) + {}_{b^-,c^+}I_\varphi f(a,d) + {}_{b^-,d^-}I_\varphi f(a,c)\right]\Bigg| \\
\supseteq\ & \frac{(b-a)(d-c)}{\Omega(1,1)}(ID)\int_0^1\int_0^1 ts\left|\Omega(t,s)-\Omega(t,1-s)-\Omega(1-t,s)+\Omega(1-t,1-s)\right| dsdt \\
&\qquad\times\left(\frac{\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|}{4}\right) \\
&+\frac{\Delta(1)}{4\Omega(1,1)}\left[{}_{a^+}I_\varphi f(b,d) + {}_{a^+}I_\varphi f(b,c) + {}_{b^-}I_\varphi f(a,d) + {}_{b^-}I_\varphi f(a,c)\right] \\
&+\frac{\Lambda(1)}{4\Omega(1,1)}\left[{}_{c^+}I_\varphi f(b,d) + {}_{d^-}I_\varphi f(b,c) + {}_{c^+}I_\varphi f(a,d) + {}_{d^-}I_\varphi f(a,c)\right].
\end{aligned}$$

proof :

Using Lemma 5.2 and the convexity of $\left|\frac{\partial^2}{\partial t\partial s}f\right|$, then we have

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+\frac{1}{4\Omega(1,1)}\left[{}_{a^+,c^+}I_\varphi f(b,d) + {}_{a^+,d^-}I_\varphi f(b,c) + {}_{b^-,c^+}I_\varphi f(a,d) + {}_{b^-,d^-}I_\varphi f(a,c)\right]\Bigg| \\
\supseteq\ & \frac{(b-a)(d-c)}{4\Omega(1,1)} \\
&\times(ID)\int_0^1\int_0^1 \Big\{\left[\Omega(t,s)-\Omega(t.1-s)-\Omega(1-t,s)+\Omega(1-t,1-s)\right] \\
&\qquad\qquad\times\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b, sc+(1-s)d)\Big\}\, dsdt \\
&+\frac{\Delta(1)}{4\Omega(1,1)}\left[{}_{a^+}I_\varphi f(b,d) + {}_{a^+}I_\varphi f(b,c) + {}_{b^-}I_\varphi f(a,d) + {}_{b^-}I_\varphi f(a,c)\right] \\
&+\frac{\Lambda(1)}{4\Omega(1,1)}\left[{}_{c^+}I_\varphi f(b,d) + {}_{d^-}I_\varphi f(b,c) + {}_{c^+}I_\varphi f(a,d) + {}_{d^-}I_\varphi f(a,c)\right]
\end{aligned}$$

$$
\begin{aligned}
\supseteq \quad & \frac{(b-a)(d-c)}{\Omega(1,1)}(ID)\int_0^1\int_0^1 ts\left|\Lambda(1-t,s)-\Lambda(t,s)+\Lambda(t,1-s)-\Lambda(1-t,1-s)\right| dsdt \\
& \times\left(\frac{\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|}{4}\right) \\
& +\frac{\Delta(1)}{4\Omega(1,1)}\left[{}_{a^+}I_\varphi f(b,d)+{}_{a^+}I_\varphi f(b,c)+{}_{b^-}I_\varphi f(a,d)+{}_{b^-}I_\varphi f(a,c)\right] \\
& +\frac{\Lambda(1)}{4\Omega(1,1)}\left[{}_{c^+}I_\varphi f(b,d)+{}_{d^-}I_\varphi f(b,c)+{}_{c^+}I_\varphi f(a,d)+{}_{d^-}I_\varphi f(a,c)\right].
\end{aligned}
$$

This completes the proof.

Lemma 5.4

Let $f: I\times I\to\mathbb{R}$ be an interval-valued convex function such that $f(t)=\left[\underline{f}(t),\overline{f}(t)\right]$ and $a,b,c,d\in I$ with $a<b$ and $c<d$. If f and $\frac{\partial^2}{\partial t\partial s}f\in ID_{([a,b]\times[c,d])}$ and $\left|\frac{\partial^2}{\partial t\partial s}f\right|$ is convex, then the following equalities for generalized fractional integrals hold:

$$
\begin{aligned}
& 4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-f\left(a,\frac{c+d}{2}\right)-f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},c\right)-f\left(\frac{a+b}{2},d\right) \\
& +2f(a,c)+2f(a,d)+2f(b,c)+2f(b,d) \\
& +\frac{1}{\Omega(1,1)}\left[{}_{b^-,d^-}I_\varphi f(a,c)+{}_{b^-,c^+}I_\varphi f(a,d)+{}_{a^+,d^-}I_\varphi f(b,c)+{}_{a^+,c^+}I_\varphi f(b,d)\right. \\
& \qquad -{}_{b^-}I_\varphi f(a,c)-{}_{b^-}I_\varphi f(a,d)-{}_{a^+}I_\varphi f(b,c)-{}_{a^+}I_\varphi f(b,d) \\
& \qquad \left.-{}_{d^-}I_\varphi f(a,c)-{}_{c^+}I_\varphi f(a,d)-{}_{d^-}I_\varphi f(b,c)-{}_{c^+}I_\varphi f(b,d)\right] \\
= \quad & \frac{1}{\Omega(1,1)}\sum_{k=1}^{16}J_k
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= (ID)\int_0^{\frac12}\int_0^{\frac12}\Lambda_1(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,sc+(1-s)d)\,dsdt, \\
J_2 &= (ID)\int_0^{\frac12}\int_0^{\frac12}-\Lambda_1(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,sc+(1-s)d)\,dsdt, \\
J_3 &= (ID)\int_0^{\frac12}\int_0^{\frac12}-\Lambda_1(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,(1-s)c+sd)\,dsdt, \\
J_4 &= (ID)\int_0^{\frac12}\int_0^{\frac12}\Lambda_1(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,(1-s)c+sd)\,dsdt, \\
J_5 &= (ID)\int_0^{\frac12}\int_{\frac12}^{1}-\Lambda_2(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,sc+(1-s)d)\,dsdt, \\
J_6 &= (ID)\int_0^{\frac12}\int_{\frac12}^{1}\Lambda_2(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,sc+(1-s)d)\,dsdt,
\end{aligned}
$$

$$
\begin{aligned}
J_7 &= (ID)\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 \Lambda_2(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,(1-s)c+sd)\,dsdt,\\
J_8 &= (ID)\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 -\Lambda_2(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,(1-s)c+sd)\,dsdt,\\
J_9 &= (ID)\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} -\Lambda_3(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,sc+(1-s)d)\,dsdt,\\
J_{10} &= (ID)\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} \Lambda_3(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,sc+(1-s)d)\,dsdt,\\
J_{11} &= (ID)\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} \Lambda_3(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,(1-s)c+sd)\,dsdt,\\
J_{12} &= (ID)\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} -\Lambda_3(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,(1-s)c+sd)\,dsdt,\\
J_{13} &= (ID)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 \Lambda_4(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,(1-s)c+sd)\,dsdt,\\
J_{14} &= (ID)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 -\Lambda_4(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,(1-s)c+sd)\,dsdt,\\
J_{15} &= (ID)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 -\Lambda_4(t,s)\frac{\partial^2}{\partial t\partial s}f(ta+(1-t)b,sc+(1-s)d)\,dsdt,\\
J_{16} &= (ID)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 \Lambda_4(t,s)\frac{\partial^2}{\partial t\partial s}f((1-t)a+tb,sc+(1-s)d)\,dsdt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda_1(t,s)=(ID)\int_0^t\int_0^s \tfrac{\varphi((b-a)u)\varphi((d-c)\lambda)}{u\lambda}dud\lambda,\quad \Lambda_2(t,s)=(ID)\int_0^t\int_s^1 \tfrac{\varphi((b-a)u)\varphi((d-c)\lambda)}{u\lambda}dud\lambda,\\
&\Lambda_3(t,s)=(ID)\int_t^1\int_0^s \tfrac{\varphi((b-a)u)\varphi((d-c)\lambda)}{u\lambda}dud\lambda,\quad \Lambda_4(t,s)=(ID)\int_t^1\int_s^1 \tfrac{\varphi((b-a)u)\varphi((d-c)\lambda)}{u\lambda}dud\lambda.
\end{aligned}
$$

proof :

Here, we apply integration by parts, then we completes the proof.

Theorem 5.5

Let $f : I\times I\to\mathbb{R}$ be an interval-valued convex function such that $f(t)=\left[\underline{f}(t),\overline{f}(t)\right]$ and $a,b,c,d\in I$ with $a<b$ and $c<d$. If f and $\frac{\partial^2}{\partial t\partial s}f\in ID_{([a,b]\times[c,d])}$ and $\left|\frac{\partial^2}{\partial t\partial s}f\right|$ is convex, then the following inequalities for generalized fractional integral hold:

$$
\Big|4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-f\left(a,\frac{c+d}{2}\right)-f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},c\right)-f\left(\frac{a+b}{2},d\right)
$$
$$
+2f(a,c)+2f(a,d)+2f(b,c)+2f(b,d)
$$

$$+\frac{1}{\Omega(1,1)}\left[{}_{b^-,d^-}I_\varphi f(a,c)+{}_{b^-,c^+}I_\varphi f(a,d)+{}_{a^+,d^-}I_\varphi f(b,c)+{}_{a^+,c^+}I_\varphi f(b,d)\right.$$
$$-{}_{b^-}I_\varphi f(a,c)-{}_{b^-}I_\varphi f(a,d)-{}_{a^+}I_\varphi f(b,c)-{}_{a^+}I_\varphi f(b,d)$$
$$\left.-{}_{d^-}I_\varphi f(a,c)-{}_{c^+}I_\varphi f(a,d)-{}_{d^-}I_\varphi f(b,c)-{}_{c^+}I_\varphi f(b,d)\right]\Big|$$
$$\supseteq\frac{1}{\Omega(1,1)}\left[\frac{\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|}{4}\right]$$
$$\times\left[(ID)\int_0^{\frac12}\int_0^{\frac12}|\Lambda_1(t,s)|\,dtds+(ID)\int_0^{\frac12}\int_{\frac12}^1|\Lambda_2(t,s)|\,dtds\right.$$
$$\left.+(ID)\int_{\frac12}^1\int_0^{\frac12}|\Lambda_3(t,s)|\,dtds+(ID)\int_{\frac12}^1\int_{\frac12}^1|\Lambda_4(t,s)|\,dtds\right].$$

proof :

Using Lemma 5.4 and the convexity of $\left|\frac{\partial^2}{\partial t\partial s}f\right|$, then we have

$$\left|4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-f\left(a,\frac{c+d}{2}\right)-f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},c\right)-f\left(\frac{a+b}{2},d\right)\right.$$
$$+2f(a,c)+2f(a,d)+2f(b,c)+2f(b,d)$$
$$+\frac{1}{\Omega(1,1)}\left[{}_{b^-,d^-}I_\varphi f(a,c)+{}_{b^-,c^+}I_\varphi f(a,d)+{}_{a^+,d^-}I_\varphi f(b,c)+{}_{a^+,c^+}I_\varphi f(b,d)\right.$$
$$-{}_{b^-}I_\varphi f(a,c)-{}_{b^-}I_\varphi f(a,d)-{}_{a^+}I_\varphi f(b,c)-{}_{a^+}I_\varphi f(b,d)$$
$$\left.-{}_{d^-}I_\varphi f(a,c)-{}_{c^+}I_\varphi f(a,d)-{}_{d^-}I_\varphi f(b,c)-{}_{c^+}I_\varphi f(b,d)\right]\Big|$$
$$\supseteq\frac{1}{\Omega(1,1)}\left\{(ID)\int_0^{\frac12}\int_0^{\frac12}\Lambda_1(t,s)\left[ts\frac{\partial^2}{\partial t\partial s}f(a,c)+t(1-s)\frac{\partial^2}{\partial t\partial s}f(a,d)\right.\right.$$
$$\left.+(1-t)s\frac{\partial^2}{\partial t\partial s}f(b,c)+(1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(b,d)\right]dtds$$
$$+(ID)\int_0^{\frac12}\int_{\frac12}^1\Lambda_2(t,s)\left[-(1-t)s\frac{\partial^2}{\partial t\partial s}f(a,c)-(1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(a,d)\right.$$
$$\left.-ts\frac{\partial^2}{\partial t\partial s}f(b,c)-t(1-s)\frac{\partial^2}{\partial t\partial s}f(b,d)\right]dtds$$
$$+(ID)\int_{\frac12}^1\int_0^{\frac12}\Lambda_3(t,s)\left[-t(1-s)\frac{\partial^2}{\partial t\partial s}f(a,c)-ts\frac{\partial^2}{\partial t\partial s}f(a,d)\right.$$
$$\left.-(1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(b,c)-(1-t)s\frac{\partial^2}{\partial t\partial s}f(b,d)\right]dtds$$
$$+(ID)\int_{\frac12}^1\int_{\frac12}^1\Lambda_4(t,s)\left[(1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(a,c)+(1-t)s\frac{\partial^2}{\partial t\partial s}f(a,d)\right.$$
$$\left.\left.+t(1-s)\frac{\partial^2}{\partial t\partial s}f(b,c)+ts\frac{\partial^2}{\partial t\partial s}f(b,d)\,dtds\right]\right\}$$

$$\supseteq \frac{1}{\Omega(1,1)}\left[\frac{\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|}{4}\right]$$
$$\times\left[(ID)\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}|\Lambda_1(t,s)|\,dtds+(ID)\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}|\Lambda_2(t,s)|\,dtds\right.$$
$$\left.+(ID)\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}|\Lambda_3(t,s)|\,dtds+(ID)\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}|\Lambda_4(t,s)|\right]dtds.$$

Conclusion: In this work, the author established Hermite-Hadamard type inequalities via generalized fractional integral. Furthermore, the author extend the inequalities on interval-valued coordinated.

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