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Generalized Stability Additive λ-Functional Inequalities With $3k$-Variable in α-Homogeneous F-Spaces

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Abstract. In this paper, we study to solve two additive λ-functional inequalities with $3k$-variables in α-homogeneous F spaces. Then we will show that the solutions of the first and second inequalities are additive mappings.

1. Introduction

Let **X** and **Y** be a normed spaces on the same field $\mathbb{K}$, and $f : \mathbf{X} \to \mathbf{Y}$. We use the notation $\|\cdot\|$ for all the norm on both **X** and **Y**. In this paper, we investisgate some additive λ-functional inequality in α-homogeneous F-spaces.

In fact, when **X** is a α_1-homogeneous F-spaces and that **Y** is a α_2-homogeneous F-spaces we solve and prove the Hyers-Ulam-Rassias type stability of two forllowing additive α-functional inequality.

$$\begin{aligned}&\Big\|f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)\Big\|_Y\\&\leq\Big\|\lambda\Big(f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f\big(z_j\big)\Big)\Big\|_Y\end{aligned} \tag{1.1}$$

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and

$$\Big\|f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f\big(z_j\big)\Big\|_Y$$
$$\leq\Big\|\lambda\Big(f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)\Big)\Big\|_Y \qquad (1.2)$$

where λ is a fixed complex number with $\big|\lambda\big|<1$, $\alpha_1,\alpha_2\in\mathbb{R}^+,\alpha_1,\alpha_2\leq 1$ and m is a fixed integer with $m>1$.

The Hyers-Ulam stability was first investigated for functional equation of Ulam in [19] concerning the stability of group homomorphisms.
The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The Hyers [9] gave firts affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers'Theorem was generalized by Aoki [1] additive mappings and by Rassias [18] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The Hyers-Ulam stability for functional inequalities have been investigated such as in [5], [10], [13], [16], [17], [18]. Gilány showed that is if satisfies the functional inequality

$$\big\|2f\big(x\big)+2f\big(y\big)-f\big(x-y\big)\big\|\leq\big\|f\big(x+y\big)\big\| \qquad (1.3)$$

Then f satisfies the Jordan-von Newman functional equation

$$2f\big(x\big)+2f\big(y\big)=f\big(x+y\big)+f\big(x-y\big) \qquad (1.4)$$

. Gilányi [8] and Fechner [5] proved the Hyers-Ulam stability of the functional inequality (1.3).
Next Chookil [16] proved the of additive β-functional inequalities in non-Archimedean Banach spaces and in complex Banach spaces, and Harin Lee[a] [11] proved the Hyers-Ulam stability of additive β-functional inequalities in ρ-homogeneous F space.

Recently, the author has studied the additive inequalities of mathematicians around the world, on spaces complex Banach spaces , non-Archimedan Banach spaces or additive β-functional inequalities in p-homogeneous F-space..
So in this paper, we solve and proved the Hyers-Ulam stability for two **ff**-functional inequalities (1.1)-(1.2), ie the α-functional inequalities with $3k$-variables. Under suitable assumptions on spaces **X** and **Y**, we will prove that the mappings satisfying the α-functional inequatilies (1.1) or (1.2). Thus, the results in this paper are generalization of those in [2], [11] for α-functional inequatilies with

$3k$- variables.

The paper is organized as follows: In section preliminarier we remind a basic property such as We only redefine the solution definition of the equation of the additive function and F^*-space .

Section 3: is devoted to prove the Hyers-Ulam stability of the addive λ- functional inequalities (1.1) when when $\mathbf{X}$ is a α_1-homogeneous F-spaces and that $\mathbf{Y}$ is a α_2-homogeneous F-spaces.
Section 4: is devoted to prove the Hyers-Ulam stability of the addive λ- functional inequalities (1.2) when when $\mathbf{X}$ is a α_1-homogeneous F-spaces and that $\mathbf{Y}$ is a α_2-homogeneous F-spaces.

2. Preliminaries

2.1. F^*- spaces.

Definition 2.1.

Let $\mathbf{X}$ be a $\big($complex$\big)$ linear space. A nonnegative valued function $\Big\|\cdot\Big\|$ is an F-norm if it satisfies the following conditions:

(1) $\Big\|x\Big\| = 0$ if and only if $x = 0$;

(2) $\Big\|\lambda x\Big\| = \Big\|x\Big\|$ for all $x \in X$ and all λ with $\Big|\lambda\Big| = 1$;

(3) $\Big\|x + y\Big\| \leq \Big\|x\Big\| + \Big\|y\Big\|$ for all $x, y \in X$;

(4) $\Big\|\lambda_n x\Big\| \to 0,\ \lambda_n \to 0$;

(5) $\Big\|\lambda_n x\Big\| \to 0,\ x_n \to 0$.

Then $\Bigg(\mathrm{X}, \Big\|\cdot\Big\|\Bigg)$ is called an F^*-space. An F-space is a complete F^*-space. An F-norm is called β-homgeneous $\Big(\beta > 0\Big)$ if $\Big\|tx\Big\| = \Big|t\Big|^{\beta}\Big\|x\Big\|$ for all $x \in \mathbf{X}$ and for all $t \in \mathbb{C}$ and $\Big(\mathrm{X}, \Big\|\cdot\Big\|\Big)$ is called α-homogeneous F-space.

2.2. **Solutions of the inequalities.** The functional equation

$$f(x + y) = f(x) + f(y)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

3. Hyers-Ulam-Rassias stability Additive λ-functional inequalities (1.1) in α-homogeneous F-spaces

Now, we first study the solutions of (1.1). Note that for these inequalities, when **X** is a α_1-homogeneous F-spaces and that **Y** is a α_2-homogeneous F-spaces. Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Lemma 3.1. *Let $m \in \mathbb{N}$ and a mapping $f : \mathbf{X} \to \mathbf{Y}$ satilies*

$$\begin{aligned}&\Big\|f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)\Big\|_Y\\&\leq\Big\|\lambda\Big(f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)-\sum_{j=1}^{k}f\big(z_j\big)\Big)\Big\|_Y\end{aligned}\tag{3.1}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, then $f : \mathbf{X} \to \mathbf{Y}$ is additive

Proof. Assume that $f : \mathbf{G} \to \mathbf{Y}$ satisfies (3.1).
We replacing $\big(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k\big)$ by$\big(0, ..., 0, 0, ..., 0, 0, ..., 0\big)$ in (3.1), we have

$$\Big\|2kf\big(0\big)\Big\|\leq\Big\|\lambda\big(2k-1\big)f\big(0\big)\Big\|_Y\leq 0$$

therefore

$$\Big(\Big|2k\Big|^{\alpha_2}-\Big|\lambda\big(2k-1\big)\Big|^{\alpha_2}\Big)\Big\|f\big(0\big)\Big\|_Y\leq 0$$

So $f\big(0\big) = 0$.
Replacing $\big(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k\big)$ by $\big(0, ..., 0, 0, ..., 0, z, 0, ..., 0)\big)$, in (3.1), we get

$$\Big\|f\big(-z\big)-f\big(-z\big)\Big\|_Y\leq 0$$

and so f is an odd mapping. Replacing $\big(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k\big)$
by$\big(x_1, ..., x_k, y_1, ..., y_k, m\cdot\frac{x_1+y_1}{2k} - v_1, ..., m\cdot\frac{x_k+y_k}{2k} - v_k\big)$ in (3.1), we have

$$\begin{aligned}&\Big\|f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}v_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f\big(v_j\big)\Big\|_Y\\&\leq\Big\|\lambda\Big(f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}v_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-v_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)\Big)\Big\|_Y\end{aligned}\tag{3.2}$$

for all $x_1, ..., x_k, y_1, ..., y_k, m\frac{x_1+y_1}{2k} - v_1, ..., m\frac{x_k+y_k}{2k} - v_k \in \mathbf{G}$. From (3.1) and (3.2) we infer that

$$\begin{aligned}
&\left\| f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f(z_j)\right\|_Y \\
&\le \left\|\lambda\Big(f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)\Big)\right\|_Y \\
&\le \left\|\lambda^2\Big(f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f(z_j)\Big)\right\|_Y
\end{aligned} \tag{3.3}$$

and so

$$f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)=\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)+\sum_{j=1}^{k}f(z_j)$$

for all $x_j, y_j, z_j \in \mathbf{G}$ for $j = 1 \to n$, as we expected. □

Theorem 3.2. *Let $r > \frac{\alpha_2}{\alpha_1}, m \in \mathbb{Z}, m > 1$, θ be nonngative real number, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that*

$$\begin{aligned}
&\left\| f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)\right\|_Y \\
&\le \left\|\lambda\Big(f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f(z_j)\Big)\right\|_Y \\
&+\theta\Big(\sum_{j=1}^{k}\|x_j\|_X^r+\sum_{j=1}^{k}\|y_j\|_X^r+\sum_{j=1}^{k}\|z_j\|_X^r\Big)
\end{aligned} \tag{3.4}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique mapping $\phi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-h(x)\right\|_Y \le \frac{\sum_{q=1}^{m-1}\left(q^{\alpha_1 r}+2k^{\alpha_1 r}\right)}{(1-|\lambda|^{\alpha_2})(m^{\alpha_1 r}-m^{\alpha_2})}\theta\|x\|_X^r. \tag{3.5}$$

for all $x \in \mathbf{X}$

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (3.4).

Replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by$(0, ..., 0, 0, ..., 0, 0, ..., 0)$ in (3.4), we have

$$\left\|2kf(0)\right\|_Y \le \left\|\lambda(2k-1)f(0)\right\|_Y \le 0$$

therefore

$$\left(|2k|^{\alpha_2}-|\lambda(2k-1)|^{\alpha_2}\right)\left\|f(0)\right\|_Y \le 0$$

So$f(0) = 0$.

Next we

replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(kx, 0, ..., 0, kx, 0, ..., 0, 0, ..., 0)$ in (3.4), we get

$$\left\| f\big((m+1)x\big) - f\big(mx\big) - f\big(x\big) \right\|_{Y} \leq 2k^{\alpha_1 r}\theta \left\| x \right\|_{X}^{r} \tag{3.6}$$

for all $x \in \mathbf{X}$. Thus for $q \in \mathbb{N}$,
we replacing $\big(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k\big)$ by$\big(kx, 0, ..., 0, kx, 0, ..., 0, qx, 0, ..., 0\big)$ in (3.4), we have

$$\begin{aligned} &\left\| f\big((m-q+1)x\big) - f\big((m-q)x\big) - f\big(x\big) \right\|_{Y} \\ &\leq \left\| \lambda\big(f\big((q+1)x\big) - f\big(qx\big) - f\big(x\big) \right\|_{Y} + \theta\big(2k^{\alpha_1 r} + q^{\alpha_1 r}\big)\left\| x \right\|_{X}^{r} \end{aligned} \tag{3.7}$$

for all $x \in \mathbf{X}$.
For (3.6) and (3.7)

$$\begin{aligned} &\sum_{q=1}^{m-1} \left\| f\big((m-q+1)x\big) - f\big((m-q)x\big) - f\big(x\big) \right\|_{Y} \\ &\leq \sum_{q=1}^{m-1} \left\| \lambda\big(f\big((q+1)x\big) - f\big(qx\big) - f\big(x\big) \right\|_{Y} + \theta\Big(\sum_{q=1}^{m-1} \big(2k^{\alpha_1 r} + q^{\alpha_1 r}\big)\left\| x \right\|_{X}^{r}\Big) \end{aligned} \tag{3.8}$$

for all $x \in \mathbf{X}$.
From (3.7) and (3.8) and triangle inequality, we have

$$\begin{aligned} &\big(1 - |\lambda|^{\alpha_2}\big)\left\| f\big(mx\big) - mf\big(x\big) \right\|_{Y} \\ &= \big(1 - |\lambda|^{\alpha_2}\big)\sum_{q=1}^{m-1} \left\| f\big((q+1)x\big) - f\big(qx\big) - f\big(x\big) \right\|_{Y} \\ &\leq \sum_{q=1}^{m-1} \big(1 - |\lambda|^{\alpha_2}\big)\left\| \big(f\big((q+1)x\big) - f\big(qx\big) - f\big(x\big) \right\|_{Y} \\ &\leq \sum_{q=1}^{m-1} \left\| f\big((q+1)x\big) - f\big(qx\big) - f\big(x\big) \right\|_{Y} - \sum_{q=1}^{m-1} \left\| \lambda\big(f\big((q+1)x\big) - f\big(qx\big) - f\big(x\big)\big) \right\|_{Y} \\ &\leq \theta\Big(\sum_{q=1}^{m-1} \big(2k^{\alpha_1 r} + q^{\alpha_1 r}\big)\left\| x \right\|_{X}^{r}\Big) \end{aligned} \tag{3.9}$$

for all $x \in \mathbf{X}$. from

$$\sum_{q=1}^{m-1} \left\| f\big((m-q+1)x\big) - f\big((m-q)x\big) - f\big(x\big) \right\|_{Y} = \sum_{q=1}^{m-1} \left\| \big(f\big((q+1)x\big) - f\big(qx\big) - f\big(x\big) \right\|_{Y}$$

Since $|\lambda| < 1$, the mapping f satisfies the inequalities

$$\left\|f(mx) - mf(x)\right\|_Y \leq \frac{\theta\Big(\sum_{q=1}^{m-1}\left(2k^{\alpha_1 r} + q^{\alpha_1 r}\right)\left\|x\right\|_X^r\Big)}{\left(1 - |\lambda|^{\alpha_2}\right)}$$

for all $x \in \mathbf{X}$.

Therefore

$$\left\|f(x) - mf\left(\frac{x}{m}\right)\right\|_Y \leq \frac{\theta\Big(\sum_{q=1}^{m-1}\left(2k^{\alpha_1 r} + q^{\alpha_1 r}\right)\left\|x\right\|_X^r\Big)}{\left(1 - |\lambda|^{\alpha_2}\right)m^{\alpha_1 r}} \tag{3.10}$$

for all $x \in X$. So

$$\begin{aligned}\left\|m^l f\left(\frac{x}{m^l}\right) - m^p f\left(\frac{x}{m^p}\right)\right\|_Y &\leq \sum_{j=l}^{p-1}\left\|m^j f\left(\frac{x}{m^j}\right) - m^{j+1} f\left(\frac{x}{m^{j+1}}\right)\right\|_Y \\ &\leq \frac{\theta\Big(\sum_{q=1}^{m-1}\left(2k^{\alpha_1 r} + q^{\alpha_1 r}\right)\Big)}{\left(1 - |\lambda|\right)m^{\alpha_1 r}}\sum_{j=l}^{p-1}\frac{m^{\alpha_2 j}}{m^{\alpha_1 r j}}\left\|x\right\|_X^r \end{aligned} \tag{3.11}$$

for all nonnegative integers p, l with $p > l$ and all $x \in \mathbf{X}$. It follows from (3.11) that the sequence $\{m^n f(\frac{x}{m^n})\}$ is a cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\{m^n f(\frac{x}{m^n})\}$ coverges.

So one can define the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ by $\phi(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n})$ for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.11), we get (3.5).

It follows from (3.4) that

$$\begin{aligned}&\left\|\phi\Big((m+1)\sum_{j=1}^{k}\frac{x_j + y_j}{2k} - \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k}\phi\Big(m\frac{x_j + y_j}{2k} - z_j\Big) - \sum_{j=1}^{k}\phi\left(\frac{x_j + y_j}{2k}\right)\right\|_Y \\ &= \lim_{n\to\infty}\left\|m^n\Big(f\Big(\frac{(m+1)}{m^n}\sum_{j=1}^{k}\frac{x_j + y_j}{2k} - \frac{1}{m^n}\sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f\Big(\frac{m}{m^n}\frac{x_j + y_j}{2k} - \frac{1}{m^n} z_j\Big) \right.\\ &\left. - \sum_{j=1}^{k} f\left(\frac{1}{m^n}\frac{x_j + y_j}{2k}\right)\Big)\right\|_Y \\ &\leq \lim_{n\to\infty}\left\|m^n\lambda\Big(\Big(f\Big(\frac{1}{m^n}\sum_{j=1}^{k}\frac{x_j + y_j}{2k} + \frac{1}{m^n}\sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f\left(\frac{1}{m^n}\frac{x_j + y_j}{2k}\right) - \sum_{j=1}^{k} f\left(\frac{1}{m^n} z_j\right)\Big)\right\|_Y \\ &+ \lim_{n\to\infty}\frac{m^{\alpha_2 n}}{m^{\alpha_1 n r}}\theta\Big(\sum_{j=1}^{k}\|x_j\|_X^r + \sum_{j=1}^{k}\|y_j\|_X^r + \sum_{j=1}^{k}\|z_j\|_X^r\Big) \\ &= |\lambda|^{\alpha_2}\left\|\phi\Big(\sum_{j=1}^{k}\frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k}\phi\Big(\frac{x_j + y_j}{2k}\Big) - \sum_{j=1}^{k}\phi(z_j)\right\|_Y \end{aligned} \tag{3.12}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$.

$$\Big\|\phi\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}\phi\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}\phi\big(\frac{x_j+y_j}{2k}\big)\Big\|_Y$$
$$\leq |\lambda|^{\alpha_2}\Big\|\phi\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}\phi\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}\phi(z_j)\Big\|_Y$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. So by lemma 3.1 it follows that the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is additive. Now we need to prove uniqueness, Suppose $\phi' : \mathbf{X} \to \mathbf{Y}$ is also an additive mapping that satisfies (3.5). Then we have

$$\begin{aligned}\Big\|\phi(x)-\phi'(x)\Big\|_Y &= m^{\alpha_2 n}\|\phi(\frac{x}{m^n})-\phi'(\frac{x}{m^n})\|_Y\\ &\leq m^{\alpha_2 n}\Big(\Big\|\phi(\frac{x}{m^n})-f(\frac{x}{m^n})\Big\|_Y+\|\phi'(\frac{x}{m^n})-f(\frac{x}{m^n})\Big\|_Y\Big)\\ &\leq \frac{2.m^{\alpha_2 n}\cdot\sum_{q=1}^{m-1}(q^{\alpha_1 r}+2k^{\alpha_1 r})}{(1-|\lambda|^{\alpha_2})m^{\alpha_1 nr}(m^{\alpha_1 r}-m^{\alpha_2})}\theta\|x\|_X^r\end{aligned} \tag{3.13}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $\phi(x) = \phi'(x)$ for all $x \in \mathbf{X}$.This proves thus the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is a unique mapping satisfying(3.5) as we expected.

□

Theorem 3.3. *Let $r < \frac{\alpha_2}{\alpha_1}, m \in \mathbb{Z}, m > 1$, θ be nonngative real number, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that*

$$\begin{aligned}&\Big\|f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)\Big\|_Y\\ &\leq \Big\|\lambda\Big(f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)-\sum_{j=1}^{k}f(z_j)\Big)\Big\|_Y\\ &+\theta\Big(\sum_{j=1}^{k}\|x_j\|_X^r+\sum_{j=1}^{k}\|y_j\|_X^r+\sum_{j=1}^{k}\|z_j\|_X^r\Big)\end{aligned} \tag{3.14}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique mapping $\phi : \mathbf{X} \to \mathbf{Y}$ such that

$$\Big\|f(x)-\phi(x)\Big\|_Y \leq \frac{\sum_{q=1}^{m-1}(q^{\alpha_1 r}+2k^{\alpha_1 r})}{(1-|\lambda|^{\alpha_2})(m^{\alpha_2}-m^{\alpha_1 r})}\theta\|x\|_X^r. \tag{3.15}$$

for all $x \in \mathbf{X}$.

The rest of the proof is similar to the proof of Theorem 3.2.

4. Hyers-Ulam-Rassias stability Additive λ-functional inequalities (1.2) in α-homogeneous F-spaces

Additive β-functional inequality in complex Banach space Now, we study the solutions of (1.2). Note that for these inequalities, when $\mathbf{X}$ is a α_1-homogeneous F-spaces and that $\mathbf{Y}$ is a α_2-homogeneous F-spaces

. Under this setting, we can show that the mapping satisfying (1.2) is additive. These results are give in the following.

Lemma 4.1. *Let $m \in \mathbb{N}$ and a mapping $f : \mathbf{Y} \to \mathbf{Y}$ satisfies*

$$\begin{aligned}&\Big\|f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)-\sum_{j=1}^{k}f\big(z_j\big)\Big\|_Y\\&\leq\Big\|\lambda\Big(f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)\Big)\Big\|_Y\end{aligned}\tag{4.1}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, then $f : \mathbf{X} \to \mathbf{Y}$ is additive

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (4.1).
Replacing $\big(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k\big)$ by$\big(0, ..., 0, 0, ..., 0, 0, ..., 0\big)$ in (4.1), we have

$$\Big\|\big(2k-1\big)f\big(0\big)\Big\|_Y \leq \Big\|k\lambda f\big(0\big)\Big\|_Y \leq 0$$

$$\Big(\Big|2k-1\Big|^{\alpha_2}-\Big|\lambda k\Big|^{\alpha_2}\Big)\Big\|f\big(0\big)\Big\|_Y \leq 0$$

So $f\big(0\big) = 0$. Replacing $\big(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k\big)$ by $\big(0, ..., 0, 0, ..., 0, -z, 0, ..., 0)\big)$, in (4.1), we get

$$\Big\|f\big(-z\big)-f\big(-z\big)\Big\|_Y \leq 0$$

and so f is an odd mapping. Replacing $\big(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k\big)$
by$\big(x_1, ..., x_k, y_1, ..., y_k, m \cdot \frac{x_1+y_1}{2k} - v_1, ..., m \cdot \frac{x_k+y_k}{2k} - v_k\big)$ in (4.1), we have

$$\begin{aligned}&\Big\|f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}v_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-v_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)\Big\|_Y\\&\leq\Big\|\lambda\Big(f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}v_j\Big)-\sum_{j=1}^{k}f\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f\big(v_j\big)\Big)\Big\|_Y\end{aligned}\tag{4.2}$$

for all $x_1, ..., x_k, y_1, ..., y_k, m\frac{x_1+y_1}{2k} - v_1, ..., m\frac{x_k+y_k}{2k} - v_k \in \mathbf{G}$. From (4.1) and (4.2) we infer that

$$\begin{aligned}
&\left\| f\Big(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} v_j\Big) - \sum_{j=1}^{k} f\Big(\frac{x_j + y_j}{2k}\Big) - \sum_{j=1}^{k} f(v_j) \right\|_Y \\
&\leq \left\| \lambda\Big(f\Big((m+1)\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \sum_{j=1}^{k} v_j\Big) - \sum_{j=1}^{k} f\Big(m\frac{x_j + y_j}{2k} - v_j\Big) - \sum_{j=1}^{k} f\big(\frac{x_j + y_j}{2k}\big)\Big) \right\|_Y \\
&\leq \left\| \lambda^2\Big(f\Big(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} v_j\Big) - \sum_{j=1}^{k} f\Big(\frac{x_j + y_j}{2k}\Big) - \sum_{j=1}^{k} f(v_j)\Big) \right\|_Y \qquad (4.3)
\end{aligned}$$

and so

$$f\Big(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j\Big) = \sum_{j=1}^{k} f\Big(\frac{x_j + y_j}{2k}\Big) + \sum_{j=1}^{k} f(z_j)$$

for all $x_j, y_j, z_j \in \mathbf{G}$ for $j = 1 \to n$, as we expected. □

Theorem 4.2. *Let $r > \frac{\alpha_2}{\alpha_1}, m \in \mathbb{Z}, m > 1$, θ be nonngative real number, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that*

$$\begin{aligned}
&\left\| f\Big(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f\big(\frac{x_j + y_j}{2k}\big) - \sum_{j=1}^{k} f(z_j) \right\|_Y \\
&\leq \left\| \lambda\Big(f\Big((m+1)\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f\Big(m\frac{x_j + y_j}{2k} - z_j\Big) - \sum_{j=1}^{k} f\big(\frac{x_j + y_j}{2k}\big)\Big) \right\|_Y \\
&+ \theta\Big(\sum_{j=1}^{k} \|x_j\|_X^r + \sum_{j=1}^{k} \|y_j\|_X^r + \sum_{j=1}^{k} \|z_j\|_X^r\Big) \qquad (4.4)
\end{aligned}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique mapping $\phi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\| f(x) - h(x) \right\|_Y \leq \frac{\sum_{q=1}^{m-1} (q^{\alpha_1 r} + 2k^{\alpha_1 r})}{(1 - |\lambda|^{\alpha_2})(m^{\alpha_1 r} - m^{\alpha_2})} \theta \|x\|_X^r. \qquad (4.5)$$

for all $x \in \mathbf{X}$

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (4.4).
Replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by$(0, ..., 0, 0, ..., 0, 0, ..., 0)$ in (4.4), we have e

$$\left\| 2kf(0) \right\|_Y \leq \left\| \lambda(2k-1)f(0) \right\|_Y \leq 0$$

therefore

$$\Big(\left|2k\right|^{\alpha_2} - \left|\lambda(2k-1)\right|^{\alpha_2}\Big)\left\| f(0) \right\|_Y \leq 0$$

So$f(0) = 0$.
Next we
replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(kx, 0, ..., 0, kx, 0, ..., 0, 0, ..., 0)$ in (4.4), we get

$$\left\|f\left((m+1)x\right)-f\left(mx\right)-f\left(x\right)\right\|_{Y}\leq 2k^{\alpha_1 r}\theta\left\|x\right\|_{X}^{r} \tag{4.6}$$

for all $x\in\mathbf{X}$. Thus for $q\in\mathbb{N}$,
we replacing $\left(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k\right)$ by $\left(kx,0,...,0,kx,0,...,0,qx,0,...,0\right)$ in (4.4), we have

$$\begin{aligned}&\left\|f\left((m-q+1)x\right)-f\left((m-q)x\right)-f\left(x\right)\right\|_{Y}\\&\qquad\leq\left\|\lambda\left(f\left((q+1)x\right)-f\left(qx\right)-f\left(x\right)\right\|_{Y}+\theta\left(2k^{\alpha_1 r}+q^{\alpha_1 r}\right)\left\|x\right\|_{X}^{r}\end{aligned} \tag{4.7}$$

for all $x\in\mathbf{X}$.
For (4.6) and (4.7)

$$\begin{aligned}&\sum_{q=1}^{m-1}\left\|f\left((m-q+1)x\right)-f\left((m-q)x\right)-f\left(x\right)\right\|_{Y}\\&\qquad\leq\sum_{q=1}^{m-1}\left\|\lambda\left(f\left((q+1)x\right)-f\left(qx\right)-f\left(x\right)\right\|_{Y}+\theta\left(\sum_{q=1}^{m-1}\left(2k^{\alpha_1 r}+q^{\alpha_1 r}\right)\left\|x\right\|_{X}^{r}\right)\end{aligned} \tag{4.8}$$

for all $x\in\mathbf{X}$.
From (4.7) and (4.8) and triangle inequality, we have

$$\begin{aligned}&\left(1-|\lambda|^{\alpha_2}\right)\left\|f\left(mx\right)-mf\left(x\right)\right\|_{Y}\\&=\left(1-|\lambda|^{\alpha_2}\right)\sum_{q=1}^{m-1}\left\|f\left((q+1)x\right)-f\left(qx\right)-f\left(x\right)\right\|_{Y}\\&\leq\sum_{q=1}^{m-1}\left(1-|\lambda|^{\alpha_2}\right)\left\|\left(f\left((q+1)x\right)-f\left(qx\right)-f\left(x\right)\right\|_{Y}\\&\leq\sum_{q=1}^{m-1}\left\|f\left((q+1)x\right)-f\left(qx\right)-f\left(x\right)\right\|_{Y}-\sum_{q=1}^{m-1}\left\|\lambda\left(f\left((q+1)x\right)-f\left(qx\right)-f\left(x\right)\right)\right\|_{Y}\\&\leq\theta\left(\sum_{q=1}^{m-1}\left(2k^{\alpha_1 r}+q^{\alpha_1 r}\right)\left\|x\right\|_{X}^{r}\right)\end{aligned} \tag{4.9}$$

for all $x\in\mathbf{X}$. from

$$\sum_{q=1}^{m-1}\left\|f\left((m-q+1)x\right)-f\left((m-q)x\right)-f\left(x\right)\right\|_{Y}=\sum_{q=1}^{m-1}\left\|\left(f\left((q+1)x\right)-f\left(qx\right)-f\left(x\right)\right\|_{Y}$$

Since $|\lambda| < 1$, the mapping f satisfies the inequalities

$$\left\| f(mx) - mf(x) \right\| \leq \frac{\theta\Big(\sum_{q=1}^{m-1} \left(2k^{\alpha_1 r} + q^{\alpha_1 r} \right) \left\| x \right\|_{X}^{r} \Big)}{\left(1 - |\lambda|^{\alpha_2}\right)}$$

for all $x \in \mathbf{X}$.
therefore

$$\left\| f(x) - mf\left(\frac{x}{m}\right) \right\|_{Y} \leq \frac{\theta\Big(\sum_{q=1}^{m-1} \left(2k^{\alpha_1 r} + q^{\alpha_1 r} \right) \left\| x \right\|_{X}^{r} \Big)}{\left(1 - |\lambda|^{\alpha_2}\right) m^{\alpha_1 r}} \tag{4.10}$$

for all $x \in X$. So

$$\begin{aligned} \left\| m^{l} f\left(\frac{x}{m^{l}}\right) - m^{p} f\left(\frac{x}{m^{p}}\right) \right\|_{Y} &\leq \sum_{j=l}^{p-1} \left\| m^{j} f\left(\frac{x}{m^{j}}\right) - m^{j+1} f\left(\frac{x}{m^{j+1}}\right) \right\|_{Y} \\ &\leq \frac{\theta\Big(\sum_{q=1}^{m-1} \left(2k^{r} + q^{r} \right) \Big)}{\left(1 - |\lambda|^{\alpha_2}\right) m^{\alpha_1 r}} \sum_{j=l}^{p-1} \frac{m^{\alpha_2 j}}{m^{\alpha_1 r j}} \left\| x \right\|_{X}^{r} \end{aligned} \tag{4.11}$$

for all nonnegative integers p, l with $p > l$ and all $x \in \mathbf{X}$. It follows from (4.11) that the sequence $\{m^n f(\frac{x}{m^n})\}$ is a cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\{m^n f(\frac{x}{m^n})\}$ coverges.

So one can define the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ by $\phi(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n})$ for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.11), we get (4.5).

It follows from (4.4) that

$$\begin{aligned} &\left\| \phi\Big(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \Big) - \phi\Big(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} \Big) - \sum_{j=1}^{k} \phi(z_j) \right\|_{Y} \\ &= \lim_{n\to\infty} \left\| m^{n} \Big(f\Big(\frac{1}{m^{n}} \sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \frac{1}{m^{n}} \sum_{j=1}^{k} z_j \Big) - f\Big(\frac{1}{m^{n}} \sum_{j=1}^{k} \frac{x_j + y_j}{2k} \Big) - \sum_{j=1}^{k} f\Big(\frac{1}{m^{n}} z_j \Big) \Big) \right\|_{Y} \\ &+ \lim_{n\to\infty} \frac{m^{\alpha_2 n}}{m^{\alpha_1 n r}} \theta\Big(\sum_{j=1}^{k} \|x_j\|_{X}^{r} + \sum_{j=1}^{k} \|y_j\|_{X}^{r} + \sum_{j=1}^{k} \|z_j\|_{X}^{r} \Big) \\ &\leq \lim_{n\to\infty} |\lambda|^{\alpha_2} \left\| m^{n} \Big(f\Big(\frac{(m+1)}{m^{n}} \sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \frac{1}{m^{n}} \sum_{j=1}^{k} z_j \Big) - \sum_{j=1}^{k} f\Big(\frac{m}{m^{n}} \frac{x_j + y_j}{2k} - \frac{1}{m^{n}} z_j \Big) \right. \\ &\left. - \sum_{j=1}^{k} f\Big(\frac{1}{m^{n}} z_j \Big) \Big) \right\|_{Y} \\ &= |\lambda|^{\alpha_2} \left\| \phi\Big((m+1) \sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \Big) - \sum_{j=1}^{k} \phi\Big(m \frac{x_j + y_j}{2k} - z_j \Big) - \sum_{j=1}^{k} \phi(z_j) \right\|_{Y} \end{aligned} \tag{4.12}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. So

$$\left\|\phi\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}\phi\Big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}\phi\big(z_j\big)\right\|_{Y}$$
$$\leq |\lambda|^{\alpha_2}\left\|\phi\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}\phi\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}\phi\big(\frac{x_j+y_j}{2k}\big)\right\|_{Y}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. So by lemma 4.1 it follows that the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is additive. Now we need to prove uniqueness ,Suppose $\phi' : \mathbf{X} \to \mathbf{Y}$ is also an additive mapping that satisfies (4.5) . Then we have

$$\begin{aligned}\Big\|\phi(x)-\phi'(x)\Big\|_{Y} &= m^{\alpha_2 n}\|\phi(\frac{x}{m^n})-\phi'(\frac{x}{m^n})\|_{Y}\\ &\leq m^{\alpha_2 n}\Big(\Big\|\phi(\frac{x}{m^n})-f(\frac{x}{m^n})\Big\|_{Y}+\|\phi'(\frac{x}{m^n})-f(\frac{x}{m^n})\Big\|_{Y}\Big)\\ &\leq \frac{2.m^{\alpha_2 n}\cdot\sum_{q=1}^{m-1}\big(q^{\alpha_1 r}+2k^{\alpha_1 r}\big)}{\big(1-|\lambda|^{\alpha_2}\big)m^{n\alpha_1 r}(m^{\alpha_1 r}-m^{\alpha_2})}\theta\|x\|_{X}^{r}\end{aligned} \tag{4.13}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $\phi(x) = \phi'(x)$ for all $x \in \mathbf{X}$.This proves thus the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is a unique mapping satisfying(4.5) as we expected.

□

Theorem 4.3. *Let $r < \frac{\alpha_2}{\alpha_1}, m \in \mathbb{Z}, m > 1$, θ be nonngative real number, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that*

$$\begin{aligned}&\left\|f\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\Big)-\sum_{j=1}^{k}f\big(z_j\big)\right\|_{Y}\\ &\leq \left\|\lambda\Big(f\Big((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\Big)-\sum_{j=1}^{k}f\Big(m\frac{x_j+y_j}{2k}-z_j\Big)-\sum_{j=1}^{k}f\big(\frac{x_j+y_j}{2k}\big)\Big)\right\|_{Y}\\ &+\theta\Big(\sum_{j=1}^{k}\|x_j\|_{X}^{r}+\sum_{j=1}^{k}\|y_j\|_{X}^{r}+\sum_{j=1}^{k}\|z_j\|_{X}^{r}\Big)\end{aligned} \tag{4.14}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique mapping $\phi : \mathbf{X} \to \mathbf{Y}$ such that

$$\Big\|f(x)-h(x)\Big\|_{Y} \leq \frac{\sum_{q=1}^{m-1}\big(q^{\alpha_1 r}+2k^{\alpha_1 r}\big)}{\big(1-|\lambda|^{\alpha_2}\big)(m^{\alpha_2}-m^{\alpha_1 r})}\theta\|x\|_{X}^{r}. \tag{4.15}$$

for all $x \in \mathbf{X}$.

The proof is similar to theorem 4.2.

5. Conclusion

In this paper, I have shown that the solutions of the first and second $k - variable$ β-functional inequalities are additive mappings. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper , which are the generalization of the results [2], [11].

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