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Some Invariant Point Results Using Simulation Function

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Abstract. Through this article, we establish an invariant point theorem by defining generalized Z_s-contractions in relation to the simulation function in S-metric space. In this article, we generalized the results of Nihal Tas, Nihal Yilmaz Ozgur and N.Mlaiki. In addition to that, we bestow an example which supports our results.

1. Introduction

Fixed point is also known as an invariant point. Banach principle of contraction [2] on metric space plays very important role in the field of invariant point theory and non linear analysis. In 1922, Stefan Banach initiated the concept of contraction and established well known Banach contraction theorem. In the year 2006, B Sims and Mustafa [9], established theory on G-metric spaces, that is an extension of metric spaces and established some properties. Later, A.Aliouche, S.Sedghi and N.Shobe [13] initiated S-metric spaces, it is a generalization of G-metric spaces in the year 2012. In 2014, S.Radojevic, N.V.Dung and N.T.Hieu [4] proved by examples that S-metric space is not a generalization of G-metric space and vice versa. Invariant points of various contractive maps on S-metric spaces were studied in [[1], [3], [6]- [8], [11]]. In 2015, F.Khajasteh, Satish Shukla and S.Radenovic [5] introduced simulation function and the concept of Z-contration in relation to simulation function and proved an invariant point theorem which generalizes the Banach Contraction principle. Very recently, Murat Olgun, O.Bicer and T.Alyildiz [10] defined generalized Z-contraction in relation to the simulation function and proved an invariant point theorem.

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In the year 2019, Nihal Tas, Nihal Ylimaz Ozgur and Nabil Mlaiki [8] proved an invariant point theorem by employing the collection of simulation mappings on S-metric spaces. In this article, we generalized the results of Nihal Tas , Nihal Yilmaz Ozgur and N.Mlaiki.

2. Preliminaries

Definition 2.1. *[13] Let $X \neq \emptyset$, then a mapping $S{:}X^3 \to [0,\infty)$ is said to be an S-metric on X if:*
(S1) $S(\xi,\vartheta,w) > 0$ for all $\xi,\vartheta,w \in X$ with $\xi \neq \vartheta \neq w$.
(S2) $S(\xi,\vartheta,w) = 0$ if $\xi = \vartheta = w$.
(S3) $S(\xi,\vartheta,w) \leq [S(\xi,\xi,a) + S(\vartheta,\vartheta,a) + S(w,w,a)]$
$\forall \xi,\vartheta,w,a \in X$. Then we call (X,S) is an S-metric space.

Example 2.1. *[13] Define $S{:}X^3 \to [0,\infty)$ by $S(\xi,\vartheta,w) = d(\xi,\vartheta) + d(\xi,w) + d(\vartheta,w)$ for any $\xi,\vartheta,w \in X$, where (X,d) be a metric space. Then (X,S) is an S-metric space.*

Example 2.2. *[4] Suppose X=R, Collection of all real numbers and let $S(\xi,\vartheta,w) = |\vartheta + w - 2\xi| + |\vartheta - w|$ for all $\xi,\vartheta,w \in X$. Then (X,S) is an S-metric space.*

Example 2.3. *[12] Suppose X=R, Collection of all real numbers and let $S(\xi,\vartheta,w) = |\xi - w| + |\vartheta - w|$ for all $\xi,\vartheta,w \in X$. Then (X,S) is an S-metric space.*

Example 2.4. *Suppose X=[0,1] and $S{:}X^3 \to [0,\infty)$ be defined by*

$$S(\xi,\vartheta,w) = \begin{cases} 0 & if\, \xi = \vartheta = w \\ max\{\xi,\vartheta,w\} & otherwise \end{cases}.$$

Then (X,S) is an S-metric space.

Lemma 2.1. *[13] In the S-metric space, we observe $S(\xi,\xi,\vartheta) = S(\vartheta,\vartheta,\xi)$.*

Lemma 2.2. *[4] In the S-metric space, we observe*
(i) $S(\xi,\xi,\vartheta) \leq 2S(\xi,\xi,w) + S(\vartheta,\vartheta,w)$ and
(ii) $S(\xi,\xi,\vartheta) \leq 2S(\xi,\xi,w) + S(w,w,\vartheta)$

Definition 2.2. *[13] Let (X,S) be a S-metric space. We have:*
(i) If $S(\xi_n,\xi_n,\xi) \to 0$ as $n \to \infty$. ,then we say sequence $\{\xi_n\} \in X$ converges to $\xi \in X$. i.e., for every $\epsilon > 0$, it can be found a natural number n_0 so that to each $n \geq n_0$, $S(\xi_n,\xi_n,\xi) < \epsilon$ and we indicate it by $\lim_{n\to\infty} \xi_n = \xi$.
(ii) a sequence $\{\xi_n\} \in X$ is known as Cauchy sequence if to each $\epsilon > 0$, it can be found $n_0 \in N$ so that $S(\xi_n,\xi_n,\xi_m) < \epsilon$ for every n,m$\geq n_0$.
(iii)If each Cauchy sequence of X is convergent, then say X is complete.

Definition 2.3. *[13] A self map h is defined on S-metric space (X,S) is known as an S-contraction if we get a constant $0 \leq \tau < 1$ so that*
$S(h(\xi),h(\xi),h(\vartheta)) \leq \tau S(\xi,\xi,\vartheta)$ for all $\xi,\vartheta \in X$.

Definition 2.4. *[5] We say that a mapping $\gamma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation mapping if:*
(γ1) $\gamma(0, 0) = 0$
(γ2) $\gamma(p, q) < q - p$ for p,q > 0
(γ3) If $\{p_n\}, \{q_n\}$ are sequences of (0,∞) so that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$, then $\lim_{n\to\infty} sup\, \gamma(p_n, q_n) < 0$.

We indicate Z as the collection of all simulation mappings. For example, $\gamma(p, q) = \tau q - p$ for $0 \leq \tau < 1$ belongning to Z.

Definition 2.5. *[5] Let h be a self map on a metric space (X,d) and $\gamma \in$Z. Then h is known as a Z-contraction in relation to γ if:*

$$\gamma(d(h\xi, h\vartheta), d(\xi, \vartheta)) \geq 0 \text{ for all } \xi, \vartheta \in X.$$

By considering the Definition 2.5. It is concluded that each Banach contraction becomes Z-contraction in relation to $\gamma(p, q) = \tau q - p$ with $0 \leq \tau < 1$. Further, it can be established from the definition of the simulation mapping that $\gamma(p, q) < 0$ for each p $\geq q > 0$. Hence, assume that h is a Z-contraction in relation to $\gamma \in$ Z then

$$d(h\xi, h\vartheta) < d(\xi, \vartheta) \text{ for all distinct } \xi, \vartheta \in X.$$

Theorem 2.1. *[5] In complete metric space (X,d), each Z-contraction has a unique invariant point and furthermore the invariant point is the limit of every Picard's sequence.*

3. Main Results

Definition 3.1. *[13] Let h be a self map on an S-metric space X and $\gamma \in$Z. We say that h is a contraction if we find a constant $0 \leq L < 1$ such that*

$$S(h\xi, h\xi, h\vartheta) \leq LS(\xi, \xi, \vartheta) \text{ for all } \xi, \vartheta \in X.$$

Nihal Tas, N.Y.Ozgur and Nabil Mlaiki [8] defined the Z_s-contraction as follows.

Definition 3.2. *[8] Let h be a self map on an S-metric space (X, S) and $\gamma \in$Z. Then h is said to be a Z_s-contraction in relation to γ if*

$$\gamma(S(h\xi, h\xi, h\vartheta), S(\xi, \xi, \vartheta)) \geq 0 \text{ for all } \xi, \vartheta \in X$$

Nihal Tas, N.Y.Ozgur and Nabil Mlaiki [8] proved the following theorem.

Theorem 3.1. *[8] Let h be a self map on an S-metric space (X, S). Then h has a unique invariant point a$\in$ X and the invariant point is the limit of the Picard sequence $\{\xi_n\}$, whenever h is a Z_s-contraction in relation to γ.*

Definition 3.3. *Let h be a self map on an S-metric space (X, S) and $\gamma \in Z$. Then h is said to be generalized Z_s-contraction in relation to γ if*

$$\gamma(S(h\xi, h\xi, h\vartheta), M(\xi, \xi, \vartheta)) \geq 0 \; for \; all \; \xi, \vartheta \in X \tag{3.1}$$

where $M(\xi, \xi, \vartheta) = max\{S(\xi, \xi, \vartheta), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\}$

Example 3.1. *Let h be a contraction on (X,S). If we take L∈[0,1) and γ(p,q) = Lq-p for all* $0 \leq p, q < \infty$*, then a contraction h is a* Z_s*-contraction in relation to* γ*. In fact, consider* $p = S(h\xi, h\xi, h\vartheta)$ *and* $q = M(\xi, \xi, \vartheta)$*. Since h is a contraction, we obtain :*

$$S(h\xi, h\xi, h\vartheta) \leq LS(\xi, \xi, \vartheta) \leq LM(\xi, \xi, \vartheta)$$
$$\implies LM(\xi, \xi, \vartheta) - S(h\xi, h\xi, h\vartheta) \geq 0$$
$$\implies \gamma(S(h\xi, h\xi, h\vartheta), M(\xi, \xi, \vartheta)) \geq 0.$$

for all $\xi, \vartheta \in X$*. Therefore, h is a generalized* Z_s*-contraction in relation to* γ*.*

Example 3.2. *Consider a complete S-metric space (X,S), where X = [0,1] and* $S : X^3 \to [0, \infty)$ *by* $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$*. Define h:X → X by*

$$h\xi = \begin{cases} \frac{2}{5}, \text{ for } \xi \in [0, \frac{2}{3}) \\ \frac{1}{5}, \text{ for } \xi \in [\frac{2}{3}, 1) \end{cases}$$

Now we prove that h be a generalized Z_s*-contraction in relation to* γ*, where* γ *is defined by* $\gamma(p, q) = \frac{6}{7}q - p$*. Now we get*

$$\begin{aligned} S(h\xi, h\xi, h\vartheta) &\leq \frac{3}{7}[S(\xi, \xi, h\xi) + S(\vartheta, \vartheta, h\vartheta)] \\ &\leq \frac{6}{7}max\{S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta)\} \\ &\leq \frac{6}{7}M(\xi, \xi, \vartheta) \end{aligned}$$

for all $\xi, \vartheta \in X$*.*
That is, we have

$$\gamma(S(h\xi, h\xi, h\vartheta), M(\xi, \xi, \vartheta)) = \frac{6}{7}M(\xi, \xi, \vartheta) - d(h\xi, h\xi, h\vartheta) \geq 0.$$

for all $\xi, \vartheta \in X$*.*

Definition 3.4. *Let* (X, S) *be an S-metric space. Then we say that a mapping h:X → X is asymptotically regular at* $\xi \in X$ *if* $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$

By the following lemma, we can conclude that a generalized Z_s-contraction is asymptotically regular at each point of X.

Lemma 3.1. *If* $h : X \to X$ *is a generalized* Z_s*-contraction in relation to* γ*, then h is an asymptotically regular at each point* $\xi \in X$*.*

Proof. Let $\xi \in X$. If for some m∈N, we have $h^m\xi = h^{m-1}\xi$, that is, h$\vartheta = \vartheta$, where $\vartheta = h^{m-1}\xi$, then $h^n\vartheta = h^{n-1}h\vartheta = h^{n-1}\vartheta$ = ... = h$\vartheta = \vartheta$ for each n∈N. Therefore, we have:

$$\begin{aligned} S(h^n\xi, h^n\xi, h^{n+1}\xi) &= S(h^{n-m+1}h^{m-1}\xi, h^{n-m+1}h^{m-1}\xi, h^{n-m+2}h^{m-1}\xi) \\ &= S(h^{n-m+1}\vartheta, h^{n-m+1}\vartheta, h^{n-m+2}\vartheta) \\ &= S(\vartheta, \vartheta, \vartheta) \\ &= 0 \end{aligned}$$

Hence

$$\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$$

Now, we assume that $h^n\xi \neq h^{n+1}\xi$, for each n∈N.
From the condition(γ2) and the generalized Z_s-contraction property, we get:

$$0 \leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi)) \tag{3.2}$$

Where

$$\begin{aligned} M(h^n\xi, h^n\xi, h^{n-1}\xi) &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^n\xi, h^n\xi, hh^n\xi), S(h^{n-1}\xi, h^{n-1}\xi, hh^{n-1}\xi), \\ &\frac{1}{2}[S(h^n\xi, h^n\xi, hh^{n-1}\xi) + S(h^{n-1}\xi, h^{n-1}\xi, hh^n\xi)]\} \\ &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^n\xi, h^n\xi, h^{n+1}\xi), S(h^{n-1}\xi, h^{n-1}\xi, h^n\xi), \\ &\frac{1}{2}[S(h^n\xi, h^n\xi, h^n\xi) + S(h^{n-1}\xi, h^{n-1}\xi, h^{n+1}\xi)\} \\ &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)\} \end{aligned}$$

If $S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) > S(h^n\xi, h^n\xi, h^{n-1}\xi)$ then, we get
$M(h^n\xi, h^n\xi, h^{n-1}\xi) = S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)$
From equation (3.2) we have,

$$\begin{aligned} 0 &\leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)) \\ &< S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) - S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) = 0 \end{aligned}$$

which is a contradiction.
Hence $M(h^n\xi, h^n\xi, h^{n-1}\xi) = S(h^n\xi, h^n\xi, h^{n-1}\xi)$.
Using generalized Z_s-contractive property, we get

$$\begin{aligned} 0 &\leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi)) \\ &= \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^n\xi, h^n\xi, h^{n-1}\xi)) \\ &< S(h^n\xi, h^n\xi, h^{n-1}\xi) - S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) \end{aligned}$$

i.e., $S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) < S(h^n\xi, h^n\xi, h^{n-1}\xi)$ for all n∈N.
Then $\{S(h^n\xi, h^n\xi, h^{n-1}\xi)\}$ is a nonnegative reals of decreasing sequence and so it should be convergent. Suppose $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = \eta \geq 0$. If $\eta > 0$, then from the condition ($\gamma 3$) and the generalized Z_s-contraction property, we get

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} sup\ \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi) \\ &= \lim_{n\to\infty} sup\ \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^n\xi, h^n\xi, h^{n-1}\xi) < 0 \end{aligned}$$

which is a contradiction. It should be $\eta = 0$.
Therefore $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$.
Hence, h is asymptotically regular at each point $\xi \in X$. □

Lemma 3.2. *The Picard sequence $\{\xi_n\}$ so that $h\xi_{n-1} = \xi_n$, to each n∈N the initial point $\xi_0 \in X$ is a bounded sequence, whenever h is a generalized Z_s-contraction in relation to γ.*

Proof. Consider $\{\xi_n\}$ be the Picard sequence in X with initial value ξ_0. Now we claim that $\{\xi_n\}$ is a bounded sequence.
Assume that $\{\xi_n\}$ is unbounded. Let $\xi_{n+m} \neq \xi_n$, for each m,n∈N.
Since $\{\xi_n\}$ is unbounded, we can find a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ so that $n_1 = 1$ and to each k∈N, n_{k+1} is the smallest integer so that
$S(\xi_{n_k+1}, \xi_{n_k+1}, \xi_{n_k}) > 1$ and $S(\xi_m, \xi_m, \xi_{n_k}) \leq 1$ for $n_k \leq m \leq n_{k+1} - 1$
Hence, from the lemma (2.2), we obtain

$$\begin{aligned} 1 &< S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}) \\ &\leq 2S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + S(\xi_{n_k}, \xi_{n_k}, \xi_{n_{k+1}-1}) \\ &\leq 2S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + 1 \end{aligned}$$

Letting k→ ∞ and using lemma (3.1), we have

$$\lim_{n\to\infty} S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}) = 1$$

$$\begin{aligned} &1 < S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}) \leq M(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1}) \\ &= max\{S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1}), S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_{k+1}}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\ &\frac{1}{2}[S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_{k+1}})]\} \\ &= max\{S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_{k+1}-1}), S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_{k+1}}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\ &\frac{1}{2}[S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_{k+1}})]\} \end{aligned}$$

$$\leq max\{2S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k})+S(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_k}),S(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_{k+1}}),$$
$$S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k}),\frac{1}{2}[S(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_k})+S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_{k+1}})]\}$$
$$\leq max\{2S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k})+1,S(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_{k+1}}),$$
$$S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k}),\frac{1}{2}[1+2S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k})+S(\xi_{n_k},\xi_{n_k},\xi_{n_{k+1}})]\}$$

Letting n$\rightarrow\infty$, we get

$$1\leq\lim_{k\rightarrow\infty}M(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_k-1})\leq 1.$$

That is $\lim_{k\rightarrow\infty}M(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_k-1})=1$

From the condition($\gamma3$) and the generalized Z_s-contraction property, we obtain

$$0\leq\lim_{k\rightarrow\infty}sup\ \gamma(S(\xi_{n_{k+1}},\xi_{n_{k+1}},\xi_{n_k}),M(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_k-1}))$$
$$=\lim_{k\rightarrow\infty}sup\ \gamma(S(\xi_{n_{k+1}},\xi_{n_{k+1}},\xi_{n_k}),S(\xi_{n_{k+1}-1},\xi_{n_{k+1}-1},\xi_{n_k-1}))<0$$

which is a contradiction. Hence our assumption is wrong.

Therefore $\{\xi_n\}$ is bounded. □

Theorem 3.2. *Let h be a self map defined on complete S-metric space (X, S). Then h has a unique invariant point $a\in X$ and Picard sequence $\{\xi_n\}$ converges to the invariant element a, whenever h is a generalized Z_s-contraction in relation to γ.*

Proof. Let the Picard sequence $\{\xi_n\}$ be defined as h$\xi_{n-1}=\xi_n$, $\forall n\in$N and $\xi_0\in X$. Now, we claim that $\{\xi_n\}$ be a cauchy sequence. To get this, Consider

$$T_n=sup\{S(\xi_i,\xi_i,\xi_j):i,j\geq n\}.$$

Clearly $\{T_n\}$ be a nonnegative reals of decreasing sequence. Hence, we can find $\tau\geq 0$ so that $\lim_{n\rightarrow\infty}T_n=\tau$. Now we prove that τ = 0. If possible suppose that $\tau>0$. From the definition of T_n, for each k$\in$N, we can find m_k,n_k so that $k\leq n_k<m_k$ and

$$T_k-\frac{1}{k}<S(\xi_{m_k},\xi_{m_k},\xi_{n_k})\leq T_k$$

Therefore, we get $\lim_{n\rightarrow\infty}S(\xi_{m_k},\xi_{m_k},\xi_{n_k})=\tau$.

From the lemma (2.2), lemma (3.1) and generalized Z_s-contraction property, we get

$$S(\xi_{m_k},\xi_{m_k},\xi_{n_k})\leq S(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1})$$
$$\leq 2S(\xi_{m_k-1},\xi_{m_k-1},\xi_{m_k})+S(\xi_{n_k-1},\xi_{n_k-1},\xi_{m_k})$$
$$\leq 2S(\xi_{m_k-1},\xi_{m_k-1},\xi_{m_k})+2S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k})+S(\xi_{m_k},\xi_{m_k},\xi_{n_k})$$

Letting as k$\rightarrow\infty$, we have

$$\lim_{k\rightarrow\infty}S(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1})=\tau$$

$$\begin{aligned}
&S(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}) \le M(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1})\\
&= max\{S(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}), S(\xi_{m_k-1},\xi_{m_k-1},h\xi_{m_k-1}), S(\xi_{n_k-1},\xi_{n_k-1},h\xi_{n_k-1}),\\
&\frac{1}{2}[S(\xi_{m_k-1},\xi_{m_k-1},h\xi_{n_k-1}) + S(\xi_{n_k-1},\xi_{n_k-1},h\xi_{m_k-1})]\}\\
&= max\{S(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}), S(\xi_{m_k-1},\xi_{m_k-1},\xi_{m_k}), S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k}),\\
&\frac{1}{2}[S(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k}) + S(\xi_{n_k-1},\xi_{n_k-1},\xi_{m_k})]\}\\
&\le max\{S(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}), S(\xi_{m_k-1},\xi_{m_k-1},\xi_{m_k}), S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k}),\\
&\frac{1}{2}[2S(\xi_{m_k-1},\xi_{m_k-1},\xi_{m_k}) + S(\xi_{m_k},\xi_{m_k},\xi_{n_k}) +\\
&2S(\xi_{n_k-1},\xi_{n_k-1},\xi_{n_k}) + S(\xi_{n_k},\xi_{n_k},\xi_{m_k})]\}
\end{aligned}$$

Letting k $\to \infty$, we get

$$\lim_{k\to\infty} M(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}) = \tau.$$

From the condition ($\gamma3$) and the generalized Z_s-contraction property, we have

$$0 \le \lim_{k\to\infty} sup\ \gamma(S(\xi_{m_k},\xi_{m_k},\xi_{n_k}), M(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1})) < 0$$

This is a contraction, Hence, $\tau = 0$.

That is $\{\xi_n\}$ is a cauchy sequence in the complete S-metric space X, we can find $\eta \in X$ so that $\lim_{n\to\infty} \xi_n = \eta$.

Now we verify that, η is an invariant point of h.

If suppose h$\eta \neq \eta$, then $S(\eta,\eta,h\eta) = S(h\eta,h\eta,\eta) > 0$.

Now,

$$\begin{aligned}
M(\xi_n,\xi_n,\eta) =& max\{S(\xi_n,\xi_n,\eta), S(\xi_n,\xi_n,h\xi_n), S(\eta,\eta,h\eta),\\
&\frac{1}{2}[S(\xi_n,\xi_n,h\eta) + S(\eta,\eta,h\xi_n)]\}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} M(\xi_n,\xi_n,\eta) &= max\{S(\eta,\eta,\eta), S(\eta,\eta,\eta), S(\eta,\eta,h\eta), \frac{1}{2}[S(\eta,\eta,h\eta) + S(\eta,\eta,\eta)]\}\\
&= S(\eta,\eta,h\eta)
\end{aligned}$$

From the conditions $(\gamma2), (\gamma3)$ and Z_s-contraction property, we get

$$0 \le \lim_{n\to\infty} sup\ \gamma(S(h\xi_n,h\xi_n,h\eta), M(\xi_n,\xi_n,\eta)) < 0$$

This is contradiction. Hence $S(\eta,\eta,h\eta) = 0 \implies$ h$\eta = \eta$.

Hence, η is a invariant point of h.

Now we claim that η is unique. Suppose α is an element in X such that $\alpha \neq \eta$ and h$\alpha = \alpha$.

Now,

$$\begin{aligned}M(\eta,\eta,\alpha) &= max\{S(\eta,\eta,\alpha), S(\eta,\eta,h\eta), S(\alpha,\alpha,h\alpha), \frac{1}{2}[S(\eta,\eta,h\alpha)+S(\alpha,\alpha,h\eta]\} \\ &= max\{S(\eta,\eta,\alpha), S(\eta,\eta,\eta), S(\alpha,\alpha,\alpha), \frac{1}{2}[S(\eta,\eta,\alpha)+S(\alpha,\alpha,\eta)]\} \\ &= S(\eta,\eta,\alpha)\end{aligned}$$

From the condition ($\gamma 2$) and Z_s-contraction property, we get

$$\begin{aligned}0 &\le \gamma(S(h\eta,h\eta,h\alpha), M(\eta,\eta,\alpha)) = \gamma(S(h\eta,h\eta,h\alpha), S(\eta,\eta,\alpha)) \\ &< S(\eta,\eta,\alpha) - S(\eta,\eta,\alpha) = 0,\end{aligned}$$

This is a contradiction. It should be $\eta = \alpha$. □

Example 3.3. *Consider a complete S-metric space (X, S), where $X = [0, \frac{1}{4}]$ and $S : X^3 \to [0,\infty)$ by $S(\xi,\vartheta,w) = |\xi - w| + |\xi - 2\vartheta + w|$. Define h: $X \to X$ by $h\xi = \frac{\xi}{1+\xi}$. From example 2.9 in [5], we have h be a Z-contraction in relation to $\gamma \in Z$, where $\gamma(p,q) = \frac{q}{q+\frac{1}{4}} - p$, for any p,q$\in [0,\infty)$ Therefore for all $\xi,\vartheta \in X$, we get*

$$\begin{aligned}0 &\le \gamma(S(h\xi,h\xi,h\vartheta), S(\xi,\xi,\vartheta)) \\ &= \frac{S(\xi,\xi,\vartheta)}{S(\xi,\xi,\vartheta)+\frac{1}{4}} - S(h\xi,h\xi,h\vartheta) \\ &\le \frac{M(\xi,\xi,\vartheta)}{M(\xi,\xi,\vartheta)+\frac{1}{4}} - S(h\xi,h\xi,h\vartheta) \\ &= \gamma(S(h\xi,h\xi,h\vartheta), M(\xi,\xi,\vartheta))\end{aligned}$$

Thus, h is generalized Z_s-contraction in relation to γ, for some $\gamma \in Z$. So, by using Theorem 3.2, h has a unique invariant point a=0.

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