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On the Frictional Contact Problem of $p(x)$-Kirchhoff Type

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Abstract. In this article we consider a class of frictional contact problem of $p(x)$-Kirchhoff type, on a bounded domain $\Omega \subseteq \mathbb{R}^2$. Using an abstract Lagrange multiplier technique and the Schauder's fixed point theorem we establish the existence of weak solutions. Furthermore, we also obtain the uniqueness of the solution assuming that the datum f_1 satisfies a suitable monotonicity condition.

1. Introduction

The purpose of this work is to investigate the existence of weak solutions for the boundary value problem

$$
\begin{aligned}
-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)div\Big(|\nabla u|^{p(x)-2}\nabla u\Big) &= f_1(x,u) \quad &&\text{in} \quad \Omega\\
u &= 0 &&\text{on} \quad \Gamma_1\\
M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= f_2(x) &&\text{on} \quad \Gamma_2\\
\Big|M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}\Big| &\le g(x), &&\\
M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= -g\frac{u}{|u|}, \quad \text{if} \quad u\neq 0 &&\text{on} \quad \Gamma_3
\end{aligned}
\tag{1.1}
$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth enough boundary Γ, partitioned in three parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that meas $(\Gamma_i) > 0$, $(i = 1, 2, 3)$; $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$, $f_2 : \Gamma_2 \to \mathbb{R}$, $g : \Gamma_3 \to \mathbb{R}$ and $M : [0, +\infty[\to [m_0, +\infty[$ are given functions, $p \in C(\overline{\Omega})$.

The study of the $p(x)$- Kirchhoff type equations with nonlinear boundary conditions of different class

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have attracted expensive attention in recent years, we refer to some interesting works [1,6,13,16] and references therein. One reason of such interest is that various real fields require PDE problems with variable exponent, for example, electrorheological fluids and image restoration. The other reason is that the nonlocal problems with variable exponent, in addition to their contributions to the modeling of many physical and biological phenomena, raise greater mathematical difficulties due to their nonlinearities; see for example [2,15,19]. Cojocaru-Matei [5] studied the unique solvability of problem (1.1) in the case $M(s) = 1$, $f_1(x,u) \equiv f_1(x)$, $p =$ constant ≥ 2, which models the antiplane shear deformation of a nonlinearly elastic cylindrical body in frictional contact on Γ_3 with a rigid foundation; see, e.g. [18]. They used a technique involving dual Lagrange multipliers, this allow to write efficient algorithms to approximate the weak solutions; see [14]. For our situation, the behavior of the material is described by the Hencky-type constitutive law:

$$\sigma(x) = ktr\varepsilon(u(x))I_3 + \mu(x)\|\varepsilon^D(u(x))\|^{\frac{p(x)-2}{2}}\varepsilon^D(u(x))$$

where σ is the Cauchy stress tensor, tr is the trace of a Cartesian tensor of second order,$\sigma(x)$ ε is the infinitesimal strain tensor, u is the displacement vector,I_3 is the identity tensor, k,μ are material parameters, p is a given function;ε^D is the *desviator* of the tensor ε defined by $\varepsilon^D = \varepsilon - \frac{1}{3}(tr\varepsilon)I_3$ where $tr\varepsilon = \sum_{i=1}^{3}\varepsilon_{ii}$; see for instance [12]. If, the Lamé coefficient is given by

$$\mu = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)$$

we obtain our mechanical problem (1.1).

Thanks to the above mentioned research articles, we consider the problem (1.1), under appropriate assumptions on M and f_1, and establish the existence of a unique weak solution of this problem via Lagrange multipliers and the Schauder fixed point theorem. In this sense, we generalize the main result in [5]. Also, we state a simple uniqueness result under suitable monotonicity condition on f_1.

The paper is designed as follows. In Section 2, we introduce the mathematical preliminaries and give several important properties of $p(x)$-Kirchhoff-Laplace operator. We deliver a weak variational formulation with Lagrange multipliers in a dual space. Section 3, is devoted to the proofs of main results.

2. Preliminaries

For the reader's convenience, we point out some basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. In this context we refer the reader to [8,17] for details. Firstly we state some basic properties of spaces $W^{1,p(x)}(\Omega)$ which will be used later. Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$

$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([11]). *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*

Proposition 2.2 ([11]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$. For any $u \in L^{p(x)}(\Omega)$, then*

1. *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
2. *$|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*
3. *if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;*
4. *if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
5. *$\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;*
6. *$\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.*

Proposition 2.3 ([9, 11]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.4 ([11]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$|\int_\Omega uv\,dx| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

We introduce the following closed space of $W^{1,p(x)}(\Omega)$

$$X = \{v \in W^{1,p(x)}(\Omega) : \gamma u = 0 \quad \text{a. e. on} \quad \Gamma_1\} \tag{2.1}$$

where γ denotes the Sobolev trace operator and $\Gamma_1 \subseteq \Gamma$, meas $(\Gamma_1) > 0$, therefore X is a separable reflexive Banach space. Now, we denote

$$\|u\|_X = |\nabla u|_{p(x)}, \quad u \in X.$$

This functional represents a norm on X.

Proposition 2.5 ([3]). *There exists $c > 0$ such that*

$$\|u\|_{1,p(x)} \leq C\|u\|_X \quad \text{for all } u \in X.$$

Then, the norms $\|.\|_X$ and $\|.\|_{1,p(x)}$ are equivalent on X.

We write

$$L(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx.$$

Proposition 2.6. *The functional $L : X \to \mathbb{R}$ is convex. The mapping $L' : X \to X'$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty} L'(u_n)(u_n - u) \leq 0 \text{ implies } u_n \to u,$$

where X' is the dual space of X.

Proof. This result is obtained in a similar manner as the one given in [10], thus we omit the details. $\square$

Now, we define the spaces

$$S = \Big\{ u \in W^{\frac{1}{p'(x)},p(x)}(\Gamma) : \exists v \in X \quad \text{such that} \quad u = \gamma v \quad \text{a.e on} \quad \Gamma \Big\} \tag{2.2}$$

which is a real reflexive Banach space, $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ for all $x \in \Omega$, and

$$Y = S', \text{ the dual of the space S.} \tag{2.3}$$

Let us introduce a bilinear form

$$b : X \times Y \longrightarrow \mathbb{R} \quad : b(v, \mu) = \langle\, \mu, \gamma v \,\rangle_{Y\times S}\,, \tag{2.4}$$

a Lagrange multiplier $\lambda \in Y$,

$$\langle\, \lambda, z \,\rangle = -\int_{\Gamma_3} M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}z d\Gamma \quad , \quad \forall z \in S$$

and the set of Lagrange multipliers

$$\Lambda = \Big\{ u \in Y : \langle\, \mu, z \,\rangle \leqslant \int_{\Gamma_3} g(x)|z(x)| \quad , \quad \forall z \in S \Big\}. \tag{2.5}$$

From $(1.1)_4$ we deduce that $\lambda \in \Lambda$.

Let u be a regular enough function satisfying problem (1.1). After some computations we get (by using density results)

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla v dx = \int_\Omega f_1(x,u)v dx$$

$$+\int_{\Gamma_2} f_2(x)\gamma v d\Gamma + M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\int_{\Gamma_3}|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}\gamma v d\Gamma \tag{2.6}$$

for all $v \in X$, where u satisfies $(1.1)_5$ on Γ_3

Now, we write problem (2.6) as an abstract mixed variational problem (by means a Lagrange multipliers technique)

We define the following operators:

$$\begin{aligned} & i)\ A : X \to X', \text{given by} \\ & \langle\ Au, v\ \rangle = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla v dx,\ \ u, v \in X. \\ & ii)\ F : X \to X', \text{given by} \\ & \langle\ F(u), v\ \rangle = \int_\Omega f_1(x, u)v dx + \int_{\Gamma_2} f_2(x)\gamma v dx \quad , \quad u, v \in X. \end{aligned} \tag{2.7}$$

So, we are led to the following variational formulation of problem (1.1)

Problem 1. Find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} \langle\ Au, v\ \rangle + b(v, \lambda) &= \langle\ F(u), v\ \rangle \quad , \quad \forall v \in X \\ b(u, \mu - \lambda) &\leq 0 \quad \forall \mu \in \Lambda \subseteq Y \end{aligned} \tag{2.8}$$

To solve this problem, we will apply the Schauder fixed point theorem.

Firstly, we "freeze" the state variable u on the function F, that is we fix $w \in X$ such that $f = F(w) \in X'$.

So, we are led to the following abstract mixed variational problem.

Problem 2. Given $f \in X'$ find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} \langle\ Au, v\ \rangle + b(v, \lambda) &= \langle\ f, v\ \rangle \quad , \quad \forall v \in X \\ b(u, \mu - \lambda) &\leq 0 \quad \forall \mu \in \Lambda \subseteq Y. \end{aligned} \tag{2.9}$$

The unique solvability of Problem 2 is given under the following generalized assumptions.

Let $(X, \|\|_X)$ and $(Y, \|\|_Y)$ be two real reflexive Banach space.

- **(B_1):** $A : X \to X'$ is hemicontinuous;
- **(B_2):** $\exists h : X \to \mathbb{R}$ such that
  - **(a)** $h(tw) = t^\gamma h(w)$ with $\gamma > 1$, $\forall t > 0, w \in X$;
  - **(b)** $\langle\ Au - Av, u - v\ \rangle_{X\times X} \geq h(v - u),\ \forall u, v \in X$;
  - **(c)** $\forall (x_\nu) \subseteq X : x_\nu \rightharpoonup x \text{ in} X \Longrightarrow h(x) \leq \lim\limits_{\nu\to\infty} \sup h(x_\nu)$
- **(B_3):** A is coercive.
- **(B_4):** The form $b : X \times Y$ es bilinear, and
  - **(i)** $\forall (u_\nu) \subseteq X : u_\nu \rightharpoonup u$ in $X \Longrightarrow b(u_\nu, \lambda_\nu) \to b(u, \lambda)$
  - **(ii)** $\forall (\lambda_\nu) \subseteq Y : \lambda_\nu \rightharpoonup y$ in $Y \Longrightarrow b(v_\nu, \lambda_\nu) \to b(v, \lambda)$
  - **(iii)** $\exists \widehat{\alpha} > 0 : \inf\limits_{\substack{\mu\in I \\ u\neq 0}} \sup\limits_{\substack{v\in X \\ v\neq 0}} \dfrac{b(v, \mu)}{|v|_X|\mu|_Y} \geq \widehat{\alpha}$

(B_5): Λ is a bounded closed convex subset of Y such that $0_Y \in \Lambda$.

(B_6): $\exists C_1 > 0, q > 0 : h(v) \geq C_1 \|v\|_X^q \quad , \quad \forall v \in X$.

Theorem 2.1. *Assume (B_1) - (B_6). Then there exists a unique solution $(u, \lambda) \in X \times \Lambda$ of Problem 2.*

Proof. See [5].

To solve Problem 1, we start by stating the following assumptions on M , f_1 , f_2 and g

(A_1) $M : [0, +\infty[\to [m_0, +\infty[$ is a locally Lipschitz-continuous and nondecreasing function; $m_0 > 0$.

(A_2) $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying

$$|f_1(x,t)| \leq c_1 + c_2 |t|^{\alpha(x)-1} \ , \ \forall (x,t) \in \Omega \times \mathbb{R},$$

$\alpha \in C_+(\overline{\Omega})$with $\alpha(x) < p^*(x), \ \alpha^+ < p^-$.

(A_3) $f_2 \in L^{p'(x)}(\Gamma_2), \ g \in L^{p'(x)}(\Gamma_3), \ g(x) \geq 0$ a.e on Γ_3.

We have the following properties about the operator A.

Proposition 2.7. *If (A_1) holds, then*

(i) *A is locally Lipschitz continuous.*

(ii) *A is bounded, strictly monotone. Furthermore*

$$\langle Au - Av, u - v \rangle \geq k_p \|u - v\|_X^{\hat{p}}$$

where

$$\hat{p} = \begin{cases} p^- & \text{if } \|u - v\|_X > 1, \\ p^+ & \text{if } \|u - v\|_X \leq 1. \end{cases}$$

So, we can take $h(v) = k_p \|v\|_X^{\hat{p}}$.

(iii) $\frac{\langle Au, u \rangle}{\|u\|_X} \to +\infty \quad$ *as $\|u\|_X \to +\infty$.*

Proof. (i) Assume that M is Lipschitz in $[0, R_1]$ with Lipschitz constant L_M, $R_1 > 0$. We have, for $u, v, w \in B(0, R_1)$

$$\begin{aligned} \langle Au - Av, w \rangle =& [M(L(u)) - M(L(v))] \int_\Omega |\nabla u|^{p(x)-2} \nabla u . \nabla v \, dx \\ &+ M(L(v)) \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v \right) . \nabla w \, dx. \end{aligned}$$

Using the Lipschitz continuity of M, the Holder inequality and the inequality $\langle ||x|^{\alpha-2}x - |y|^{\alpha-2}y, x - y \rangle| \leq c|x-y| \left(|x| + |y| \right)^{\alpha-2}, \quad \forall x, y \in \mathbb{R}^n, 2 \leq \alpha < +\infty$, we get

$$|\langle Au - Av, w \rangle| \leq C \|u - v\|_X \|w\|_X,$$

which implies $\|Au - Av\|_{X'} \leq C \|u - v\|_X$.

ii)The functional $S : X \to X'$ defined by

$$\langle Su, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla v\, dx, \qquad \forall u, v \in X, \tag{2.10}$$

is bounded (See [10]). Then

$$\langle Su, v\rangle = M(L(u))\langle Su, v\rangle \qquad \forall u, v \in X. \tag{2.11}$$

Hence, since M is continuous and L is bounded (see Proposition 2.6), A is bounded.

To obtain that A is strictly monotone we develop the same arguments to those in [7], we omit it.

To establish the inequality in ii), we apply Lemma 3 in [4] to obtain

$$\begin{aligned}\langle Au - Av, u - v\rangle \geq &\int_\Omega \Big(M(L(u))|\nabla u|^{p(x)-2}\nabla u - M(L(v))|\nabla v|^{p(x)-2}\nabla v\Big).(\nabla v - \nabla u)\, dx\\ \geq & m_0 \int_\Omega \frac{1}{p(x)}(|\nabla u - \nabla u|^{p(x)})\, dx \geq \frac{m_0}{p^+}\int_\Omega |\nabla u - \nabla u|^{p(x)}\, dx\\ \geq & \frac{m_0}{p^+}\|u - v\|_X^{\hat{p}}.\end{aligned}$$

iii)For $u \in X$ with $\|u\|_X > 1$ we have

$$\begin{aligned}\frac{\langle Au, u\rangle}{\|u\|_X} = &\frac{M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\Big)\int_\Omega |\nabla u|^{p(x)}\, dx}{\|u\|}\\ \geq & m_0\|u\|_X^{p^- -1} \to +\infty \text{ as } \|u\|_X \to +\infty.\end{aligned}$$

□

Proposition 2.8. *The form $b : X \times Y \to \mathbb{R}$ defined in (2.4) is bilinear and, it verifies i), ii) and iii) in assumption (B_4). Moreover*

$$b(u, \mu) \leq \int_{\Gamma_3} g(x)|u(x)|\, d\Gamma \text{ for all } \mu \in \Lambda; \tag{2.12}$$

$$b(u, \lambda) = \int_{\Gamma_3} g(x)|u(x)|\, d\Gamma \tag{2.13}$$

$$b(u, \mu - \lambda) \leq 0 \quad \text{for all } \mu \in \Lambda. \tag{2.14}$$

Moreover, Λ is a bounded closed convex subset of Y such that $0_Y \in \Lambda$.

Proof. The assertions i), ii), iii) and Λ bounded are word for word as [5], Theorem 3, pags 138-139.

It is obvious to check (2.12). To justify (2.13), we have to show that, a.e. $x \in \Omega$

$$-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u(x)|^{p(x)-2}\frac{\partial u(x)}{\partial \nu}u(x) = g(x)|u(x)|$$

In fact, let $x \in \Omega$. If $|u(x)| = 0$, then

$$-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\Big)|\nabla u(x)|^{p(x)-2}\frac{\partial u(x)}{\partial \nu}u(x) = 0 = g(x)|u(x)| \text{ on } \Gamma_3.$$

Otherwise, if $|u(x)| \neq 0$,then

$$\begin{aligned} -M(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx)|\nabla u(x)|^{p(x)-2}\frac{\partial u(x)}{\partial \nu}u(x) =& g(x)\frac{(u(x))^2}{|u(x)|} \\ =& g(x)|u(x)| \text{ on } \Gamma_3 \end{aligned}$$

Furthermore, for all $\mu \in \Lambda$:

$$b(u, \mu - \lambda) = b(u, \mu) - b(u, \lambda) = \langle\, \mu, \gamma u \,\rangle_{Y\times S} - \langle\, \lambda, \gamma u \,\rangle_{Y\times S}\,. \tag{2.15}$$

Hence, thanks to (2.12), (2.13) and (2.15), we obtain (2.14). □

3. Existence and uniqueness of solutions

We are ready to solve problem 1. For this, we consider the Banach spaces X and Y given in (2.1) and (2.3) respectively, the form $b : X \times Y \to \mathbb{R}$ defined in (2.4) and the set Λ in (2.5).

Theorem 3.1. *Suppose $(A_1) - (A_3)$ hold. Then problem 1 admits a solution $(u, \lambda) \in X \times \Lambda$.*

Proof. We apply the Schauder fixed point theorem.

As has been said before, we "freeze" the state variable u on the function F , that is, we fix $w \in X$ and consider the problem:

Find $u \in X$ and $\lambda \in \Lambda$ such that

$$\langle\, Au, v \,\rangle + b(v, \lambda) \;=\; \langle\, f, v \,\rangle \quad , \quad \forall v \in X \tag{3.1}$$

$$b(u, \mu - \lambda) \;\leq\; 0 \quad \forall \mu \in \Lambda \subseteq Y. \tag{3.2}$$

with $f = F(w) \in X'$. Note that by the hypotheses on α and f_1, given in (A_2), we have $f_1(w) \in L^{\alpha'(x)}(\Omega) \hookrightarrow X'$.

By theorem (2.1), problem (3.1)-(3.2) has a unique solution $(u_w, \lambda_w) \in X \times \Lambda$.

Here we drop the subscript w for simplicity. Setting $v = u$ in (3.1) and $\mu = 0_Y$ in (3.2), using proposition 2.7 ii), we get

$$k_p\|u\|_X^{\hat{p}} \leq (2C_1C_\alpha\|w\|_X^{\sigma} + 2C_2C_\alpha|\Omega| + c_p|f_2|_{p'(x),\Gamma_2})\|u\|_X \tag{3.3}$$

where

$$\sigma = \begin{cases} \alpha^- & \text{if } \|w\|_X > 1, \\ \alpha^+ & \text{if } \|w\|_X \leq 1, \end{cases}$$

and C_χ is the embedding constant of $X \hookrightarrow L^{\chi(x)}(\Omega)$.

Then

$$\|u\|_X \leq [C(1 + \|w\|_X)]^{\frac{1}{\hat{p}-1}}\,.$$

Therefore, either $\|u\|_X \leq 1$ or

$$\|u\|_X \leq [C(1 + \|w\|_X)]^{\frac{1}{p^- - 1}}\,. \tag{3.4}$$

Since $p^- > \alpha^+ + 1$, we have

$$t^{p^- - 1} - Ct^{\sigma} - C \to +\infty \qquad \text{as } t \to +\infty$$

Hence, there is some $\bar{R}_1 > 0$ such that

$$\bar{R_1}^{p^- - 1} - C\bar{R_1}^{\sigma} - C \geq 0 \tag{3.5}$$

From (3.4) and (3.5) we infer that if $\|w\|_X \leq \bar{R}_1$ then $\|u\|_X \leq \bar{R}_1$.

Thus there exists $R_1 = \min\{1, \bar{R}_1\}$ such that

$$\|u\|_X \leq R_1 \quad \text{for all } u \in X. \tag{3.6}$$

For this constant, define K as

$$K = \{v : v \in L^{\alpha(x)}(\Omega), \|v\|_X \leq R_1\}$$

which is a nonempty, closed, convex subset of $L^{\alpha(x)}(\Omega)$. We can define the operator

$$T : K \to L^{\alpha(x)}(\Omega), \qquad Tw = u_w$$

where u_w is the first component of the unique pair solution of the problem (3.1)-(3.2), $(u_w, \lambda_w) \in X \times \Lambda$

From (3.6) $\|Tw\|_X \leq R_1$, for every $w \in K$, so that $T(K) \subseteq K$.

Moreover, if $(u_\nu)_{\nu \geq 1}$ $(u_{w_\nu} \equiv u_\nu)$ is a bounded sequence in K, then from (3.6) is also bounded in X. Consequently, from the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, $(Tw_\nu)_{\nu \geq 1}$ is relatively compact in $L^{\alpha(x)}(\Omega)$ and hence, in K.

To prove the continuity of T , let $(w_\nu)_{\nu \geq 1}$ be a sequence in K such that

$$w_\nu \to w \quad \text{strongly in} L^{\alpha(x)}(\Omega) \tag{3.7}$$

and suppose $u_\nu = Tw_\nu$. The sequence $\{(u_\nu, \lambda_\nu)\}_{\nu \geq 1}$ satisfies

$$\begin{aligned} \langle\ Au_\nu, v\ \rangle + b(v, \lambda_\nu) \quad &= \quad \langle\ F(w_\nu), v\ \rangle \quad , \quad \forall v \in X \\ b(u_\nu, \mu - \lambda_\nu) \quad &\leq \quad 0 \quad \forall \mu \in \Lambda. \end{aligned}$$

Using (3.6)-(3.7) we can extract a subsequence (u_{ν_k}) of (u_ν) and a subsequence (w_{ν_k}) of (w_ν) such that

$$\begin{gathered} u_{\nu_k} \to u^* \text{weakly in} X, \\ u_{\nu_k} \to u^* \text{ strongly in } L^{\alpha(x)}(\Omega) \text{ and a.e. in } \Omega, \\ w_{\nu_k} \to w \quad \text{a.e. in } \Omega, \\ L(u_{\nu_k}) \to t_0, \quad \text{for some } t_0 \geq 0, \end{gathered} \tag{3.8}$$

and in view of continuity of M

$$M(L(u_{\nu_k})) \to M(t_0). \tag{3.9}$$

We shall show that $u^* = Tw$. To this end, by choosing $u_{\nu_k} - u^*$ as a test function, we have

$$\begin{aligned} \langle\, Au_{\nu_k}, u_{\nu_k} - u^* \,\rangle + b(u_{\nu_k} - u^*, \lambda_\nu) &= \langle\, F(w_{\nu_k}), u_{\nu_k} - u^* \,\rangle \\ \langle\, Au^*, u_{\nu_k} - u^* \,\rangle + b(u_{\nu_k} - u^*, \lambda^*) &= \langle\, F(w), u_{\nu_k} - u^* \,\rangle . \end{aligned} \tag{3.10}$$

Then

$$\begin{aligned} &[M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega |\nabla u^*|^{p(x)-2} \nabla u^* .(\nabla u_{\nu_k} - \nabla u^*)\, dx + \\ &M(L(u_{\nu_k})) \int_\Omega (|\nabla u^*|^{p(x)-2} \nabla u^* - |\nabla u_{\nu_k}|^{p(x)-2} \nabla u_{\nu_k}).(\nabla u_{\nu_k} - \nabla u^*)\, dx + \\ &b(u_{\nu_k} - u^*, \lambda^* - \lambda_{\nu_k}) = \langle\, F(w) - F(w_{\nu_k}), u_{\nu_k} - u^* \,\rangle . \end{aligned} \tag{3.11}$$

Since $b(u_{\nu_k} - u^*, \lambda^* - \lambda_{\nu_k}) \geq 0$, by the inequality $|x|^{p-2}x - |y|^{p-2}y \geq C|x - y|^p$, $p \geq 2$, we obtain

$$\begin{aligned} &m_0 C_p \int_\Omega |\nabla u_{\nu_k} - \nabla u^*|^{p(x)}\, dx + [M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega |\nabla u^*|^{p(x)-2} \nabla u^* .(\nabla u_{\nu_k} - \nabla u^*)\, dx \\ &\leq |\langle\, F(w_{\nu_k}) - F(w), u_{\nu_k} - u^* \,\rangle| \end{aligned} \tag{3.12}$$

But, using (3.8) we get

$$\begin{aligned} &|[M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega |\nabla u^*|^{p(x)-2} \nabla u^* .(\nabla u_{\nu_k} - \nabla u^*)\, dx| \\ &\leq \frac{\vartheta_{\nu_k}}{p^-} |\int_\Omega |\nabla u^*|^{p(x)-2} \nabla u^* .(\nabla u_{\nu_k} - \nabla u^*)\, dx| \to 0 \ \text{ as } k \to \infty, \end{aligned} \tag{3.13}$$

where $\vartheta_{\nu_k} = \max\{\|u_{\nu_k}\|_X^{p^-}, \|u_{\nu_k}\|_X^{p^+}\} + \max\{\|u^*\|_X^{p^-}, \|u^*\|_X^{p^+}\}$ is bounded.

Also, by (A_2), (3.8) and the compact embedding of $X \hookrightarrow L^{\alpha(x)}(\Omega)$ we deduce, thanks to the Krasnoselki theorem, the continuity of the Nemytskii operator

$$\begin{aligned} N_{f_1} : L^{\alpha(x)}(\Omega) &\to L^{\alpha'(x)}(\Omega) \\ w &\longmapsto N_{f_1}(w), \end{aligned} \tag{3.14}$$

given by $(N_{f_1}(w))(x) = f_1(x, w(x)), \quad x \in \Omega.$

Hence

$$\|f_1(w_{\nu_k}) - f_1(w)\|_{\alpha'(x)} \to 0$$

It follows from the definition of F and the above convergence that

$$|\langle\, F(w_{\nu_k}) - F(w), u_{\nu_k} - u^* \,\rangle| \to 0 \tag{3.15}$$

Thus, from (3.12)-(3.15) we conclude that

$$u_{\nu_k} \to u^* \quad \text{ in } X$$

Since the possible limit of the sequence $(u_\nu)_{\nu \geq 1}$ is uniquely determined, the whole sequence converges toward $u^* \in X$

Therefore, from (3.7) and the continuous embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we get $u^* = Tw \equiv u_w$.

On the other hand

$$\begin{aligned}\frac{b(v,\lambda)}{\|v\|_X} &= \frac{\langle F(w),v\rangle - \langle Au,v\rangle}{\|v\|_X} \le \frac{\langle F(w),v\rangle}{\|v\|_X} + \|Au\|_{X'} \\ &\le \frac{1}{\|v\|_X}\left[\int_\Omega f_1(x,w)v\,dx + \int_{\Gamma_2} f_2(x)\gamma v\,d\Gamma\right] + L_A\|u\|_X + \|A0\|_{X'} \\ &\le C(\|f_1(w)\|_{\alpha'(x)} + \|f_2\|_{p'(x),\Gamma_2} + \|A0\|_{X'} + 1)\end{aligned} \tag{3.16}$$

Next, using the boundedness of the operator N_{f_1} and the sequence $(u_\nu)_{\nu\ge1}$, and the inf-sup property of the form b, we get $\|\lambda_\nu\|_Y \le C$. It follows that up to a subsequence

$$\lambda_\nu \to \lambda_0 \quad \text{weakly in } Y$$

for some $\lambda_0 \in Y$.

So (u^*,λ^*) and (u^*,λ_0) are solutions of problem (3.1)-(3.2).Then, by the uniqueness $\lambda_0 = \lambda^* \equiv \lambda_w$. This shows the continuity of T.

To prove that T is compact, let $(w_\nu)_{\nu\ge1} \subseteq K$ be bounded in $L^{\alpha(x)}(\Omega)$ and $u_\nu = T(w_\nu)$. Since $(w_\nu)_{\nu\ge1} \subseteq K$, $\|w_\nu\|_X \le C$ and then, up to a subsequence again denoted by $(w_\nu)_{\nu\ge1}$ we have

$$w_\nu \to w \quad \text{weakly in } X$$

By the compact embedding Xinto $L^{\alpha(x)}(\Omega)$, it follows that

$$w_\nu \to w \text{ strongly in } L^{\alpha(x)}(\Omega).$$

Now, following the same arguments as in the proof of the continuity of T we obtain

$$u_\nu = T(w_\nu) \to T(w) = u \quad \text{strongly in } X$$

Thus

$$T(w_\nu) \to T(w) \;\text{ strongly in }\; L^{\alpha(x)}(\Omega).$$

Hence, we can apply the Schauder fixed point theorem to obtain that T possesses a fixed point. This gives us a solution of $(u,\lambda_0) \in X \times \Lambda$ of Problem 1, which concludes the proof. □

Next, we consider the uniqueness of solutions of (2.8). To this end, we also need the following hypothesis on the nonlinear term f_1.

(A4) There exists $b_0 \ge 0$ such that

$$(f_1(x,t) - f_1(x,s))(t-s) \le b_0|t-s|^{p(x)} \quad \text{a.e.} \quad x\in\Omega, \forall t,s\in\mathbb{R}.$$

Our uniqueness result reads as follows.

Theorem 3.2. *Assume that* $(A1)-(A4)$ *hold. If, in addition* $2 \le p$ *for all* $x \in \bar{\Omega}$, *then* (2.8) *has a unique weak solution provided that*

$$\frac{k_p}{b_0\lambda_*^{-1}} < 1,$$

where

$$\lambda_* = \inf_{u \in X \setminus \{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx} > 0.$$

Proof. Theorem 3.1 gives a weak solution $(u, \lambda) \in X \times \Lambda$. Let $(u_1, \lambda_1), (u_2, \lambda_2)$ be two solutions of (2.8). Considering the weak formulation of u_1 and u_2 we have

$$\begin{aligned} \langle\, Au_i, v \,\rangle + b(v, \lambda_i) &= \langle\, F(u_i), v \,\rangle \quad , \quad \forall v \in X \\ b(u_i, \mu - \lambda_i) &\leq 0 \quad \forall \mu \in \Lambda \subseteq Y \quad i = 1, 2. \end{aligned} \tag{3.17}$$

By choosing $v = u_1 - u_2$, $\mu = \lambda_2$ if $i = 1$ and $\mu = \lambda_1$ if $i = 2$, we have

$$\begin{aligned} \langle\, Au_1 - Au_2, u_1 - u_2 \,\rangle + b(u_1 - u_2, \lambda_1 - \lambda_2) &= \langle\, F(u_1) - F(u_2), u_1 - u_2 \,\rangle \ , \forall v \in X \\ b(u_1 - u_2, \lambda_2 - \lambda_1) &\leq 0 \quad \forall \mu \in \Lambda \subseteq Y. \end{aligned} \tag{3.18}$$

It gives

$$\langle\, Au_1 - Au_2, u_1 - u_2 \,\rangle = \langle\, F(u_1) - F(u_2), u_1 - u_2 \,\rangle + b(u_1 - u_2, \lambda_2 - \lambda_1)$$

Thus, using (3.18) and repeating the argument used in the proof of Proposition 2.7, ii) we get

$$\begin{aligned} k_p \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)}\, dx &\leq |\, \langle\, f_1(u_1) - f_1(u_2), u_1 - u_2 \,\rangle \,| \\ &\leq |\int_\Omega (f_1(x, u_1) - f_1(x, u_2))(u_1 - u_2)\, dx| \\ &\leq |\int_\Omega |u_1 - u_2|^{p(x)}\, dx \leq b_0 \lambda_*^{-1} \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)}\, dx \end{aligned}$$

Consequently when $\frac{k_p}{b_0 \lambda_*^{-1}} < 1$, it follows that $u_1 = u_2$. This completes the proof. □

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