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$M_\varphi A$-h-Convexity and Hermite-Hadamard Type Inequalities

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Abstract. We investigate a family of $M_\varphi A$-h-convex functions, give some properties of it and several inequalities which are counterparts to the classical inequalities such as the Jensen inequality and the Schur inequality. We give the weighted Hermite-Hadamard inequalities for an $M_\varphi A$-h-convex function and several estimations for the product of two functions.

1. Preliminaries

It is known that the classical convexity can be generalized to an MN-convexity, where M and N are means which is described in [8]. The other direction of generalization leads to the concept of h-convexity, [13]. It is interesting to see properties of a function which definition combines some elements of MN-convexity and of h-convexity.

Let M and N be two means in two variables. We say that a function $f : I \to \mathbb{R}$ is MN-convex if

$$f(M(x,y)) \leq N(f(x), f(y))$$

for every $x, y \in I$.

In this paper we will focus on a somewhat special type of means.

Let φ be a continuous, strictly monotone function defined on the interval I. By M_φ we denote a quasi-arithmetic mean:

$$M_\varphi(x,y;t,1-t) := \varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)), \quad x,y \in I, t \in [0,1].$$

It is obvious that the power mean M_p corresponds to $\varphi(x) = x^p$ if $p \neq 0$ and to $\varphi(x) = \log x$ if $p = 0$.

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Let φ and ψ be two continuous, strictly monotone functions defined on intervals I and K respectively. Let $h : J \to \mathbb{R}$ be a non-negative function, $(0,1) \subseteq J$ and let $f : I \to K$ such that $h(t)\psi(f(x)) + h(1-t)\psi(f(y)) \in \psi(K)$ for all $x, y \in I$, $t \in (0,1)$. We say that a function f is $M_\varphi M_\psi$-h-convex if

$$f(M_\varphi(x, y; t, 1-t)) \leq M_\psi(f(x), f(y); h(t), h(1-t))$$

for all $x, y \in I$ and all $t \in (0,1)$. Especially, a function $f : I \to \mathbb{R}$ is called $M_\varphi A$-h-convex if

$$f(M_\varphi(x, y; t, 1-t)) \leq h(t)f(x) + h(1-t)f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in (0,1)$. The $M_\varphi M_\psi$-h-concavity is defined on a natural way.

Some particular cases of $M_\varphi M_\psi$-h-convex functions are recently investigated. If $h(t) = t$, then the $M_\varphi M_\psi$-h-convexity collapses to the $M_\varphi M_\psi$-convexity which is described in [8]. If M_φ, M_ψ are an arithmetic mean (A), a geometric mean (G) or a harmonic mean (H), then we can find several results. For example, AA-h-convexity or simply h-convexity firstly appeared in [13]. An HA-h-convexity or harmonic-h-convexity is described in [2] and [10]. HG-h-convexity investigated in [10] and AG-h-convexity or log-h-convexity in [9]. AM_p-h-convexity or (h, p)-convexity is described in [6] while some properties of M_pA-h-convex functions are given in [4]. Also, we have to mention article [1] devoted to the MN-h-convexity where $M, N \in \{A, G, H\}$.

The aim of this paper is to give several statements about $M_\varphi A$-h-convex functions primarly related to the Hermite-Hadamard inequality and the Jensen inequality. The following section is devoted to the properties of $M_\varphi A$-h-convex functions. Also in that section we give counterparts to the Jensen and the Schur inequality and some related results. In the third section we prove several inequalities of Hermite-Hadamard type.

2. Properties of $M_\varphi A$-h-convex functions and Jensen-type inequalities

Proposition 2.1. *Let φ be a continuous, strictly monotone function defined on the interval I. Let h be a non-negative function defined on the interval J, $(0,1) \subseteq J$. A function f is $M_\varphi A$-h-convex (concave) on I if and only if the function $f \circ \varphi^{-1}$ is h-convex (concave) on $\varphi(I)$.*

Proof. Let us suppose that$f$ is $M_\varphi A$-h-convex on I and let $u, v \in \varphi(I)$, $t \in (0,1)$. Since φ is continuous and strictly monotone on I, there exist $x, y \in I$ such that $u = \varphi(x)$, $v = \varphi(y)$. Then

$$\begin{aligned} (f \circ \varphi^{-1})(tu + (1-t)v) &= (f \circ \varphi^{-1})(t\varphi(x) + (1-t)\varphi(y))) \\ &= f(M_\varphi(x, y; t, 1-t)) \leq h(t)f(x) + h(1-t)f(y) \\ &= h(t)f(\varphi^{-1}(u)) + h(1-t)f(\varphi^{-1}(v)) \\ &= h(t)(f \circ \varphi^{-1})(u) + h(1-t)(f \circ \varphi^{-1})(v) \end{aligned}$$

which means that $f \circ \varphi^{-1}$ is h-convex. The second case is proved similarly. $\square$

Proposition 2.2. *Let φ be a continuous, strictly monotone function defined on the interval I. Let h, h_1, h_2 be non-negative functions defined on the interval J, $(0,1) \subseteq J$.*

(i) Let h_1 and h_2 have a property

$$h_2(t) \le h_1(t), \quad t \in (0,1).$$

If $f : I \to [0, \infty)$ is $M_\varphi A$-h_2-convex, then f is an $M_\varphi A$-h_1-convex function.

(ii) If f, g are $M_\varphi A$-h-convex functions, $\lambda > 0$, then $f + g$ and λf are $M_\varphi A$-h-convex.

(iii) Let $f, : I \to [0, \infty)$ be similarly ordered functions on I, i.e.

$$(f(x) - f(y))(g(x) - g(y)) \ge 0, \quad x, y \in I$$

and $h(t) + h(1-t) \le c$ for all $t \in (0,1)$, where $h = \max\{h_1, h_2\}$ and c is a fixed positive number. If f is $M_\varphi A$-h_1-convex and g is $M_\varphi A$-h_2-convex, then the product fg is $M_\varphi A$-h-convex.

Proof. The proof is based on the known results for h-convex functions and characterization given in Proposition 2.1. Let us prove part (i). If f is $M_\varphi A$-h_2-convex, then $f \circ \varphi^{-1}$ is h_2-convex. Then, using Proposition 8 from [13], we get that $f \circ \varphi^{-1}$ is h_1-convex, i.e. f is $M_\varphi A$-h_1-convex.

Other parts are proved similarly by applying Propositions 9 and 10 from [13]. □

The following theorem gives a counterpart of the Schur inequality.

Theorem 2.1. *Let h be a non-negative supermultiplicative function defined on the interval J, $(0,1) \subseteq J$. Let φ be a continuous, strictly monotone function defined on the interval I. Let $f : I \to [0, \infty)$ be $M_\varphi A$-h-convex.*

If φ is increasing, then for any $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ and $\varphi(x_3) - \varphi(x_2)$, $\varphi(x_3) - \varphi(x_1)$, $\varphi(x_2) - \varphi(x_1) \in J$ the following holds

$$h(\varphi(x_3) - \varphi(x_2)) f(x_1) - h(\varphi(x_3) - \varphi(x_1)) f(x_2) + h(\varphi(x_2) - \varphi(x_1)) f(x_3) \ge 0. \tag{2.1}$$

If φ is decreasing, then for any $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ and $\varphi(x_2) - \varphi(x_3)$, $\varphi(x_1) - \varphi(x_3)$, $\varphi(x_1) - \varphi(x_2) \in J$ the following holds

$$h(\varphi(x_2) - \varphi(x_3)) f(x_1) - h(\varphi(x_1) - \varphi(x_3)) f(x_2) + h(\varphi(x_1) - \varphi(x_2)) f(x_3) \ge 0. \tag{2.2}$$

Proof. Let assume that φ is increasing. For $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ we have

$$u_1 := \varphi(x_i) < u_2 := \varphi(x_2) < u_3 := \varphi(x_3).$$

Since a function $g := f \circ \varphi^{-1}$ is h-convex, using Proposition 16 from [13], we get

$$h(u_3 - u_2) g(u_1) - h(u_3 - u_1) g(u_2) + h(u_2 - u_1) g(u_3) \ge 0$$

and after appropriate substitutions we obtain inequality (2.1). Inequality (2.2) is proved in a similar way. □

The following theorem is a counterpart of the discrete Jensen inequality and its converse for an $M_\varphi A$-h-convex function.

Theorem 2.2. *Let $h : J \to \mathbb{R}$ be a non-negative supermultiplicative function, $(0,1) \subseteq J$. Let φ be a continuous, strictly monotone function defined on the interval I. Let $f : I \to [0,\infty)$ be a $M_\varphi A$-h-convex function. Let $w_1, \dots, w_n$ be non-negative real numbers such that $W_n = \sum_{i=1}^n w_i \neq 0$ and $\frac{w_i}{W_n} \in J$, $i = 1, \dots, n$.*

(i) Then for all $x_1, \dots, x_n \in I$ the following holds

$$f\left(\varphi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^n w_i\varphi(x_i)\right)\right) \leq \sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) f(x_i).$$

(ii) Then for all $x_1, \dots, x_n \in (a,b) \subseteq I$ the following holds

$$\begin{aligned}\sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) f(x_i) \quad &\leq \quad f(a)\sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) h\left(\frac{\varphi(b)-\varphi(x_i)}{\varphi(b)-\varphi(a)}\right) \\ &\quad + f(b)\sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) h\left(\frac{\varphi(x_i)-\varphi(a)}{\varphi(b)-\varphi(a)}\right).\end{aligned}$$

Proof. Since f is a $M_\varphi A$-h-convex function, then $f \circ \varphi^{-1}$ is h-convex on $\varphi(I)$ and using the Jensen inequality for h-convex functions and its converse ([13, Theorems 19 and 21]), we get the above results. □

The following result is a property of subadditivity for an index set function. Let K be a finite non-empty set of positive integers. Let us define the index set function F by

$$F(K) = h(W_K)f\left(\varphi^{-1}\left(\frac{1}{W_K}\sum_{i\in K} w_i\varphi(x_i)\right)\right) - \sum_{i\in K} h(w_i)f(x_i),$$

where $w_i \in J$, $W_K := \sum_{i\in K} w_i \in J$, $x_i \in I$.

Theorem 2.3. *Let $h : J \to \mathbb{R}$ be a non-negative supermultiplicative function and let M and K be finite non-empty sets of positive integers with $M \cap K = \emptyset$. Let $w_i > 0$, $(i \in M \cup K)$ be such that $W_K, W_M, W_{M\cup K} \in J$. Let φ be a continuous, strictly monotone function defined on the interval I.*

If $f : I \to [0,\infty)$ is $M_\varphi A$-h-convex, then the following inequality holds

$$F(M \cup K) \leq F(M) + F(K).$$

Furthermore, if $M_k := \{1, \dots, k\}$, $k = 2, \dots, n$ and $W_{M_k} \in J$, then

$$F(M_n) \leq F(M_{n-1}) \leq \ldots \leq F(M_2) \leq 0$$

and

$$F(M_n) \leq \min_{1\leq i<j\leq n}\Big\{h(w_i+w_j)f\left(\varphi^{-1}\left(\frac{w_i\varphi(x_i)+w_j\varphi(x_j)}{w_i+w_j}\right)\right) \\ -h(w_i)f(x_i)-h(w_j)f(x_j)\Big\}.$$

Proof. Let us consider the following difference

$$\begin{aligned}
&F(M)+F(K)-F(M\cup K)\\
&=h(W_M)f\left(\varphi^{-1}\left(\frac{1}{W_M}\sum_{i\in M}w_i\varphi(x_i)\right)\right)+h(W_K)f\left(\varphi^{-1}\left(\frac{1}{W_K}\sum_{i\in K}w_i\varphi(x_i)\right)\right)\\
&-h(W_{M\cup K})f\left(\varphi^{-1}\left(\frac{1}{W_{M\cup K}}\sum_{i\in M\cup K}w_i\varphi(x_i)\right)\right).
\end{aligned}$$

Since numbers $u:=\frac{1}{W_M}\sum_{i\in M}w_i\varphi(x_i)$ and $v:=\frac{1}{W_K}\sum_{i\in K}w_i\varphi(x_i)$ belong to $\varphi(I)$, there exist numbers $x,y\in I$ such that $\varphi(x)=u$, $\varphi(y)=v$. Using a definition of the $M_\varphi A$-h-convexity for $t=\frac{W_M}{W_{M\cup K}}$, $1-t=\frac{W_K}{W_{M\cup K}}$, and x,y and supermultiplicativity of h, we get

$$f\left(M_\varphi(x,y;\frac{W_M}{W_{M\cup K}},\frac{W_K}{W_{M\cup K}})\right)\leq\frac{h(W_M)}{h(W_{M\cup K})}f(x)+\frac{h(W_K)}{h(W_{M\cup K})}f(y) \tag{2.3}$$

and inequality $F(M)+F(K)-F(M\cup K)\geq 0$ follows from (2.3) immediately. □

Remark 2.1. If $M_\varphi=A$, then the above results related to the Jensen inequality, its converse and to the index set function for an h-function were proved in [13].

If $M_\varphi=H$, then the Jensen type inequality for HA-h-convex function is given in [2]. If $M_\varphi=M_p$ and $h(t)=t$, then the Jensen inequality for M_pA-convex was proved in [4]. If $M_\varphi\in\{A,G,H\}$, then results from this section are given in [1].

3. Hermite-Hadamard type inequality and related results

Counterparts of the Hermite-Hadamard inequality appear in the study of every kind of convexity. Namely, in the classical convexity, the left-hand side or the right-hand side of the Hermite-Hadamard inequality are equivalent to the definition of convexity. The Hermite-Hadamard inequality for an h-convex function was proved in [3] and [11] and has the following form.

If h is an integrable function, $h(\frac{1}{2})\neq 0$, then for an integrable h-convex function $f:[a,b]\to\mathbb{R}$, the following sequence of inequalities hold:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq[f(a)+f(b)]\int_0^1 h(x)\,dx. \tag{3.1}$$

This section begins with the weighted Hermite-Hadamard inequality for an $M_\varphi A$-h-convex function. This result is usually called the Hermite-Hadamard-Féjer inequality.

Theorem 3.1. *Let h be a non-negative function defined on the interval J, $(0,1) \subseteq J$, $h(\frac{1}{2}) \neq 0$ and φ be a differentiable, strictly monotone function defined on $[a,b]$.*

Let $w : [a,b] \to [0,\infty)$ be a function such that $w\varphi' \in L([a,b])$ and

$$w\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right) = w\left(\varphi^{-1}((1-t)\varphi(a)+t\varphi(b))\right) \tag{3.2}$$

for all $t \in (0,1)$. If f is $M_\varphi A$-h-convex, $fw\varphi' \in L([a,b])$, then

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_a^b w(x)\varphi'(x)\,dx \\
&\leq \int_a^b f(x)w(x)\varphi'(x)\,dx \\
&\leq [f(a)+f(b)]\int_a^b h\left(\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}\right) w(x)\varphi'(x)\,dx,
\end{aligned} \tag{3.3}$$

provided that all integrals exist. Moreover,

$$\begin{aligned}
\frac{1}{2h(\frac{1}{2})} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) &\leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)\,dx \\
&\leq [f(a)+f(b)]\int_0^1 h(x)\,dx,
\end{aligned} \tag{3.4}$$

provided that all integrals exist.

Proof. Let us prove the first inequality in (3.3). Since φ is continuous, strictly monotone, then for fixed $t \in (0,1)$ there exist $u, v \in [a,b]$ such that $\varphi(u) = t\varphi(a)+(1-t)\varphi(b)$ and $\varphi(v) = (1-t)\varphi(a)+t\varphi(b)$. Then, we get

$$\begin{aligned}
&f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \\
&= f\left(\varphi^{-1}\left(\frac{1}{2}[t\varphi(a)+(1-t)\varphi(b)] + \frac{1}{2}[(1-t)\varphi(a)+t\varphi(b)]\right)\right) \\
&\leq h\left(\frac{1}{2}\right) f(u) + h\left(\frac{1}{2}\right) f(v).
\end{aligned}$$

Multiplying the above inequality with $w\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)$, integrating over $[0,1]$ and using condition (3.2), we get

$$\begin{aligned}
&f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_0^1 w\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\,dt \\
&\leq h\left(\frac{1}{2}\right)\int_0^1 f(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b)))w\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\,dt \\
&+h\left(\frac{1}{2}\right)\int_0^1 f(\varphi^{-1}((1-t)\varphi(a)+t\varphi(b)))w\left(\varphi^{-1}((1-t)\varphi(a)+t\varphi(b))\right)\,dt \\
&= \frac{2h(\frac{1}{2})}{\varphi(b)-\varphi(a)}\int_a^b f(x)w(x)\varphi'(x)\,dx
\end{aligned}$$

and the first inequality in (3.3) is proved.

Multiplying inequality (1.1) with $w\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)$ and integrating, we get

$$\int_0^1 f(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b)))w\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\,dt$$
$$\leq \int_0^1 [h(t)f(a)+h(1-t)f(b)]w\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\,dt.$$

Using condition (3.2) and substitution $t\varphi(a)+(1-t)\varphi(b)=\varphi(x)$, we get the second inequality in (3.3).

The last inequality follows from (3.3) for the particular weight $w(t)=1$. □

Remark 3.1. Some particular results related to the non-weighted Hermite-Hadamard inequality are know. If $h(t)=t$, then the counterpart of the Hermite-Hadamard inequality (3.4) is given in [7]. The Hermite-Hadamard inequality (3.4) for a HA-h-convex function is given in [10], see also [15]. Inequality (3.4) for an $M_\varphi A$-convex function is given in [14] and for $M_p A$-h-convex is given in [5].

Corollary 3.1. *Let h be a non-negative function defined on the interval J, $(0,1)\subseteq J$. Let $w:[a,b]\to[0,\infty)$, $[a,b]\subset(0,\infty)$ be a function such that*

$$w(a^t b^{1-t})=w(a^{1-t}b^t)$$

for all $t\in(0,1)$. If f is GA-h-convex, then

$$\frac{1}{2h(\frac{1}{2})}f(\sqrt{ab})\int_a^b\frac{w(x)}{x}dx \leq \int_a^b f(x)\frac{w(x)}{x}\,dx \leq [f(a)+f(b)]\int_a^b h\left(\frac{\log b/x}{\log b/a}\right)\frac{w(x)}{x}\,dx,$$

provided that all integrals exist. Furthermore,

$$\frac{1}{2h(\frac{1}{2})}f(\sqrt{ab})\leq\int_a^b\frac{f(x)}{x}\,dx\leq[f(a)+f(b)]\int_0^1 h(t)\,dt,$$

provided that all integrals exist.

Proof. Putting in inequalities (3.3) and (3.4) $\varphi(x)=\log x$, we get the required results. □

The following theorem contains estimations for the integral mean of the product of two $M_\varphi A$-h-convex functions.

Theorem 3.2. *Let φ be a differentiable, strictly monotone function defined on the interval $[a,b]$. Let $h_i, i=1,2$ be non-negative functions defined on the interval J_i, $(0,1)\subseteq J_i$, and let $f,g:[a,b]\to[0,\infty)$.*

If f is $M_\varphi A$-h_1-convex and g is $M_\varphi A$-h_2-convex, then the following hold:

(i)

$$\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)g(x)\varphi'(x)\,dx$$
$$\le M(a,b)\int_0^1 h_1(t)h_2(t)\,dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt \tag{3.5}$$

(ii)

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right) g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right)$$
$$-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)g(x)\varphi'(x)\,dx$$
$$\le M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt + N(a,b)\int_0^1 h_1(t)h_2(t)\,dt, \tag{3.6}$$

where $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$.

(iii)

$$\frac{1}{2(\varphi(b)-\varphi(a))^2}\int_a^b\int_a^b\int_0^1 \varphi'(x)\varphi'(y)f(M_\varphi(x,y;t,1-t))g(M_\varphi(x,y;t,1-t))\,dt\,dy\,dx$$
$$\le \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)g(x)\varphi'(x)\,dx\int_0^1 h_1(t)h_2(t)\,dt$$
$$+[M(a,b)+N(a,b)]\int_0^1 h_1(t)\,dt\int_0^1 h_2(t)\,dt\int_0^1 h_1(t)h_2(1-t)\,dt \tag{3.7}$$

(iv)

$$\frac{1}{\varphi(b)-\varphi(a)}\int_a^b\int_0^1 \varphi'(x)f(M_\varphi(x,\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right);t,1-t))\times$$
$$\times g(M_\varphi(x,\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right);t,1-t))\,dt\,dx$$
$$\le \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)g(x)\varphi'(x)\,dx\int_0^1 h_1(t)h_2(t)\,dt$$
$$+[M(a,b)+N(a,b)]\left\{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)h_2(t)\,dt\right.$$
$$\left.+\left[h_1\left(\frac{1}{2}\right)\int_0^1 h_2(t)\,dt + h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)\,dt\right]\int_0^1 h_1(t)h_2(1-t)\,dt\right\}, \tag{3.8}$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b), \quad N(a,b)=f(a)g(b)+f(b)g(a)$$

and provided that all integrals exist.

Proof. (i) Since f is $M_\varphi A$-h_1-convex and g is $M_\varphi A$-h_2-convex, we get

$f(M_\varphi((a,b;t,1-t)) \le h_1(t)f(a)+h_1(1-t)f(b)$ and $g(M_\varphi(a,b;t,1-t)) \le h_2(t)g(a)+h_2(1-t)g(b)$.

Multiplying these two inequalities and integrating it over $[0,1]$, we obtain

$$\int_0^1 f(M_\varphi(a,b;t,1-t))g(M_\varphi(a,b;t,1-t))\,dt$$
$$\leq M(a,b)\int_0^1 h_1(t)h_2(t)\,dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt$$

and after a substitution $M_\varphi(a,b;t,1-t)=x$, we get inequality (3.5).

(ii) Since
$$\frac{\varphi(a)+\varphi(b)}{2}=\frac{1}{2}(t\varphi(a)+(1-t)\varphi(b))+\frac{1}{2}((1-t)\varphi(a)+t\varphi(b))$$
for $t\in(0,1)$ and since f is $M_\varphi A$-h_1-convex and g is $M_\varphi A$-h_2-convex, we get

$$f\left(M_\varphi(u,v;\frac{1}{2},\frac{1}{2})\right)\leq h_1\left(\frac{1}{2}\right)[f(u)+f(v)]$$

and

$$g\left(M_\varphi(u,v;\frac{1}{2},\frac{1}{2})\right)\leq h_2\left(\frac{1}{2}\right)[g(u)+g(v)],$$

where $\varphi(u)=t\varphi(a)+(1-t)\varphi(b)$ and $\varphi(v)=(1-t)\varphi(a)+t\varphi(b)$. Multiplying these inequalities, we obtain

$$f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right)g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right)=f\left(M_\varphi(u,v;\frac{1}{2},\frac{1}{2})\right)g\left(M_\varphi(u,v;\frac{1}{2},\frac{1}{2})\right)$$
$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big\{f(u)g(u)+f(v)g(v)+f(u)g(v)+f(v)g(u)\Big\}$$
$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big\{[f(u)g(u)+f(v)g(v)]+f(a)g(a)[h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]$$
$$+f(a)g(b)[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]+f(b)g(a)[h_1(1-t)h_2(1-t)+h_1(t)h_2(t)]$$
$$+f(b)g(b)[h_1(1-t)h_2(t)+h_1(t)h_2(1-t)]\Big\},$$

where in the last inequality we used the $M_\varphi A$-h-convexity again. Integrating the above inequality and using into account that

$$\int_0^1 f(u)g(u)\,dt=\int_0^1 f(v)g(v)\,dt=\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)g(x)\varphi'(x)\,dx$$

$$\int_0^1 h_1(t)h_2(1-t)\,dt=\int_0^1 h_1(1-t)h_2(t)\,dt,\int_0^1 h_1(t)h_2(t)\,dt=\int_0^1 h_1(1-t)h_2(1-t)\,dt$$

we obtain inequality (3.6).

(iii) Since f is $M_\varphi A$-h_1-convex and g is $M_\varphi A$-h_2-convex, we get

$$f(M_\varphi(x,y;t,1-t))\leq h_1(t)f(x)+h_1(1-t)f(y)$$

and

$$g(M_\varphi(x,y;t,1-t))\leq h_2(t)g(x)+h_2(1-t)g(y).$$

Multiplying these two inequalities, then multiplying with $\varphi'(x)\varphi'(y)$ and integrating it over $[a,b]$ with respect to x and y and over $[0,1]$ with respect to t, we obtain

$$\begin{aligned}&\int_a^b\int_a^b\int_0^1 \varphi'(x)\varphi'(y)f(M_\varphi(x,y;t,1-t))g(M_\varphi(x,y;t,1-t))\,dt\,dy\,dx\\&\le 2(\varphi(b)-\varphi(a))\int_0^1 h_1(t)h_2(t)\,dt\int_a^b f(x)g(x)\varphi'(x)\,dx\\&+2\int_0^1 h_1(1-t)h_2(t)\,dt\int_a^b f(x)\varphi'(x)\,dx\int_a^b g(x)\varphi'(x)\,dx.\end{aligned}$$

Using the right-hand side of inequality (3.4) to estimate $\int_a^b f(x)\varphi'(x)\,dx$ and $\int_a^b g(x)\varphi'(x)\,dx$ and some simple transformations, we get (3.7).

(iv) In this case we begin with inequalities

$$f(M_\varphi(x,\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right);t,1-t))\le h_1(t)f(x)+h_1(1-t)f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)$$

$$g(M_\varphi(x,\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right);t,1-t))\le h_2(t)g(x)+h_2(1-t)g\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)$$

and proceed in the similar way, i.e. multiply them mutually, then multiply with $\varphi'(x)$ and integrate with respect to x and t. We get

$$\begin{aligned}&\int_a^b\int_0^1 \varphi'(x)f(M_\varphi(x,\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right);t,1-t))\times\\&\times g(M_\varphi(x,\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right);t,1-t))\,dt\,dx\\&\le\int_a^b f(x)g(x)\varphi'(x)\,dx\int_0^1 h_1(t)h_2(t)\,dt\\&+g\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_a^b f(x)\varphi'(x)\,dx\int_0^1 h_1(t)h_2(1-t)\,dt\\&+f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_a^b g(x)\varphi'(x)\,dx\int_0^1 h_1(1-t)h_2(t)\,dt\\&+f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)g\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_0^1 h_1(1-t)h_2(1-t)\,dt.\end{aligned}$$

In the next step we use the right-hand side of (3.4) to estimate $\int_a^b f(x)\varphi'(x)\,dx$ and $\int_a^b g(x)\varphi'(x)\,dx$ and definition of $M_\varphi A$-h-convexity to estimate $f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)$ and $g\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)$. After a short calculation we get the required inequality (3.8). □

Remark 3.2. Particular cases of the above results are already known. If $\varphi(x)=x$, then (3.5) and (3.6) for AA-h-convex functions are given in [11]. The above results for $M_\varphi A$-convex functions, i.e. with $h(t)=t$ are given in [14]. If $\varphi(x)=x^p$, $p\ne 0$, then (3.5) -(3.8) for M_pA-h-convex functions are given in [5].

The case $p = 0$ is not considered in [5], so, in the following corollary we give this, complementary result.

Corollary 3.2. *Let $h_i, i = 1, 2$ be non-negative functions defined on the interval J_i, $(0, 1) \subseteq J_i$, and let $f, g : [a, b] \to [0, \infty)$, $[a, b] \subset (0, \infty)$.*

If f is GA-h_1-convex and g is GA-h_2-convex, then

(i)

$$\frac{1}{\log b/a} \int_a^b f(x)g(x)\frac{dx}{x} \leq M(a, b) \int_0^1 h_1(t)h_2(t)\, dt + N(a, b) \int_0^1 h_1(t)h_2(1-t)\, dt$$

(ii)

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f(\sqrt{ab})g(\sqrt{ab}) - \frac{1}{\log b/a} \int_a^b f(x)g(x)\,\frac{dx}{x}$$
$$\leq M(a, b) \int_0^1 h_1(t)h_2(1-t)\, dt + N(a, b) \int_0^1 h_1(t)h_2(t)\, dt,$$

where $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$

(iii)

$$\frac{1}{2\log^2 b/a} \int_a^b \int_a^b \int_0^1 \frac{1}{xy} f(x^t y^{1-t})g(x^t y^{1-t})\, dt\, dy\, dx$$
$$\leq \frac{1}{\log b/a} \int_a^b f(x)g(x)\frac{dx}{x} \int_0^1 h_1(t)h_2(t)\, dt$$
$$+[M(a, b) + N(a, b)] \int_0^1 h_1(t)\, dt \int_0^1 h_2(t)\, dt \int_0^1 h_1(t)h_2(1-t)\, dt.$$

Proof. Applying the function $\varphi(x) = \log x$ in Theorem 3.2, we get results of this corollary. ☐

The following theorem also contains some estimations for the integral mean of the product of two functions, but the proofs of these inequalities are based on the following inequality:

$$\text{if } a \leq b \text{ and } c \leq d, \text{ then } ad + cb \leq bd + ac. \tag{3.9}$$

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied. Then*

(i)

$$\int_a^b \left[g(a)h_2\left(\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}\right) + g(b)h_2\left(\frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}\right)\right] f(x)\varphi'(x)\, dx$$
$$+ \int_a^b \left[f(a)h_1\left(\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}\right) + f(b)h_1\left(\frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}\right)\right] g(x)\varphi'(x)\, dx$$
$$\leq (\varphi(b)-\varphi(a)) \left[M(a, b) \int_0^1 h_1(t)h_2(t)\, dt + N(a, b) \int_0^1 h_1(t)h_2(1-t)\, dt\right]$$
$$+ \int_a^b f(x)g(x)\varphi'(x)\, dx \tag{3.10}$$

(ii)

$$\frac{1}{\varphi(b)-\varphi(a)}\int_a^b \left[h_2\left(\frac{1}{2}\right) f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right) g(x) \right.$$
$$\left. + h_1\left(\frac{1}{2}\right) g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right) f(x)\right] \varphi'(x)\,dx$$
$$\leq \frac{h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)}{\varphi(b)-\varphi(a)}\int_a^b f(x)g(x)\varphi'(x)\,dx$$
$$+h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)\left[M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt + N(a,b)\int_0^1 h_1(t)h_2(t)\,dt\right]$$
$$+\frac{1}{2} f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right) g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right), \tag{3.11}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b), \quad N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. (i) Putting in (3.9)

$$a = f\left(M_\varphi(a,b;t,1-t)\right), \quad b = h_1(t)f(a) + h_1(1-t)f(b)$$
$$c = g\left(M_\varphi(a,b;t,1-t)\right), \quad d = h_2(t)g(a) + h_2(1-t)g(b)$$

and integrating obtained inequality with respect to t, we get

$$\int_0^1 [g(a)h_2(t) + g(b)h_2(1-t)]\, f\left(M_\varphi(a,b;t,1-t)\right)\,dt$$
$$+\int_0^1 [f(a)h_1(t) + f(b)h_1(1-t)]\, g\left(M_\varphi(a,b;t,1-t)\right)\,dt$$
$$\leq M(a,b)\int_0^1 h_1(t)h_2(t)\,dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt$$
$$+\int_0^1 f\left(M_\varphi(a,b;t,1-t)\right) g\left(M_\varphi(a,b;t,1-t)\right)\,dt.$$

After substitution $u = M_\varphi(a,b;t,1-t)$ in integrals the above inequality collapses to inequality (3.10).

(ii) From $M_\varphi A$-h_i-convexity we get

$$f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right) = f\left(M_\varphi(u,v;\frac{1}{2},\frac{1}{2})\right) \leq h_1\left(\frac{1}{2}\right) f(u) + h_1\left(\frac{1}{2}\right) f(v)$$

and

$$g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right) = g\left(M_\varphi(u,v;\frac{1}{2},\frac{1}{2})\right) \leq h_2\left(\frac{1}{2}\right) g(u) + h_2\left(\frac{1}{2}\right) g(v),$$

where $\varphi(u) = (1-t)\varphi(a) + t\varphi(b)$ and $\varphi(v) = t\varphi(a) + (1-t)\varphi(b)$. Putting in (3.9)

$$a = f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right), \quad b = h_1\left(\frac{1}{2}\right)[f(u)+f(v)]$$
$$c = g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right), \quad d = h_2\left(\frac{1}{2}\right)[g(u)+g(v)]$$

and integrating with respect to t, we get

$$\begin{aligned}
&h_2\left(\frac{1}{2}\right) f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right)\int_0^1 [g(u)+g(v)]\,dt \\
&+h_1\left(\frac{1}{2}\right) g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right)\int_0^1 [f(u)+f(v)]\,dt \\
&\quad\le h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)\left[\int_0^1 f(u)g(u)\,dt+\int_0^1 f(v)g(v)\,dt\right] \\
&\quad+2h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)\left[M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt+N(a,b)\int_0^1 h_1(t)h_2(t)\,dt\right] \\
&\quad+f\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right) g\left(M_\varphi(a,b;\frac{1}{2},\frac{1}{2})\right).
\end{aligned}$$

After substitution $\varphi(x)=(1-t)\varphi(a)+t\varphi(b)$ in integrals $\int_0^1 f(u)g(u)\,dt$, $\int_0^1 f(u)\,dt$ and $\int_0^1 g(u)\,dt$, and substitution $\varphi(x)=t\varphi(a)+(1-t)\varphi(b)$ in integrals $\int_0^1 f(v)g(v)\,dt$, $\int_0^1 f(v)\,dt$ and $\int_0^1 g(v)\,dt$, we obtain inequality (3.11). □

Remark 3.3. If $h(t)=t$, results (3.10) and (3.11) are given in [14].

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References

[1] M.W. Alomari, Some Properties of $h-MN$-Convexity and Jensen's Type Inequalities, J. Interdiscip. Math. 22 (2019), 1349–1395. https://doi.org/10.1080/09720502.2019.1698402.

[2] I.A. Baloch, M. de la Sen, I. Iscan, Characterizations of Classes of Harmonic Convex Functions and Applications, Int. J. Anal. Appl. 17 (2019), 722–733. https://doi.org/10.28924/2291-8639-17-2019-722.

[3] M. Bombardelli, S. Varošanec, Properties of h-Convex Functions Related to the Hermite-Hadamard-Féjer Inequalities, Computers Math. Appl. 58 (2009), 1869–1877. https://doi.org/10.1016/j.camwa.2009.07.073.

[4] T.H. Dinh, K.T.B. Vo, Some Inequalities for Operator (p,h)-Convex Functions, Linear Multilinear Algebra, 66 (2018), 580–592. https://doi.org/10.1080/03081087.2017.1307914.

[5] Z.B. Fang, R. Shi, On the (p,h)-Convex Function and Some Integral Inequalities, J. Inequal. Appl. 2014 (2014), 45. https://doi.org/10.1186/1029-242X-2014-45.

[6] L.V. Hap, N.V. Vinh, On some Hadamard-Type Inequalities for (h,r)-Convex Functions, Int. J. Math. Anal. 7 (2013), 2067–2075. https://doi.org/10.12988/ijma.2013.28236.

[7] F.C. Mitroi, C.I. Spiridon, Hermite-Hadamard Type Inequalities of Convex Functions With Respect to a Pair of Quasi-Arithmetic Means, Math. Rep. 14 (2012), 291–295.

[8] C.P. Niculescu, L.E. Persson, Convex Functions and Their Applications: A Contemporary Approach, Springer New York, New York, NY, 2006. https://doi.org/10.1007/0-387-31077-0.

[9] M.A. Noor, F. Qi, M.U. Awan, Some Hermite–Hadamard Type Inequalities for Log-h-Convex Functions, Analysis. 33 (2013), 367–375. https://doi.org/10.1524/anly.2013.1223.

[10] M.A. Noor, K.I. Noor, M.U. Awan, et al. Some Integral Inequalities for Harmonically h-Convex Functions, U.P.B. Sci. Bull. Ser. A. 77 (2015), 5–16.

[11] M.Z. Sarikaya, A. Saglam, H. Yildirim, On Some Hadamard-Type Inequalities for h-Convex Functions, J. Math. Inequal. 2 (2008), 335–341.

[12] S. Turhan, M. Kunt, I. Iscan, Hermite-Hadamard Type Inequalities for $M_\varphi A$-Convex Functions, Int. J. Math. Model. Comput. 10 (2020), 57–75.

[13] S. Varošanec, On h-Convexity, J. Math. Anal. Appl. 326 (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086.

[14] S. Wu, M.U. Awan, M.A. Noor, et al. On a New Class of Convex Functions and Integral Inequalities, J. Inequal. Appl. 2019 (2019), 131. https://doi.org/10.1186/s13660-019-2074-y.

[15] D. Zhao, T. An, G. Ye, et al. On Hermite-Hadamard type Inequalities for Harmonical h-Convex Interval-Valued Functions, Math. Inequal. Appl. 23 (2020), 95–105. https://doi.org/10.7153/mia-2020-23-08.