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Analytical Solution of Nonlinear Fractional Gradient-Based System Using Fractional Power Series Method

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ABSTRACT. This paper adapted a reliable treatment technique, called the fractional residual power series, to the fractional gradient-based system in solving a class of nonlinear programming model in Caputo's sense. The gradient-based system has been constructed to transform the nonlinear programming problem with equality constraints to unconstrained optimization problem, and then the fractional residual power series method is implemented to obtain the essential behavior of underlying problem. The proposed methods have been applied effectively to produce optimal solution in rapidly convergent fractional series representations without linearization, or any limitations. To confirm the performance of the proposed methods, some optimization problems are tested. Further, numerical comparisons with other existing methods are also given. The results exhibit that the FRPS method is easy, simple, effective, and fully compatible with the complexity of such models.

1. INTRODUCTION

Over the past years, the topic of optimization has had the interest of many scholars in various applications of science and technology and associated with several categories of optimization problems. Besides, many effective methods have been developed to find the optimal solution to these problems. For more details see [1-6]. The gradient-based approach is one such method that is used to solve nonlinear programming (NLP) problems. Transforming the optimization problem into a system of ordinary differential equations (ODEs) is the basic idea of this method, which is equipped

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with ideal conditions to reach the optimal solutions to this problem [7-13]. Fractional calculus is a generalization of the derivatives and integrals of an arbitrary system. Recently, the subject of fractional calculus has received the attention of scientists and engineers because of its important applications in various fields, whether science or engineering [14-19]. Many real-life problems in various fields of applied science have been modeled using fractional differential equations (FDEs), which are generalizations of ODEs.

For describing the behavior of the unknowns of FDEs, many researchers usually implement some numerical or numerical analytical techniques instead. In this regard, some recent techniques are proposed for solving FDEs. Among them decomposition technique, symmetric perturbation technique, variable frequency technique, and partial differential transformation technique [20-26]. Further, more applications and promising approaches are utilized to treat the nonlinear fractional gradient-based systems of FDEs could be founded in the references [27-30]. Motivated by the existing techniques, the main contribution of this article is to transform equality constrained NLP problem to unconstrained optimization problem by identifying a penalty function, then construct a gradient-based system of FDEs. Besides the, the fractional residual power series (FRPS) technique is applied to provide us the accordance between the optimal solution of the NLP problem and the power series solution of the obtained FDEs system.

FRPS technique is one of the modern numeric-analytic techniques was initially proposed in [31] to investigate the sequential solution of fuzzy differential equations of both first and second degree. It has been used to generate accurate approximate solutions in terms of fractional series formulas for several kinds for linear and non-linear FDEs, Partial FDEs and Fuzzy FDEs (see [32-36]). This scheme is used basically the residual-error function and employed the fractional differentiation in each stage in determining the coefficient of the suggested series expansion without linearity, division, or perturbation (see [37-42]). It does not require any converting while switching from the lower order to the higher order; as a result, the method can be applied directly to the given problem by choosing an initial guess approximation. FRPS is quick and easy calculation to find series solutions via utilizing software package. Also, different Taylor series method, FRPS needs easy computation state with high reliability and less time.

The organization of this paper was to present a brief presentation of some basic and necessary definitions and properties in fractional calculus in section 2, in addition to the fact that the central problem in this paper was presented in section 3. Section 4 presents the details of the application of the proposed technique and provides a clear and simple algorithm for the basic steps of this technique. Clarifying the applicability and efficiency of the proposed technique by comparing the results derived from it for some numerical examples with the results of the fourth degree Rung-Kutta method, done in section 5. Section 6 is designed to provide some concluding observations.

2. PRELIMINARIES

In this section, we present the definition of the Riemann-Liouville fractional integral operator, the Caputo partial derivative, and some of their properties [43-49]. Throughout this paper, the symbol R denotes the set of real numbers, N the set of integers, and $\Gamma(.)$ is a gamma function. For more details, please see [45-49] and references therein.

Definition 1. Let $\alpha \geq 0$ and for each $\alpha, x \in \mathbb{R}$. The Riemann-Liouville fractional integral operator of order, for a function $u(x)$, is defined as follow

$$I^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{1-\alpha}u(s)ds\,,\ t > 0,$$

In particular, if $\alpha = 0$, then $I^{\alpha}u(t) = u(t)$.

Definition 2. The Caputo fractional derivative of a function $u(x)$ with $m-1 < \alpha \leq m,\ m \in \mathbb{N}$, is defined as

$$D^{\alpha}u = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x (x-s)^{m-\alpha-1}u^{(m)}(s)ds\,, & m-1 < \alpha < m, \quad x > 0. \\ u^{(m)}(x)\,, & \alpha = m\ . \end{cases}$$

On the other hand, the operator D^{α} has some basic properties such as, for any real number A, then we have $D^{\alpha}A = 0$, and for $u(x) = x^k,\ k \geq -1$, we have $D^{\alpha}x^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}x^{k-\alpha}$. Moreover, the Caputo fractional derivative has the linearity property, this means, for each constant γ and μ we have $D^{\alpha}[\gamma u(x) + \mu v(x)] = \gamma D^{\alpha}u(x) + \mu D^{\alpha}v(x)$.

Lemma 1. For $m-1 < \alpha \leq m$ and $u(x) \in C^m[0,\infty)$, $m \in \mathbb{N}$, then we have

1. $D^{\alpha}I^{\alpha}u(x) = u(x)$,
2. $I^{\alpha}D^{\alpha}u(x) = u(x) - \sum_{i=0}^{m-1} u^{(k)}(0^+)\frac{x^k}{k!}, x > 0$.

3. OPTIMIZATION PROBLEM

The second part presents the details of the optimization problem to be studied in this paper. We consider the following NLP problem with equality constrains

$$\min u(x) \ s.t. \ w(x) = 0, x \in \mathbb{R}^n, \tag{2}$$

where $x \in \mathbb{R}^n$ is decision variable, $u(x)$ is a vector-valued function of a real variable, and $w = \left(w_1, w_2, \dots, w_p\right)^T : \mathbb{R}^n \to \mathbb{R}^p \ (p \leq n)$ are twice continuously differentiable function such that whose gradient $\nabla w(x)$ has full rank. We assume that a feasible region of (2) is nonempty bounded set.

The penalty function can be defined as

$$P(x) = u(x) + \phi(x), \tag{3}$$

where $\phi(x)$ is the penalty term defined on $\mathbb{R}^n$ and has the following property

$$\phi(x) = \begin{cases} 0, & w(x) = 0 \\ nonnegative, & w(x) \neq 0. \end{cases} \tag{4}$$

One can be defined the penalty term $\phi(x)$ as

$$\phi(x) = \frac{1}{\mu} \|w(x)\|_\mu^\mu = \frac{1}{\mu} \sum_{i=1}^{p} |w_i(x)|^\mu, \tag{5}$$

where $\mu > 0$ is a constant.

A well-known penalty function for the problem (2) has been defined as

$$P(x, \eta_m) = u(x) + \frac{\eta_m}{\mu} \sum_{i=1}^{p} |w_i(x)|^\mu, \tag{6}$$

where the penalty parameter η_m satisfying the inequality $0 < \eta_m < \eta_{m+1}$ for all m, $\eta_m \to \infty$. It is worth mentioning that the penalty parameter η_m can be chosen randomly. Thus, $\eta_m$can be chosen in positive constant depends on the difficulty of minimizing the penalty function at every iteration. It can be shown that under some suitable conditions the solution of the equality constrains NLP problem (2) are solution of the following unconstrained optimization problem

$$\min P(x, \eta_m) = u(x) + \frac{\eta_m}{\mu} \sum_{i=1}^{p} |w_i(x)|^\mu \quad s.t \ \ x \in \mathbb{R}^n. \tag{7}$$

We assume that the unconstrained optimization problem (7) for each m, has a solution and we denote it by x_m. The main results that connect the solutions of the equality constrained NLP problem (2) and unconstrained problem (7) present in the following theorem.

Theorem 1. Let $\{\mathrm{x_m}\}$ be a sequence generated by the penalty method of the unconstrained problem (7). Then any limit point of the sequence is a solution to the equality constrained NLP problem (2).

The author in [12] showed that the unconstrained optimization problem (7) can be described by the following gradient based dynamic system of ODEs

$$\frac{dx(t)}{dt} = -\nabla_x \left(u(x) + \frac{\eta_m}{\mu} \sum_{i=1}^{p} |w_i(x)|^\mu \right), x(t_0) = b \in \mathbb{R}^n \tag{8}$$

where ∇_x is the gradient vector with respect to the $x \in \mathbb{R}^n$. We can describe the system (8) by an approach based on fractional gradient based dynamic system by the following system of FDEs

$$D_t^{\alpha} x(t) = -\nabla_x \left(u(x) + \frac{\eta_m}{\mu} \sum_{i=1}^{p} |w_i(x)|^{\mu}\right), x(t_0) = b \in \mathbb{R}^n,\ 0 < \alpha \le 1. \tag{9}$$

The system (9) has an equilibrium point x_e, if $x_e \in \mathbb{R}^n$ is satisfies the right-hand side of the system. A more convenient form of the system (9) can be expressed as follows

$$D_t^{\alpha} x_i(t) = f_i(t, \eta_m, x_1, x_2, \dots, x_n), i = 1,2,\dots,n,\ x(t_0) = b \in \mathbb{R}^n,\ 0 < \alpha \le 1. \tag{10}$$

The stable equilibrium point of the fractional system (10) is acquired with the RPS technique.

4. FRPS TECHNIQUE

This subsection devoted to applying the RPS method to derive analytic solution of the system of FDEs (10). We begin by propose the definition of fractional power series.

Definition 3. A power series (PS) expansion at $t = t_0$ of the following form

$$\sum_{i=0}^{\infty} c_i (t - t_0)^{j\alpha} = c_0 + c_1 (t - t_0)^{\alpha} + c_2 (t - t_0)^{2\alpha} + \cdots, \tag{11}$$

for $0 \le n - 1 < \alpha \le n, n \in \mathbb{N}, t \ge t_0$ where c_j's are constants called the fractional power series (FPS).

Theorem 2. Suppose that $x(t)$ has a FPS representation at $t = t_0$ of the form

$$x(t) = \sum_{i=0}^{\infty} c_j (t - t_0)^{j\alpha}, \qquad t_0 \le t \le t_0 + R. \tag{12}$$

If $x(t) \in C[t_0, t_0 + R)$, and $D_t^{i\alpha} x(t) \in C(t_0, t_0 + R)$, for $i = 0,1,2,\dots$, then the coefficients c_i in Eq. (6) will take the form $c_i = \frac{D^{i\alpha} x(t_0)}{\Gamma(i\alpha+1)}$, where $D^{i\alpha} = D^{\alpha} \cdot D^{\alpha} \cdots D^{\alpha}$ (i-times) and R is the convergent radius.

Theorem 3. If $K \in (0,1)$, such that $\|x_{i+1}(t)\| \le K \|x_i(t)\|\ \forall i \in \mathbb{N}$ and $0 < t < \mathrm{T} < 1$, then the series of numerical solutions converges to an exact solution

Proof. We notice that $\forall\ 0 < t < \mathrm{T} < 1$,

$$\|x(t) - x_i(t)\| = \left\| \sum_{m=i+1}^{\infty} x_i(t) \right\| \le \sum_{m=i+1}^{\infty} \|x_i(t)\| \le \|b\| \left\| \sum_{m=i+1}^{\infty} K^m \right\|$$

$$= \frac{K^{k+1}}{1-K} \|b\| \to 0\ as\ k \to \infty.$$

Now, to utilize the FRPS technique to solve the system (10), we substitute the FPS expansion (12) among the recurrent fractional differentiation of truncation residual functions. Suppose that the solution of the system (10) about the initial point $t = t_0$ takes the form

$$x_i(t) = \sum_{j=0}^{\infty} c_{ij}\,(t - t_0)^{j\alpha}, i = 1,2,\dots,n. \tag{13}$$

Differentiate the expansion form (13) for $t > 0$ within the interval of convergence, we get

$$D_t^{\alpha} x_i(t) = \sum_{j=1}^{\infty} \frac{c_{ij}\Gamma(j\alpha+1)}{\Gamma((j-1)\alpha+1)}(t - t_0)^{(j-1)\alpha}, i = 1,2,\dots,n. \tag{14}$$

Hence,

$$D_t^{2\alpha} x_i(t) = \sum_{j=2}^{\infty} \frac{c_{ij}\Gamma(j\alpha+1)}{\Gamma((j-2)\alpha+1)}(t - t_0)^{(j-2)\alpha}, i = 1,2,\dots,n, \tag{15}$$

where $D_t^{2\alpha} = D_t^{\alpha} D_t^{\alpha}$. Therefore, we suppose that the approximate solution of $x_i(t)$ can be constructed by the following series

$$x_{ik}(t) = \sum_{j=0}^{k} c_{ij}\,(t - t_0)^{j\alpha}, i = 1,2,\dots,n. \tag{16}$$

Since $x_i(t)$ satisfy the initial condition $x_i(t_0) = b_i, i = 1,2,\dots,n$, then $c_{i0} = b_i$ and series (16) can be rewritten by

$$x_{ik}(t) = b_i + \sum_{j=1}^{k} c_{ij}\,(t - t_0)^{j\alpha}, \qquad k = 1,2,3,\dots, i = 1,2,\dots,n. \tag{17}$$

According the RPS technique, we define the residual function, $Res_{x_i}(t)$, about $t = t_0$ for the system (10) as follows

$$Res_{x_i}(t) = D_t^{\alpha} x_i(t) + \nabla_{x_i} P(x_i, \eta), i = 1,2,\dots,n. \tag{18}$$

by assuming the penalty parameter $\eta_m = \eta$ (constant). The k-th residual function, $Res_{x_i,k}(t)$, can be defined by

$$Res_{x_i,k}(t) = D_t^{\alpha} x_{ik}(t) + \nabla_{x_{ik}} P(x_{ik}, \eta), k = 1,2,3,\dots, i = 1,2,\dots,n. \tag{19}$$

Note that, $Res_{x_i}(t) = 0$ and $Lim_{k\to\infty} Res_{x_i,k}(t) = Res_{x_i}(t)$, $i = 1,2,\dots,n$, for all $t \geq 0$. As a matter of fact, it yields the following algebraic system

$$D_t^{(k-1)\alpha} Res_{x_i,k}(t_0) = 0, \qquad k = 1,2,3,\dots, i = 1,2,\dots,n. \tag{20}$$

Through the algebraic system (20) we can obtained the coefficient $c_j . j = 1,2,3, \dots, k$ by applying the following steps.

Algorithm 1. Algorithm of finding the coefficients $\boldsymbol{c_{ij}}$ of the k-th truncated series (16).

Step 1: Substitute the k-th residual expansion (19) into (18), where

$$x_{ik}(t) = c_{i0} + \sum_{j=1}^{k} c_{ij}\,(t - t_0)^{j\alpha}, c_{ij} = \frac{D^{j\alpha} b_i}{\Gamma(j\alpha + 1)}. \tag{21}$$

Step 2: Find the formula $D_t^{(k-1)\alpha}\left(Res_{x_i,k}(t_0)\right), i = 1,2, \dots, n.$

Step 3: The coefficient c_{ij} can be obtained by solve the algebraic system

$$D_t^{(k-1)\alpha} Res_{x_i,k}(t_0) = 0, k = 1,2,3, \dots, i = 1,2, \dots, n. \tag{22}$$

Step 4: Substitute the obtained values of c_{ij} back into Eq. (17).

5. NUMERICAL IMPLEMENTATION AND RESULTS

This section is designed to apply the proposed technique, FRPS, and evaluate its performance and accuracy in deriving some numerical solutions for a number of examples and comparing the results obtained with the analytical solutions of the examples. We used the Mathematica software package to perform numerical and symbolic calculations.

Example 1. Consider the following NLP problem

$$\min u(x) = 100(x_1^2 - x_2)^2 + (x_1 - 1),$$

subject to

$$w(x) = x_1(x_1 - 4) - 2x_2 + 12 = 0. \tag{23}$$

According to (6), the correspond penalty function for the problem (23) at $\mu = 2$ is given by

$$P(x_1, x_2, \eta) = 100(x_1^2 - x_2)^2 + (x_1 - 1) + \frac{1}{2}\eta(x_1^2 - 4x_1 - 2x_2 + 12)^2, \tag{24}$$

where the penalty variable $\eta \in \mathbb{R}^+,\ \eta \to \infty$. Hence, we get the following correspond system of FDEs

$$D_t^{\alpha} x_1(t) = -400(x_1^2 - x_2)x_1 - 2(x_1 - 1)$$
$$-\eta(2x_1 - 4)(x_1^2 - 4x_1 - 2x_2 + 12),$$
$$D_t^{\alpha} x_2(t) = 200(x_1^2 - x_2) + 2\eta(x_1^2 - 4x_1 - 2x_2 + 12),$$
$$x_1(0) = x_2(0) = 0, \tag{25}$$

where $0 < \alpha \leq 1$.

To apply the FRPS technique to solve the system (25), we suppose that

$$x_1(t) = \sum_{j=0}^{\infty} c_{1j}\,(t-t_0)^{j\alpha}, x_2(t) = \sum_{j=0}^{\infty} c_{2j}\,(t-t_0)^{j\alpha}. \tag{26}$$

Thus, the k-th truncated series is

$$x_{1k}(t) = \sum_{j=1}^{k} c_{1j}\,(t-t_0)^{j\alpha}, x_{2k}(t) = \sum_{j=1}^{k} c_{2j}\,(t-t_0)^{j\alpha}. \tag{27}$$

Consequently, the k-th residual function, $Res_{x_i,k}(t)$, can be defined by

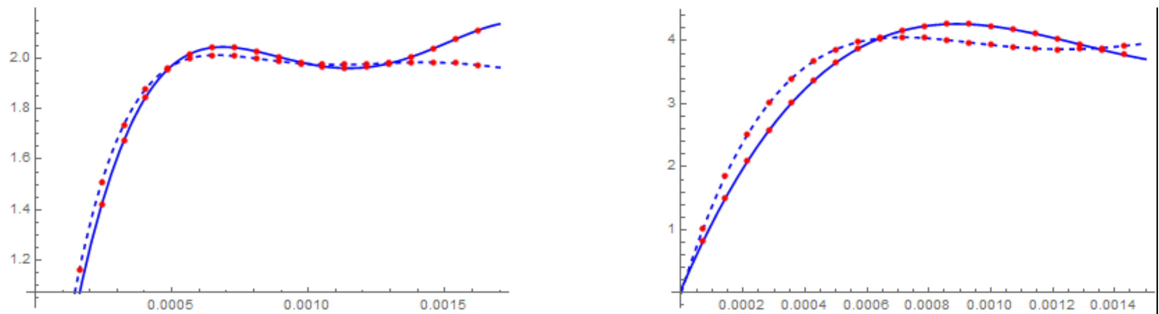
$$\begin{aligned} Res_{x_1,k}(t) &= D_t^{\alpha} x_{1k}(t) + 400(x_1^2 - x_2)x_1 + 2(x_1 - 1) \\ &\quad +\eta(2x_1 - 4)\,(x_1^2 - 4x_1 - 2x_2 + 12), \\ Res_{x_2,k}(t) &= D_t^{\alpha} x_{2k}(t) - 200(x_1^2 - x_2) - 2\eta(x_1^2 - 4x_1 - 2x_2 + 12), \end{aligned} \tag{28}$$

Finally, Algorithm 1 applied to find the coefficients c_{ij}, then getting the approximate solution of the system (25).

The efficiency of FRPS approach is introduced via establishing some numerical comparisons for the obtained results and the results obtained by Runge-Kutta approach and listed in Table 1, for problem (23), from this table we noted that the FRPS solutions are compatible with the exact solution more that the RK4 solutions. The geometric behavior of the FRPS approximate solutions are shown against the exact and RK4 solutions as in Figure 1. Clearly, the figure indicates that the exact and FRPS approximate solutions are in good agreement for different values of α, comparing with the exact and RK4 approximate solutions over the domain of α.

Table 1. Comparison of $x_1(t)$ and $x_2(t)$ for problem (23) between FRPS with RK4 solutions at $\eta = 200$ and $\alpha = 1$.

t	$FRPS\ x_1(t)$	$FRPS\ x_2(t)$	$RK4\ x_1(t)$	$RK4\ x_2(t)$	Absolute error $x_1(t)$	Absolute error $x_2(t)$
0.0000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.0005	1.971356	3.872061	1.970899	3.871887	0.000457	0.000174
0.0010	1.978467	3.908321	1.978274	3.907993	0.000193	0.000328
0.0013	1.98151	3.923481	1.981384	3.922554	0.000126	0.000927
0.0015	1.984216	3.932718	1.983132	3.930578	0.001084	0.00214
0.0020	1.9864	3.937884	1.984252	3.935654	0.002148	0.00223



(a) (b)

Figure 1. Comparison of $x_1(t)$ and $x_2(t)$ for problem (23) between FRPS with RK4 solutions at $\eta = 200$ and $\alpha = 1$: (a) $x_1(t)$, (b) $x_2(t)$, solidline: FRPS, dashline: RK4.

Example 2. Consider the following NLP problem:

$$\min u(x) = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^4 + (x_4 - x_5)^4,$$

subject to:

$$w_1(x) = x_1 + x_2^2 + x_3^3 - 2 - 3\sqrt{2} = 0$$

$$w_2(x) = x_2 - x_3^2 + x_4 + 2 - 2\sqrt{2} = 0,$$

$$w_3(x) = x_1 x_5 - 2 = 0 \qquad (29)$$

We define the penalty function for the problem (29) as

$$P(x_1, x_2, x_3, x_4, x_5, \eta) = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^4 + (x_4 - x_5)^4 + \frac{\eta}{2}\left[\left(x_1 + x_2^2 + x_3^3 - 2 - 3\sqrt{2}\right)^2 + \left(x_2 - x_3^2 + x_4 + 2 - 2\sqrt{2}\right)^2 + (x_1 x_5 - 2)^2\right], \qquad (30)$$

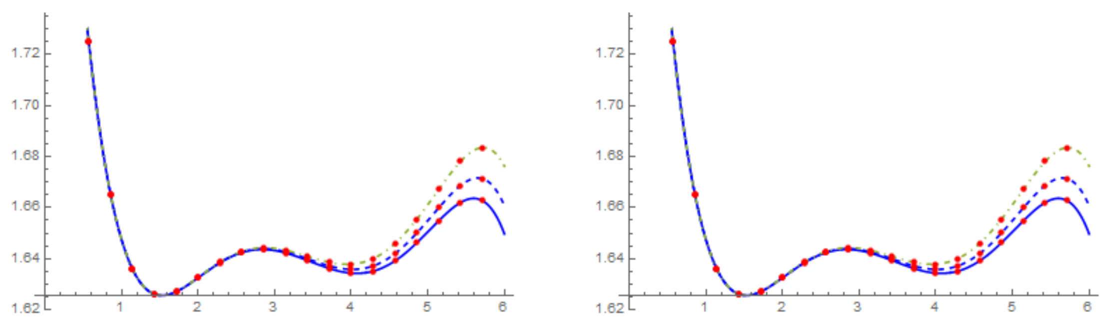
where the penalty variable $\eta \in \mathbb{R}^+$, $\eta \to \infty$. Depending on this penalty function, the correspond FDEs system can be written as

$$D_t^{\alpha} x_i(t) = \nabla_{x_i} u(x) + \eta \sum_{i=1}^{5} \nabla_{x_i} w_i(x)\, w_i(x), x_i(0) = 2, i = 1,2,3,4,5, \qquad (31)$$

where $0 < \alpha \leq 1$. We adapt the FRPS technique to the FDEs system (31) with penalty variable $\eta = 300$ at fractional order derivative $\alpha = 0.9$, we acquired solutions as shown in Table 2. Figure 2 present the obtained FRPS solutions $x_1(t)$ and $x_4(t)$ at various fractional derivative order α. Obviously, from Figure 2, the curves-FRPS approximate solutions consistent with each other and approach the exact curve with increasing fractional values to the integer-order value $\alpha = 1$.

Table 2. The obtained solution of NLP problem (29) by FRPS technique at $\eta = 300$ and $\alpha = 0.9$.

t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$	$x_5(t)$
0.000	2.00000	2.00000	2.00000	2.00000	2.00000
1.000	1.19855	1.36387	1.47429	1.64587	1.67875
2.000	1.19951	1.36981	1.47237	1.63658	1.67801
3.000	1.20181	1.36473	1.48531	1.63782	1.67947
4.000	1.21414	1.36588	1.48431	1.63842	1.68371
5.000	1.26941	1.36947	1.48997	1.64889	1.68753
6.000	1.27542	1.37842	1.49568	1.64998	1.68947



(a) (b)

Figure 2. The FRPS solutions: (a) $x_1(t)$, (b) $x_4(t)$; $\alpha = 0.9$ solid, $\alpha = 0.8$ dashed, and $\alpha = 0.7$ dot-dashed, at $\eta = 300$.

Example 3. Consider the following NLP problem:

$$min. u(t) = (x_1 - 20)^2 + (x_2 + 20)^2,$$

subject to:

$$w(t) = \frac{x_1^2}{100} + \frac{x_2^2}{4} - 1 = 0, \tag{32}$$

The penalty function of this problem can be written as

$$P(x_1, x_2, \eta) = (x_1 - 20)^2 + (x_2 + 20)^2 + \frac{\eta}{2}\left(\frac{x_1^2}{100} + \frac{x_2^2}{4} - 1\right)^2, \tag{33}$$

where the penalty variable $\eta \in \mathbb{R}^+$, $\eta \to \infty$. Therefore, we can write the correspond FDEs system as

$$D_t^{\alpha} x_1(t) = 2x_1 - 40 + \eta\left(\frac{x_1^3}{5000} + \frac{x_1 x_2^2}{200} - \frac{x_1}{50}\right),$$

$$D_t^{\alpha} x_2(t) = 2x_2 + 40 + \eta\left(\frac{x_1^2 x_2}{200} + \frac{x_2^3}{8} - \frac{x_2}{2}\right),$$

$$x_1(0) = x_2(0) = 0, \tag{34}$$

where $0 < \alpha \le 1$.

The FRPS utilize to construct the solution of this system and we get the following numerical solutions of the NLP problem (32) as shown in Table 3. It is very clear that the results obtained for example 3 indicated that the proposed method is simple, and its performance is very effective comparing with Runge-Kutta method.

Table 3. Comparison of $x_1(t)$ and $x_2(t)$ for problem (32) between FRPS with RK4 solutions $\eta = 10^6$ and $\alpha = 1$.

t	$FRPS\ x_1(t)$	$FRPS\ x_2(t)$	$RK\ x_1(t)$	$RK\ x_2(t)$	Error $x_1(t)$	Error $x_2(t)$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1.0	5.850119	−0.47496	5.847465	−0.477547	0.002654	0.002584
2.0	6.744776	−0.53968	6.741904	−0.542553	0.002872	0.002873
4.0	8.460532	−0.58119	8.457551	−0.584133	0.002981	0.002948
6.0	8.735192	−0.61163	8.732010	−0.615305	0.003182	0.003675
7.0	8.980451	−0.63668	8.977187	−0.640466	0.003264	0.003782
8.0	9.092537	−0.663578	9.090838	−0.661673	0.001699	0.001905
10.0	9.304238	−0.691245	9.303232	−0.680066	0.001006	0.018393

5. CONCLUSIONS

In this a novel paper, the corresponding system of FDEs for NLP was designed and analyzed under the meaning of Caputo derivative and then solved the target system via a modern efficient technique, named FRPS technique. The benefit of utilizing the present technique is that it gives accurate convergence approximate solution to an exact solution with needs a small size of computation to the optimal solutions to the NLP problems. Three attractive NLP are considered to validate the applicability and superiority of the FRPS scheme. Simulation data and graphical representations are discussed and indicated that the obtained results by FRPS approach are in good agreement with each other and with exact solution at various values of fractional derivative order α, as well as from the figures the results emphasized that obtained FRPS solutions more accurate from the obtained Runge-Kutta solutions versus the exact solution. Consequently, the presented technique has the ability to handle both linear and nonlinear fractional biological phenomena. In future works, the FRPS can be extended to generate accurate approximate solutions for systems of fractional partial differential equations.

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References

[1] W. Sun, Y.X. Yuan, Optimization Theory and Methods: Nonlinear Programming, Springer-Verlag, New York (2006).

[2] N. Özdemir, F. Evirgen, Solving NLP Problems With Dynamic System Approach Based on Smoothed Penalty Function, Selcuk J. Appl. Math. 10 (2009), 63–73. https://hdl.handle.net/20.500.12462/4852.

[3] H. Yamashita, A Differential Equation Approach to Nonlinear Programming, Math. Program. 18 (1980), 155–168. https://doi.org/10.1007/bf01588311.

[4] M. Al-Smadi, O.A. Arqub, D. Zeidan, Fuzzy Fractional Differential Equations Under the Mittag-Leffler Kernel Differential Operator of the ABC Approach: Theorems and Applications, Chaos Solitons Fractals. 146 (2021), 110891. https://doi.org/10.1016/j.chaos.2021.110891.

[5] A.V. Fiacco, G.P. Mccormick. Nonlinear Programming: Sequential Unconstrained Minimization Techniques, John Wiley, New York (1968).

[6] C.A. Botsaris, Differential Gradient Methods, J. Math. Anal. Appl. 63 (1978), 177–198. https://doi.org/10.1016/0022-247x(78)90114-2.

[7] M. Al-Smadi, Fractional Residual Series for Conformable Time-Fractional Sawada–Kotera–Ito, Lax, and Kaup–Kupershmidt Equations of Seventh Order, Math Methods Appl. Sci. (2021). https://doi.org/10.1002/mma.7507.

[8] A.A. Brown, M.C. Bartholomew-Biggs, ODE Versus SQP Methods for Constrained Optimization, J. Optim. Theory Appl. 62 (1989), 371–386. https://doi.org/10.1007/bf00939812.

[9] J. Schropp, I. Singer, A Dynamical Systems Approach to Constrained Minimization, Numer. Funct. Anal. Optim. 21 (2000), 537–551. https://doi.org/10.1080/01630560008816971.

[10] S. Wang, X.Q. Yang, K.L. Teo, A Unified Gradient Flow Approach to Constrained Nonlinear Optimization Problems, Comput. Optim. Appl. 25 (2003), 251–268. https://doi.org/10.1023/a:1022973608903.

[11] M. Al-Smadi, O.A. Arqub, Computational Algorithm for Solving Fredholm Time-Fractional Partial Integrodifferential Equations of Dirichlet Functions Type With Error Estimates, Appl. Math. Comput. 342 (2019), 280–294. https://doi.org/10.1016/j.amc.2018.09.020.

[12] L. Jin, L.W. Zhang, X.T. Xiao, Two Differential Equation Systems for Equality-Constrained Optimization, Appl. Math. Comput. 190 (2007), 1030–1039. https://doi.org/10.1016/j.amc.2006.11.041.

[13] M. Al-Smadi, A. Freihat, H. Khalil, S. Momani, R. Ali Khan, Numerical Multistep Approach for Solving Fractional Partial Differential Equations, Int. J. Comput. Methods. 14 (2017), 1750029. https://doi.org/10.1142/s0219876217500293.

[14] N. Andrei, Gradient Flow Algorithm for Unconstrained Optimization, ICI Technical Report, 2004.

[15] L. Jin, A Stable Differential Equation Approach for Inequality Constrained Optimization Problems, Appl. Math. Comput. 206 (2008), 186–192. https://doi.org/10.1016/j.amc.2008.09.007.

[16] N. Özdemir, F. Evirgen, A Dynamic System Approach to Quadratic Programming Problems With Penalty Method, Bull. Malays. Math. Sci. Soc. (2) 33 (2010), 79–91.

[17] S. Hasan, M. Al-Smadi, A. El-Ajou, et al. Numerical Approach in the Hilbert Space to Solve a Fuzzy Atangana-Baleanu Fractional Hybrid System, Chaos Solitons Fractals. 143 (2021) 110506. https://doi.org/10.1016/j.chaos.2020.110506.

[18] A. Qazza, R. Hatamleh, M. Al-hawari, Dirichlet Problem in the Simply Connected Domain, Bounded by Unicursal Curve, Int. J. Appl. Math. 22 (2009), 599-614.

[19] A. Qazza, R. Hatamleh, Dirichlet Problem in the Simply Connected Domain, Bounded by the Nontrivial Kind, Adv. Differ. Equ. Control Processes, 17 (2016), 177-188.

[20] R. Saadeh, M. Alaroud, M. Al-Smadi, et al. Application of Fractional Residual Power Series Algorithm to Solve Newell–Whitehead–Segel Equation of Fractional Order, Symmetry. 11 (2019), 1431. https://doi.org/10.3390/sym11121431.

[21] R. Edwan, R. Saadeh, S. Hadid, M. Al-Smadi, S. Momani, Solving Time-Space-Fractional Cauchy Problem with Constant Coefficients by Finite-Difference Method, in: D. Zeidan, S. Padhi, A. Burqan, P. Ueberholz (Eds.), Computational Mathematics and Applications, Springer Singapore, Singapore, 2020: pp. 25–46. https://doi.org/10.1007/978-981-15-8498-5_2.

[22] A. Burqan, A. El-Ajou, R. Saadeh, M. Al-Smadi, A New Efficient Technique Using Laplace Transforms and Smooth Expansions to Construct a Series Solution to the Time-Fractional Navier-Stokes Equations, Alexandria Eng. J. 61 (2022), 1069–1077. https://doi.org/10.1016/j.aej.2021.07.020.

[23] R. Saadeh, A. Qazza, A. Burqan, A New Integral Transform: ARA Transform and Its Properties and Applications, Symmetry. 12 (2020), 925. https://doi.org/10.3390/sym12060925.

[24] M. Al-Smadi, N. Djeddi, S. Momani, et al. An attractive numerical algorithm for solving nonlinear Caputo-Fabrizio fractional Abel differential equation in a Hilbert space, Adv. Differ. Equ. 2021 (2021), 271. https://doi.org/10.1186/s13662-021-03428-3.

[25] S.A. El-Wakil, A. Elhanbaly, M.A. Abdou, Adomian Decomposition Method for Solving Fractional Nonlinear Differential Equations, Appl. Math. Comput. 182 (2006), 313-324. https://doi.org/10.1016/j.amc.2006.02.055.

[26] M. Al-Smadi, H. Dutta, S. Hasan, S. Momani, On Numerical Approximation of Atangana-Baleanu-Caputo Fractional Integro-Differential Equations Under Uncertainty in Hilbert Space, Math. Model. Nat. Phenom. 16 (2021), 41. https://doi.org/10.1051/mmnp/2021030.

[27] A. Khan, A. Khan, M. Sinan, Ion Temperature Gradient Modes Driven Soliton and Shock by Reduction Perturbation Method for Electron-Ion Magneto-Plasma, Math. Model. Numer. Simul. Appl. 2 (2022), 1-12. https://doi.org/10.53391/mmnsa.2022.01.001.

[28] F. Evirgen, Conformable Fractional Gradient Based Dynamic System for Constrained Optimization Problem, Acta Phys. Pol. A. 132 (2017), 1066–1069. https://doi.org/10.12693/aphyspola.132.1066.

[29] F. Evirgen, M. Yavuz, An Alternative Approach for Nonlinear Optimization Problem With Caputo - Fabrizio Derivative, ITM Web Conf. 22 (2018), 01009. https://doi.org/10.1051/itmconf/20182201009.

[30] A. Alvarado-Sánchez, E. Rangel-Cortes, A. Hernández-Hernández, Bi-Dimensional Crime Model Based on Anomalous Diffusion With Law Enforcement Effect, Math. Model. Numer. Simul. Appl. 2 (2022), 26–40. https://doi.org/10.53391/mmnsa.2022.01.003.

[31] O. Abu Arqub. Series Solution of Fuzzy Differential Equations Under Strongly Generalized Differentiability, J. Adv. Res. Appl. Math. 5 (2013), 31-52. https://doi.org/10.5373/jaram.1447.051912.

[32] M. Alaroud, M. Al-Smadi, R.R. Ahmad, et al. Computational Optimization of Residual Power Series Algorithm for Certain Classes of Fuzzy Fractional Differential Equations, Int. J. Differ. Equ. 2018 (2018), 8686502. https://doi.org/10.1155/2018/8686502.

[33] S. Hasan, M. Al-Smadi, H. Dutta, et al. Multi-Step Reproducing Kernel Algorithm for Solving Caputo–Fabrizio Fractional Stiff Models Arising in Electric Circuits, Soft Comput. 26 (2022), 3713–3727. https://doi.org/10.1007/s00500-022-06885-4.

[34] S. Momani, N. Djeddi, M. Al-Smadi, S. Al-Omari, Numerical Investigation for Caputo-Fabrizio Fractional Riccati and Bernoulli Equations Using Iterative Reproducing Kernel Method, Appl. Numer. Math. 170 (2021), 418-434. https://doi.org/10.1016/j.apnum.2021.08.005.

[35] M. Al-Smadi, O. Abu Arqub, S. Momani, Numerical Computations of Coupled Fractional Resonant Schrödinger Equations Arising in Quantum Mechanics Under Conformable Fractional Derivative Sense, Physica Scripta, 95 (2020), 075218. https://doi.org/10.1088/1402-4896/ab96e0.

[36] A. Elsaid. Fractional Differential Transform Method Combined With the Adomian Polynomials, Appl. Math. Comput. 218 (2012), 6899-6911. https://doi.org/10.1016/j.amc.2011.12.066.

[37] A. Freihet, S. Hasan, M. Al-Smadi, et al. Construction of Fractional Power Series Solutions to Fractional Stiff System Using Residual Functions Algorithm, Adv. Differ. Equ. 2019 (2019), 95. https://doi.org/10.1186/s13662-019-2042-3.

[38] M. Al-Smadi, O. Abu Arqub, S. Hadid. An Attractive Analytical Technique for Coupled System of Fractional Partial Differential Equations in Shallow Water Waves With Conformable Derivative, Commun. Theor. Phys. 72 (2020), 085001. https://doi.org/10.1088/1572-9494/ab8a29.

[39] M. Al-Smadi, O. Abu Arqub, S. Hadid, Approximate solutions of nonlinear fractional Kundu-Eckhaus and coupled fractional massive Thirring equations emerging in quantum field theory using conformable residual power series method, Physica Scripta, 95 (2020), 105205. https://doi.org/10.1088/1402-4896/abb420.

[40] M. Al-Smadi, A. Freihat, O. Abu Arqub, et al. A Novel Multistep Generalized Differential Transform Method for Solving Fractional-Order Lü Chaotic and Hyperchaotic Systems, J. Comput. Anal. Appl. 19 (2015), 713–724.

[41] M. Alabedalhadi, M. Al-Smadi, S. Al-Omari, et al. New Optical Soliton Solutions for Coupled Resonant Davey-Stewartson System With Conformable Operator, Opt. Quant. Electron. 54 (2022), 392. https://doi.org/10.1007/s11082-022-03722-8.

[42] I. Podlubny, Fractional Differential Equation, Academic Press, San Diego, 1999.

[43] M. Alabedalhadi, M. Al-Smadi, S. Al-Omari, et al. Structure of Optical Soliton Solution For Nonliear Resonant Space-Time Schrödinger Equation In Conformable Sense With Full Nonlinearity Term, Physica Scripta, 95 (2020), 105215. https://doi.org/10.1088/1402-4896/abb739.

[44] Z. Altawallbeh, M. Al-Smadi, I. Komashynska, A. Ateiwi, Numerical Solutions of Fractional Systems of Two-Point BVPs by Using the Iterative Reproducing Kernel Algorithm, Ukr. Math J. 70 (2018), 687–701. https://doi.org/10.1007/s11253-018-1526-8.

[45] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., Amsterdam, 2006.

[46] S. Hasan, M. Al-Smadi, A. Freihet, et al.Two Computational Approaches for Solving a Fractional Obstacle System in Hilbert Space, Adv. Differ. Equ. 2019 (2019), 55. https://doi.org/10.1186/s13662-019-1996-5.

[47] S. Momani, O. Abu Arqub, A. Freihat, et al. Analytical Approximations for Fokker-Planck Equations of Fractional Order in Multistep Schemes, Appl. Comput. Math. 15 (2016), 319-330.

[48] M. Al-Smadi, O. Abu Arqub, M. Gaith, Numerical Simulation of Telegraph and Cattaneo Fractional-Type Models Using Adaptive Reproducing Kernel Framework, Math. Meth. Appl. Sci. 44 (2020), 8472-8489. https://doi.org/10.1002/mma.6998.

[49] R. Abu-Gdairi, M. Al-Smadi, G. Gumah, An Expansion Iterative Technique for Handling Fractional Differential Equations Using Fractional Power Series Scheme, J. Math. Stat. 11 (2015), 29-38. https://doi.org/10.3844/jmssp.2015.29.38.