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Computational Approach for a Singularly Perturbed Differential Equations With Mixed Shifts Using a Non-Polynomial Spline

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Abstract. In this paper, a second order singularly perturbed differential difference equation with both the negative and positive shifts is considered. A fitted non-polynomial spline approach is applied to solve the problem. Taylor series expansion process is being used to produce an approximated form of the considered problem, and then a fitted non-polynomial spline approach is devised in the form of a three-term recurrence relation. The convergence of the method is examined, and a quadratic rate of convergence is achieved. The maximum absolute error with quadratic rate of convergence of the solution is recorded. Layer profile is examined using the graphs.

1. Introduction

In applied science and engineering, there are many physical and biological problems with solutions that include boundary and interior layers for specific parameter values and these are known as singular perturbation problems(SPPs). A SPP is one in which no asymptotic expansion is always true over the interval as the perturbation parameter approaches zero. Due to the boundary or interior layer structure of the solutions,the numerical and analytical treatments of such SPPs are very difficult. There is a narrow transitional layer in which the solution changes most quickly, whereas the solution responds uniformly and slowly away from the layer. When the perturbation parameter approaches

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zero, the regularity of the solution decreases. A singularly perturbed differential-difference equation (SPDDE) is formed by multiplying the highest order derivative term of a differential equation with a small positive perturbation parameter ε and involves a delay and advance terms. For the solutions such SPDDEs in the interval of boundary conditions is a challenging in the modeling of a variety of physical and biological problems like, the initial exit time problem in neuronal variability activation models [25], oscillations of the human pupil light reflex with delayed and mixed responses [15], red cell system evaluation [27] and bifurcated gap in a hybrid optical system [3] etc. For a detailed information on the SPDDEs is given in [2], [4], [5], [6], [17], [18]. A few numerical methods are proposed to obtain valid and realistic approaches for the solution for second order SPDDEs with mixed shifts (negative and positive) with boundary constraints. In the articles [11] - [14] the authors presented asymptotic analysis of SPDDEs. In [13], [14] Taylor series is used to analyze a term containing small shift. The same authors investigated the impact of small shifts on the oscillatory solution of the problem in [11]. In [1] the authors proposed an a nonstandard exponentially fitted finite difference method(FDM) to solve SPDDEs with boundary layer on each sides of the interval. To solve SPDDE with delay, an impactful Haar wavelet collocation methodology is developed in [20]. In the articles [21], [22] the authors suggested an exponentially fitted FDM to solve SPDDEs with delay and advanced terms and turning point problems. The authors advised a fourth order FDM with a fitting factor in [10], [16] to solve the considered problem. In [8], [9] the authors developed some numerical techniques to solve SPDDEs with mixed shifts. An asymptotic expansion estimation of the solution is designed to solve SPDDEs in [23]. The author suggested successive complementary expansion method (SCEM) for solving this model in [19]. A mixed FDM is proposed to solve SPDDEs with both the shifts delay and advanced in [24].
The approach given in this study gives a novel difference method to the SPDDE with mixed shifts. In Section 2, the problem description is given and the non-polynomial spline is discussed. In Section 3, a three term difference scheme is devised using the non-polynomial spline. Convergence analysis of this computational approach is discussed in Section 4. In Section 5, the computational solutions for the test problems to illustrate the technique, along with comparisons to alternative approach and conclusions are presented.

2. Definition of the problem and Non-Polynomial Spline

Consider the following SPDDE with delay and advance terms

$$\varepsilon z''(\theta) + p(\theta) z'(\theta) + q(\theta) z(\theta-\delta) + r(\theta) z(\theta) + s(\theta) z(\theta+\eta) = f(\theta), \tag{2.1}$$

for $\theta \in (0,1)$ with the boundary constraints

$$z(\theta) = \varphi(\theta), \ -\delta \le \theta \le 0; \quad z(\theta) = \psi(\theta), 1 \le \theta \le 1+\eta. \tag{2.2}$$

where $0 < \varepsilon \ll 1$ and $p(\theta)$, $q(\theta)$, $r(\theta)$, $f(\theta)$, $\varphi(\theta)$ and $\psi(\theta)$ are sufficiently smooth functions on (0,1) and $0 < \delta,\ \eta = o(\varepsilon)$, $\delta,\ \eta$ are delay and advance shifts respectively.
The solution of equation (2.1) with (2.2) represents the layer behavior at each end of the interval if $p(\theta) - \delta q(\theta) + \eta s(\theta) > 0$ and $p(\theta) - \delta q(\theta) + \eta s(\theta) < 0$ (left end and right end). Depending on the sign of $q(\theta) + r(\theta) + s(\theta)$, existence of boundary layers in two cases reported in [21].
The Taylor's series expansions of $z(\theta - \delta)$, $z(\theta + \eta)$ in the neighborhood of the point θ, we have

$$z(\theta - \delta) \approx z(\theta) - \delta z'(\theta) + O\left(\delta^2\right) \text{ and } z(\theta + \eta) \approx z(\theta) - \eta z'(\theta) + O\left(\eta^2\right) \tag{2.3}$$

Using (2.3) in (2.1), we get

$$\varepsilon z''(\theta) + a(\theta) z'(\theta) + b(\theta) z(\theta) = f(\theta), \quad 0 < \theta < 1 \tag{2.4}$$

with the boundary constraints

$$z(0) = \varphi(0) = \varphi_0, \quad z(1) = \psi(1) = \psi_1 \tag{2.5}$$

Consider the domain $[0,1]$ and it is split into N equal length of sub-domains with constant step size h. Let $0 = \theta_0 < \theta_1 < \ldots < \theta_N = 1$ be the N grid points. Then we have, $\theta_i = ih$ and in each sub-domain $[\theta_i, \theta_{i+1}]$, $i = 1, 2, \ldots, N-1$ the non-polynomial spline is of the form

$$G_i(\theta) = a_i sin\tau(\theta - \theta_i) + b_i\left(e^{-\tau(\theta-\theta_i)} + e^{\tau(\theta-\theta_i)}\right) + c_i(\theta - \theta_i) + d_i \tag{2.6}$$

where a_i, b_i, c_i and d_i are constant coefficients, and $\tau \neq 0$ arbitrary parameter.
To derive the coefficients a_i, b_i, c_i and d_i (2.6) in terms of z_i, z_{i+1}, Q_i and Q_{i+1}, we define

$$L_i(\theta_i) = z_i, \quad L_i(\theta_{i+1}) = z_{i+1} \tag{2.7}$$

$$L_i''(\theta_i) = Q_i, \quad L_i''(\theta_{i+1}) = Q_{i+1} \tag{2.8}$$

By using the conditions (2.7) and (2.8), we calculate the coefficients in (2.6) as

$$.\begin{cases} a_i = \frac{Q_i h^2\left(e^{\omega}+e^{-\omega}\right)}{2\omega^2 sin\omega} - \frac{Q_{i+1}h^2}{2\omega^2 sin\omega} \\ b_i = \frac{Q_i h^2}{2\omega^2} \\ c_i = \frac{z_{i+1} - z_i}{h} + \frac{Q_i h^2\left(1 - e^{\omega} - e^{-\omega}\right)}{h\omega^2} + \frac{Q_{i+1}h^2}{h\omega^2} \\ d_i = y_i - \frac{Q_i h^2}{\omega^2} \end{cases} \tag{2.9}$$

where $\omega = \tau h$.
Using the first derivative of continuity, ${L_{i-1}}^{(m)}(\theta_i) = {L_i}^{(m)}(\theta_i)$, $m = 1$, we obtain the relation

$$(z_{i-1} - 2z_i + z_{i+1}) = h^2(\alpha Q_{i-1} + \beta Q_i + \gamma Q_{i+1}), i = 1, 2, \ldots, N-1 \tag{2.10}$$

where,

$$\alpha = \frac{\omega\left(e^{-\omega} + e^{\omega}\right) cos\omega + \omega\left(-e^{-\omega} + e^{\omega}\right) sin\omega + 2\left(1 - e^{\omega} - e^{-\omega}\right) sin\omega}{2\omega^2 sin\omega},$$

$$\beta = \frac{-2\omega cos\omega + 2sin\omega - \omega\left(e^{\omega} + e^{-\omega}\right) - 2\left(1 - e^{\omega} - e^{-\omega}\right) sin\omega}{2\omega^2 sin\omega},$$

$$\gamma = \frac{\omega - sin\omega}{\omega^2 sin\omega}$$

If $h \to 0$, then $\omega = h\tau \to 0$, we have $(\alpha, \beta, \gamma) \to \left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right)$,
(2.10) reduces to cubic spline [7].

3. Numerical Algorithm

At the grid points θ_i, (2.4) can be written as

$$\varepsilon z_i'' = -a(\theta_i) z_i' - b(\theta_i) z_i + f(\theta_i)$$

Using $L_i''(\theta_i) = Q_i = z_i''$ in above equation, we get

$$\varepsilon Q_j = -a_j(\theta_i) z_i' - b_j(\theta_i) z_i + f_j(\theta_i) \text{ for } j = i,\ i \pm 1 \tag{3.1}$$

Substitute (3.1) in (2.10) and then using z_j' for $j = i,\ i \pm 1$
i.e

$$z'_i \approx \frac{1}{2h}(z_{i+1} - z_{i-1}),\ z'_{i+1} \approx \frac{1}{2h}(3z_{i+1} - 4z_i + z_{i-1})$$
$$\text{and } z'_{i-1} \approx \frac{1}{2h}(-z_{i+1} + 4z_i - 3z_{i-1})$$

$$\begin{aligned}\frac{\varepsilon}{h^2}(z_{i+1} - 2z_i + z_{i-1}) &= -\alpha a_{i-1}\frac{(-z_{i+1} + 4z_i - 3z_{i-1})}{2h} - \beta a_i \frac{(-z_{i-1} + z_{i+1})}{2h} \\ &- \gamma a_{i+1}\frac{(z_{i-1} - 4z_i + 3z_{i+1})}{2h} - \alpha b_{i-1} z_{i-1} - \beta b_i z_i - \gamma b_{i+1} z_{i+1} + \\ &\left(\alpha f_{i-1} + \beta f_i + \gamma f_{i+1}\right)\end{aligned} \tag{3.2}$$

We incorporate a fitting parameter $\sigma_i(\rho)$ in the present approach to increase the solution's accuracy and manage the layer behaviour. Then, we have

$$\begin{aligned}\frac{\varepsilon\sigma_i(\rho)}{h^2}(z_{i+1} - 2z_i + z_{i-1}) &= -\alpha a_{i-1}\frac{(-z_{i+1} + 4z_i - 3z_{i-1})}{2h} - \\ \beta a_i \frac{(z_{i+1} - z_{i-1})}{2h} &- \gamma a_{i+1}\frac{(z_{i-1} - 4z_i + 3z_{i+1})}{2h} - \alpha b_{i-1} z_{i-1} - \beta b_i z_i - \\ &\gamma b_{i+1} z_{i+1} + \left(\alpha f_{i-1} + \beta f_i + \gamma f_{i+1}\right)\end{aligned} \tag{3.3}$$

On simplification, we obtain the following tridiagonal system

$$E_i z_{i-1} + F_i z_i + G_i z_{i+1} = H_i,\ i = 1, 2, \ldots,\ N-1 \tag{3.4}$$

where

$$E_i = \varepsilon\sigma_i - \frac{3\alpha h a_{i-1}}{2} - \frac{\beta h a_i}{2} + \frac{\gamma h a_{i+1}}{2} + h^2 \alpha b_{i-1},$$
$$F_i = -2\sigma_i \varepsilon + 2\alpha h a_{i-1} - 2\gamma h a_{i+1} + h^2 \beta b_i,$$

$$G_i = \varepsilon\sigma_i - \frac{\alpha h a_{i-1}}{2} + \frac{\beta h a_i}{2} + \frac{3\gamma h a_{i+1}}{2} + h^2\gamma b_{i+1} ,$$
$$H_i = h^2\left(\alpha\, f_{\ i-1} + \beta f_i \ + \gamma f_{i+1}\ \right)$$

The system (3.4) can be solved using Thomas algorithm with the constraints $z(0) = \varphi_0, \ \ z(1) = \psi_1$. To calculate the fitting parameter, we adopt the process given in [18]. The following is an approximation for the solution of the homogeneous problem of (2.1)

$$z(\theta) = z_0(\theta) + \frac{a(0)}{a(\theta)}(\alpha - z_0(0))\, e^{-\int_0^\theta\left(\frac{a(\theta)}{\varepsilon} - \frac{b(\theta)}{a(\theta)}\right)d\theta} + o(\varepsilon) \tag{3.5}$$

where $z_0(\theta)$ is the solution of $a(\theta)\, z_0{}'(\theta) + b(\theta)\, z_0(\theta) = f(\theta)\,,\ z_0(1) = \psi_1$
By using the expansion for $a(\theta)$ and $b(\theta)$ about the point zero, then (3.5) becomes

$$z(\theta) = z_0(\theta) + (\varphi_0 - z_0(0))\, e^{-\left(\frac{a(\theta)}{\varepsilon}\ \right)\theta} + o(\varepsilon)$$

From (3.5), we have

$$\lim_{h\to 0} z(ih) = z_0(0) + (\varphi_0 - z_0(0))\, e^{-a(\theta)i\rho},$$

Using these limit values in (3.3), we obtain the following fitting factor

$$\sigma_i(\rho) = \left(\alpha + \frac{\beta}{2}\right)\ a_i\rho\ Coth\left(\frac{a_i\rho}{2}\right), \text{where } \rho \ = \ \frac{h}{\varepsilon}$$

.

4. Convergence Analysis

The local error estimate for the numerical scheme of (3.4) is

$$T_i(h) = h^2\left[1 - (\alpha + \beta + \gamma)\right]\varepsilon z_i'' + h^3(\gamma - \alpha)\left[b_i' z_i + a_i' z_i' + b_i z_i' - f_i'\right]$$
$$+h^4\frac{(\gamma + \alpha)}{2}\left[b_i'' z_i + a_i'' z_i' + 2b_i' z_i' + 2a_i' z_i''\ + b_i z_i'' - f_i''\right] +$$
$$(\alpha - \gamma)\, O\left(h^5\right) + O\left(h^6\right)$$

Hence, with $(\alpha, \beta, \gamma) = \left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right)$, truncation error is of fourth order.

With the help of (2.2), the matrix form of (3.4) is

$$\left(\tilde{R} + \tilde{G}\right) Z + \tilde{M} + T(h) = O \tag{4.1}$$

where

$$\tilde{R} = \begin{bmatrix} -2\varepsilon\sigma & \varepsilon\sigma & 0 & 0 & \dots & 0 \\ \varepsilon\sigma & -2\varepsilon\sigma & \varepsilon\sigma & .. & \dots & 0 \\ 0 & .. & .. & .. & \dots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & \varepsilon\sigma & -2\varepsilon\sigma \end{bmatrix},$$

$$\tilde{G} = [x_i, v_i, w_i] = \begin{bmatrix} v_1 & w_1 & 0 & 0 & \ldots\ldots\ldots & 0 \\ x_2 & v_2 & w_2 & 0 & \ldots\ldots\ldots & 0 \\ 0 & x_3 & v_3 & w_3 & \ldots\ldots\ldots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & x_{N-1} & v_{N-1} \end{bmatrix}$$

$$x_i = -\frac{3\alpha h a_{i-1}}{2} - \frac{\beta h a_i}{2} + \frac{\gamma h a_{i+1}}{2} + h^2 \alpha b_{i-1},$$

$$v_i = 2\alpha h a_{i-1} - 2\gamma h a_{i+1} + h^2 \beta b_i,$$

$$w_i = -\frac{\alpha h a_{i-1}}{2} + \frac{\beta h a_i}{2} + \frac{3\gamma h a_{i+1}}{2} + \gamma h^2 b_{i+1}, \text{ for all } i = 1, 2, \ldots, N-1$$

$$\tilde{M} = [m_1 + (\varepsilon\sigma + x_1)\phi(0), m_2, m_3, \ldots, m_{N-2}, m_{N-1} + (\varepsilon\sigma + w_{N-1})\gamma]^T$$

$$\text{where, } m_i = h^2 (\gamma f_{i+1} + \beta f_i + \alpha f_{i-1}), \text{for } i = 1, 2, \ldots, N-1,$$

$$T(h) = o(h^4), \; Z = [Z_1, Z_2, ..., Z_{N-1}]^T, \quad T(h) = [t_1, t_2, ..., t_{N-1}]^T$$

$$\text{and } O = [0, 0, ..., 0]^T$$

are corresponding vectors of (4.1).

Let $z = [z_1, z_2, ..., z_{N-1}]^T \cong z$ which satisfies the equation

$$\left(\tilde{R} + \tilde{G}\right) z + \tilde{M} = 0 \tag{4.2}$$

Let $e_i = z_i - Z_i$, $i = 1, 2 \ldots, N-1$ denote the discretized error so that $\tilde{E} = [e_1, e_2, ..., e_{N-1}]^T = z - Z$. From (4.1) and (4.2), we obtain the error equation

$$\left(\tilde{R} + \tilde{G}\right) \tilde{E} = T(h) \tag{4.3}$$

Let $|a_i| \leq K_1, |b_i| \leq K_2$ so that, if $\tilde{Q}_{i,j}$ is the $(i,j)^{th}$ element of matrix $\tilde{G}$, then

$$\left|\tilde{Q}_{i,\ i+1}\right| = |w_i| \leq \varepsilon + h(\alpha + \beta + 3\gamma) K_1 + h^2 \alpha K_2, \; i = 1, 2, \ldots, \; N-2 \tag{4.4a}$$

$$\left|\tilde{Q}_{i,\ i-1}\right| = |u_i| \leq \varepsilon + h(3\alpha + \beta + \gamma) K_1 + h^2 \alpha K_2, i = 2, 3, \ldots, N-1 \tag{4.4b}$$

As a result, for relatively small h ($h \to 0$), we see that

$$\left|\tilde{Q}_{i,i+1}\right| < \varepsilon, \; \forall \; i = 1, 2, \ldots, \; N-1, \; \left|\tilde{Q}_{i,i-1}\right| < \varepsilon, \forall \; i = 2, 3, \ldots, N-1 \tag{4.4c}$$

Hence, $\left(\tilde{R} + \tilde{G}\right)$ is irreducible [26].

Let the sum of elements of i^{th} row of $\left(\tilde{R} + \tilde{G}\right)$ be $\mathbb{S}_i$, then we have

$$\mathbb{S}_i = -\varepsilon\sigma + \frac{3\alpha h a_{i-1}}{2} + \frac{\beta h a_i}{2} - \frac{\gamma h a_{i+1}}{2} + h^2 (\gamma b_{i+1} + \beta b_i), \text{ for } i = 1$$

$$\mathbb{S}_i = h^2 (\alpha b_{i-1} + \beta b_i + \gamma b_{i+1}), \quad \text{for } i = 2, 3, \ldots, N-2$$

$$\mathbb{S}_i = -\varepsilon\sigma + \frac{\alpha h a_{i-1}}{2} - \frac{\beta h a_i}{2} - \frac{3\gamma h a_{i+1}}{2} + h^2 (\alpha b_{i-1} + \beta b_i), \quad \text{for } i = N-1$$

Let

$$K_{1*} = \min_{1 \le i \le N-1} |a_i|, \; K_1{}^* = \max_{1 \le i \le N} |a_i|, K_{2*} = \min_{1 \le i \le N-1} |b_i|,$$

$$K_2{}^* = \max_{1 \le i \le N} |b_i| \text{ then } 0 \le K_{1*} \le K_1 \le K_1{}^*, \quad 0 \le K_{2*} \le K_2 \le K_2{}^*$$

For relatively small h, $\left(\tilde{R}+\tilde{G}\right)$ is monotone [26].

Hence $\left(\tilde{R}+\tilde{G}\right)^{-1}$ exists and $\left(\tilde{R}+\tilde{G}\right)^{-1} \ge 0$.

Thus from (4.3), we have

$$||\tilde{E}|| \le \; ||\tilde{R}+\tilde{G}||^{-1}||T|| \tag{4.5}$$

For relatively small h, we have Let $\left(\tilde{R}+\tilde{G}\right)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $\left(\tilde{R}+\tilde{G}\right)^{-1}$ and define

$$||\tilde{R}+\tilde{G}||^{-1} = \max_{1 \le i \le N-1} \sum_{k=1}^{N-1} \left(\tilde{R}+\tilde{G}\right)^{-1}_{i,k}, \text{ and } \quad ||T(h)|| = \max_{1 \le i \le N-1} |T_i|.$$

Since $(\tilde{R}+\tilde{G})^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{N-1} (\tilde{R}+\tilde{G})^{-1}_{i,k} . \mathbb{S}_k = 1$, for $i = 1(1)N-1$

$$\left(\tilde{R}+\tilde{G}\right)^{-1}_{i,\;k} \le \frac{1}{\mathbb{S}_i} < \frac{1}{h^2 K_2}, \quad i = 1. \tag{4.6a}$$

$$\left(\tilde{R}+\tilde{G}\right)^{-1}_{i,\;k} \le \frac{1}{\mathbb{S}_i} < \frac{1}{h^2 K_2}, \quad i = N-1 \tag{4.6b}$$

Further more,

$$\sum_{k=1}^{N-1} \left(\tilde{R}+\tilde{G}\right)^{-1}_{i,k} \le \frac{1}{\min\limits_{2 \le i \le N-2} \mathbb{S}_i} < \frac{1}{h^2 K_2}, \text{for } i = 2, 3, \ldots, \; N-2 \tag{4.6c}$$

With the help of (4.6a), (4.6b), (4.6c) and using (4.5), we have

$$||\tilde{E}|| \le O\left(h^2\right)$$

This illustrates the second-order convergence for the scheme (3.4) with $(\alpha, \beta, \gamma) = \left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right)$

5. Numerical Examples

To show the validity and robustness of the suggested technique, we reported the computational results of the four example problems in terms of maximum absolute errors (MAEs) with calculated rates of convergence (ROC) in the tables. Since the exact solutions are not known for considered examples, the MAEs are estimated using the double mesh approach using the formula

$$E_N \;=\; \max_{0 \le i \le N} \left| z_i{}^N - z_{2i}{}^{2N} \right|$$

where $z_i{}^N$ and $z_{2i}{}^{2N}$ are the computational solutions of the example problem for N and $2N$ grid points respectively. Further, the rate of convergence is determine by the formula

$$R_N \;=\; log_2 \left| \frac{E_N}{E_{2N}} \right|.$$

Example 5.1.

$$\varepsilon z''(\theta) + z'(\theta) - z(\theta - \delta) + z(\theta) - z(\theta + \eta) = -1,$$

subject to boundary constraints $z(\theta) = \begin{cases} 1, & -\delta \le \theta \le 0 \\ 1, & 1 \le \theta \le 1 + \eta. \end{cases}$

Example 5.2.

$$\varepsilon z''(\theta) + 2.5z'(\theta) - 2exp(\theta)\, z(\theta - \delta) - z(\theta) - \theta z(\theta + \eta) = 1,$$

subject to boundary constraints $z(\theta) = \begin{cases} 1, & -\delta \le \theta \le 0 \\ 1, & 1 \le \theta \le 1 + \eta. \end{cases}$

Example 5.3.

$$\varepsilon z''(\theta) - \left(1 + exp\left(-\theta^2\right)\right) z'(\theta) - \theta z(\theta - \delta) - \theta^2 z(\theta) -$$
$$(1.5 - exp(-\theta))\, z(\theta + \eta) = 1$$

, with boundary constraints $z(\theta) = \begin{cases} 1, & -\delta \le \theta \le 0 \\ 1, & 1 \le \theta \le 1 + \eta. \end{cases}$

Example 5.4.

$$\varepsilon z''(\theta) - \left(1 + exp\left(\theta^2\right)\right) z'(\theta) - \theta z(\theta - \delta) + \theta^2 z(\theta) -$$
$$(1 - exp(-\theta))\, z(\theta + \eta) = 1,$$

with boundary constraints $z(\theta) = \begin{cases} 1, & -\delta \le \theta \le 0 \\ -1, & 1 \le \theta \le 1 + \eta. \end{cases}$

6. Conclusion

A novel finite difference algorithm is suggested for solving SPDDE of second order with mixed shifts using a non-polynomial cubic spline with fitting factor. To represent the validity and efficiency of the method, we solved four test problems for different values N and with $\delta = 0.5\varepsilon = \eta$ and recorded computational results in the form of MAEs and ROCs. Using MATLAB, the MAEs in the solutions of the Examples 5.1, 5.2 and 5.3 are listed in comparison to the method given in [21] in Tables 1, 2, 3 and 4. Tables 5 and 6 compare the MAEs in Example 5.4 solution to the method described in [16]. The mixed shifts have no significant impact on the layer behaviour of the problems with boundary layers at the left-side and right-side of the points in the given interval shown in Figures. 1, 2, 3, 4, 5, 6, 7 and 8. Based on the results, we observe that the thickness of the layer increases as the size of the delay parameter increases and decreases as the size of the advance parameter increases. The proposed method is simple and can be easily implemented on a computer.

Table 1. MAEs of Example 5.1 with various values of ε

$\varepsilon \downarrow N \rightarrow$	2^3	2^4	2^5	2^6	2^7	2^8
Present method			$\eta = \delta = 0.5\varepsilon$			
10^{-1}	3.769e-03	9.401e-04	2.209e-04	5.439e-05	1.359e-05	3.396e-06
	2.0034	2.0889	2.0223	2.0002	1.7786	
10^{-2}	8.914e-03	4.578e-03	2.006e-03	7.512e-04	1.893e-04	4.425e-05
	0.9611	1.1905	1.4168	1.9885	2.0968	
10^{-3}	8.968e-03	5.034e-03	2.683e-03	1.386e-03	6.778e-04	2.835e-04
	0.8330	0.9076	0.9528	1.0324	1.2573	
10^{-4}	8.970e-03	5.036e-03	2.684e-03	1.388e-03	7.060e-04	3.561e-04
	0.8334	0.9075	0.9517	0.9752	0.9874	
10^{-5}	8.970e-03	5.036e-03	2.684e-03	1.388e-03	7.060e-04	3.561e-04
	0.8334	0.9075	0.9516	0.9752	0.9874	
10^{-6}	8.970e-03	5.036e-03	2.684e-03	1.388e-03	7.061e-04	3.561e-04
	0.8334	0.9075	0.9516	0.9752	0.9874	
Results in [21]			$\eta = \delta = 0.5\varepsilon$			
10^{-1}	3.658e-03	9.595e-04	2.409e-04	6.759e-05	1.776e-05	1.232e-05
10^{-2}	1.695e-02	7.297e-03	2.486e-03	6.964e-04	1.776e-04	2.616e-05
10^{-3}	2.020e-02	1.047e-02	5.210e-03	2.461e-03	1.057e-03	3.771e-04
10^{-4}	2.052e-02	1.079e-02	5.520e-03	2.769e-03	1.363e-03	6.539e-04
10^{-5}	2.061e-02	1.088e-02	5.608e-03	2.858e-03	1.453e-03	7.417e-04
10^{-6}	1.951e-02	9.783e-03	4.513e-03	1.762e-03	3.577e-04	3.729e-04

Table 2. MAEs in Example 5.2

$\varepsilon \downarrow N \rightarrow$	10^1	10^2	10^3	10^4
Present method		$\eta = 0.5\varepsilon$	$\delta = 0.7\varepsilon$	
10^{-1}	1.243e-02	1.581e-04	1.585e-06	1.585e-08
10^{-2}	2.466e-02	1.819e-03	2.077e-05	2.080e-07
10^{-3}	2.488e-02	3.337e-03	1.914e-04	2.157e-06
10^{-4}	2.490e-02	3.341e-03	3.458e-04	1.924e-05
Results in [21]		$\eta = 0.5\varepsilon$	$\delta = 0.7\varepsilon$	
10^{-1}	1.533e-02	1.917e-04	1.921e-06	1.917e-08
10^{-2}	2.817e-02	1.865e-03	2.024e-05	2.026e-07
10^{-3}	2.853e-02	3.389e-03	1.919e-04	2.162e-06
10^{-4}	2.857e-02	3.395e-03	3.463e-04	1.925e-05

Table 3. MAEs and ROCs in Example 5.2

$\varepsilon \downarrow N \rightarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Present method			$\eta = \delta = 0.5\varepsilon$			
2^{-3}	1.150e-03	2.919e-04	7.324e-05	1.832e-05	4.583e-06	1.145e-06
	1.9789	1.9948	1.9987	1.9996	1.9999	
2^{-4}	2.582e-03	6.718e-04	1.697e-04	4.253e-05	1.064e-05	2.660e-06
	1.9423	1.9851	1.9961	1.9990	1.9997	
2^{-5}	5.108e-03	1.437e-03	3.712e-04	9.358e-05	2.344e-05	5.864e-06
	1.8290	1.9533	1.9880	1.9969	1.9992	
2^{-6}	8.048e-03	2.759e-03	7.653e-04	1.968e-04	4.955e-05	1.241e-05
	1.5444	1.8501	1.9592	1.9896	1.9973	
2^{-7}	9.448e-03	4.293e-03	1.438e-03	3.959e-04	1.016e-04	2.556e-05
	1.1381	1.5751	1.8615	1.9623	1.9904	
2^{-8}	9.619e-03	5.019e-03	2.221e-03	7.353e-04	2.015e-04	5.165e-05
	0.9384	1.1762	1.5946	1.8674	1.9640	
2^{-9}	9.641e-03	5.099e-03	2.590e-03	1.130e-03	3.718e-04	1.017e-04
	0.9191	0.9770	1.1965	1.6038	1.8703	
Results in [21]			$\eta = \delta = 0.5\varepsilon$			
2^{-3}	1.378e-03	3.486e-04	8.742e-05	2.187e-05	5.469e-06	1.367e-06
2^{-4}	2.880e-03	7.458e-04	1.881e-04	4.714e-05	1.179e-05	2.948e-06
2^{-5}	5.477e-03	1.526e-03	3.930e-04	9.902e-05	2.480e-05	6.204e-06
2^{-6}	8.487e-03	2.862e-03	7.898e-04	2.028e-04	5.105e-05	1.278e-05
2^{-7}	9.922e-03	4.413e-03	1.466e-03	4.024e-04	1.031e-04	2.596e-05
2^{-8}	1.009e-02	5.148e-03	2.252e-03	7.424e-04	2.032e-04	5.206e-05
2^{-9}	1.011e-02	5.228e-03	2.624e-03	1.138e-03	3.736e-04	1.021e-04

Table 4. MAEs and ROCs in Example 5.3

$\varepsilon \downarrow N \rightarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Present method			$\eta = \delta = 0.5\varepsilon$			
2^{-3}	5.102e-04	1.261e-04	3.161e-05	7.897e-06	1.973e-06	4.934e-07
	2.0165	1.9960	2.0010	2.0002	2.0000	
2^{-4}	1.174e-03	2.837e-04	7.153e-05	1.784e-05	4.457e-06	1.114e-06
	2.0489	1.9878	2.0034	2.0008	2.0002	
2^{-5}	2.676e-03	6.223e-04	1.550e-04	3.864e-05	9.656e-06	2.413e-06
	2.1045	2.0049	2.0042	2.0007	2.0002	
2^{-6}	4.201e-03	1.413e-03	3.222e-04	8.159e-05	2.024e-05	5.067e-06
	1.5715	2.1330	1.9815	2.0111	1.9978	
2^{-7}	4.818e-03	2.205e-03	7.283e-04	1.642e-04	4.195e-05	1.041e-05
	1.1270	1.5986	2.1483	1.9692	2.0107	
2^{-8}	4.980e-03	2.520e-03	1.132e-03	3.700e-04	8.300e-05	2.129e-05
	0.9823	1.1550	1.6131	2.1563	1.9629	
2^{-9}	4.997e-03	2.604e-03	1.290e-03	5.736e-04	1.865e-04	4.172e-05
	0.9401	1.0127	1.1700	1.6206	2.1604	
Results in [21]			$\eta = \delta = 0.5\varepsilon$			
2^{-3}	8.434e-04	2.112e-04	5.284e-05	1.321e-05	3.303e-06	8.260e-07
2^{-4}	4.172e-03	1.047e-03	2.640e-04	6.602e-05	1.650e-05	4.127e-06
2^{-5}	1.858e-02	4.743e-03	1.190e-03	2.980e-04	7.452e-05	1.864e-05
2^{-6}	6.074e-02	1.988e-02	5.080e-03	1.275e-03	3.192e-04	7.981e-05
2^{-7}	1.111e-01	6.451e-02	2.061e-02	5.270e-03	1.323e-03	3.311e-04
2^{-8}	1.297e-01	1.176e-01	6.658e-02	2.101e-02	5.372e-03	1.349e-03
2^{-9}	1.310e-01	1.372e-01	1.212e-01	6.766e-02	2.122e-02	5.425e-03

Table 5. MAEs and ROCs in Example 5.4

$\varepsilon \downarrow N \rightarrow$	2^5	2^6	2^7	2^8	2^9
Present method			$\eta = \delta = 0.5\varepsilon$		
2^{-3}	4.896e-03	1.129e-03	2.774e-04	6.934e-05	1.731e-05
	2.1154	2.0257	2.0005	2.0018	
2^{-4}	1.114e-02	2.544e-03	5.859e-04	1.434e-04	3.587e-05
	2.1312	2.1187	2.0302	1.9992	
2^{-5}	1.898e-02	5.677e-03	1.295e-03	2.979e-04	7.289e-05
	1.7412	2.1315	2.1208	2.0309	
2^{-6}	2.371e-02	9.579e-03	2.864e-03	6.537e-04	1.501e-04
	1.3076	1.7414	2.1317	2.1221	
2^{-7}	2.448e-02	1.188e-02	4.811e-03	1.438e-03	3.282e-04
	1.0425	1.3047	1.7415	2.1319	
2^{-8}	2.449e-02	1.224e-02	5.950e-03	2.411e-03	7.210e-04
	1.0000	1.0414	1.3032	1.7416	
Results in [16]			$\eta = \delta = 0.5\varepsilon$		
2^{-3}	8.354e-03	2.013e-03	4.986e-04	1.249e-04	3.121e-05
2^{-4}	1.719e-02	4.378e-03	1.041e-03	2.571e-04	6.429e-05
2^{-5}	2.517e-02	8.889e-03	2.238e-03	5.290e-04	1.303e-04
2^{-6}	3.154e-02	1.294e-02	4.516e-03	1.131e-03	2.664e-04
2^{-7}	4.478e-02	1.622e-02	6.559e-03	2.276e-03	5.686e-04
2^{-8}	7.878e-02	2.317e-02	8.224e-03	3.301e-03	1.142e-03

Table 6. MAE in Example 5.4 with $\varepsilon = 0.1$

$N \rightarrow$	10^1	10^2	10^3	10^4
Present method				
$\delta \downarrow$		$\eta = 0.5\varepsilon$		
0.00	8.123e-02	6.690e-03	3.573e-04	2.225e-05
0.05	8.066e-02	6.582e-03	3.529e-04	2.194e-05
0.09	8.019e-02	6.495e-03	3.494e-04	2.171e-05
$\eta \downarrow$		$\delta = 0.5\varepsilon$		
0.00	8.051e-02	6.527e-03	3.508e-04	2.180e-05
0.05	8.066e-02	6.582e-03	3.529e-04	2.194e-05
0.09	8.077e-02	6.626e-03	3.546e-04	2.206e-05
Results in [16]				
$\delta \downarrow$		$\eta = 0.5\varepsilon$		
0.00	9.109e-02	1.112e-02	6.382e-04	4.004e-05
0.05	9.047e-02	1.095e-02	6.306e-04	3.950e-05
0.09	8.996e-02	1.082e-02	6.244e-04	3.906e-05
$\eta \downarrow$		$\delta = 0.5\varepsilon$		
0.00	9.604e-02	1.116e-02	6.458e-04	3.924e-05
0.05	9.621e-02	1.124e-02	6.494e-04	3.950e-05
0.09	9.634e-02	1.131e-02	6.522e-04	3.970e-05

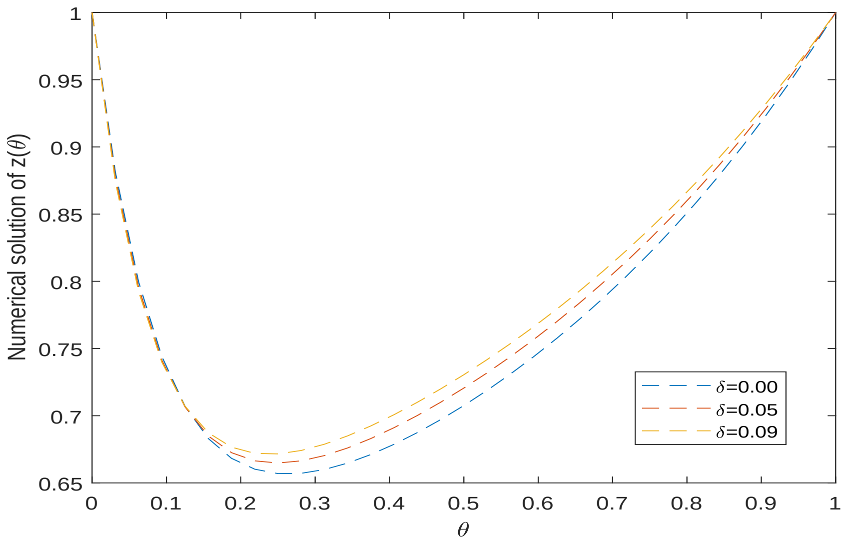

Figure 1. Layer profile in Example 5.1 with δ , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

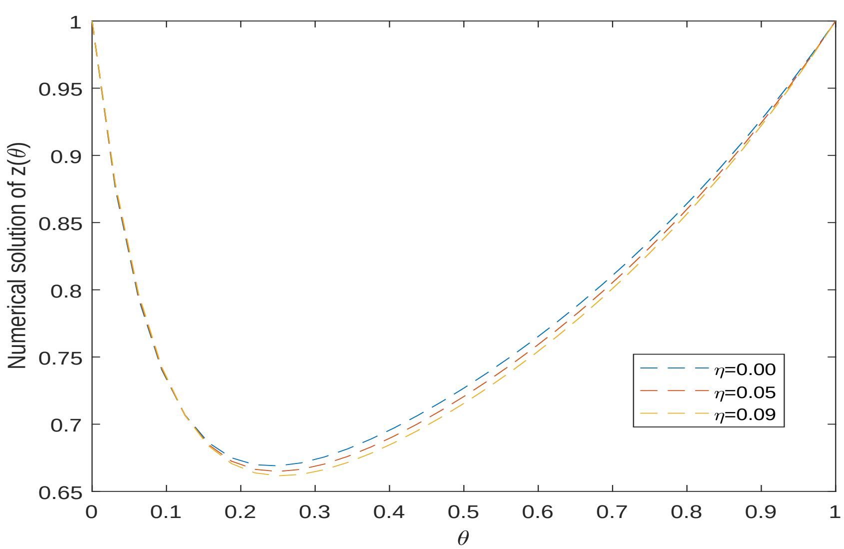

Figure 2. Layer profile of Example 5.1 with η , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

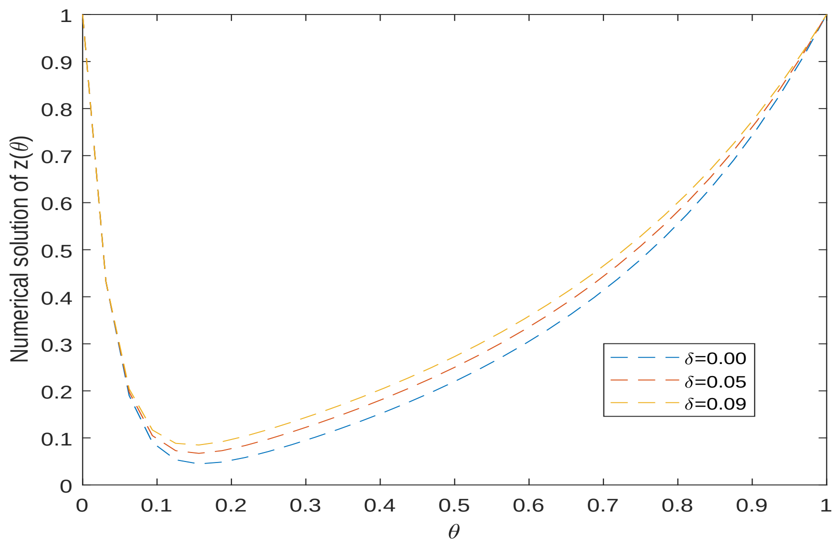

Figure 3. Layer profile of Example 5.2 with δ , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

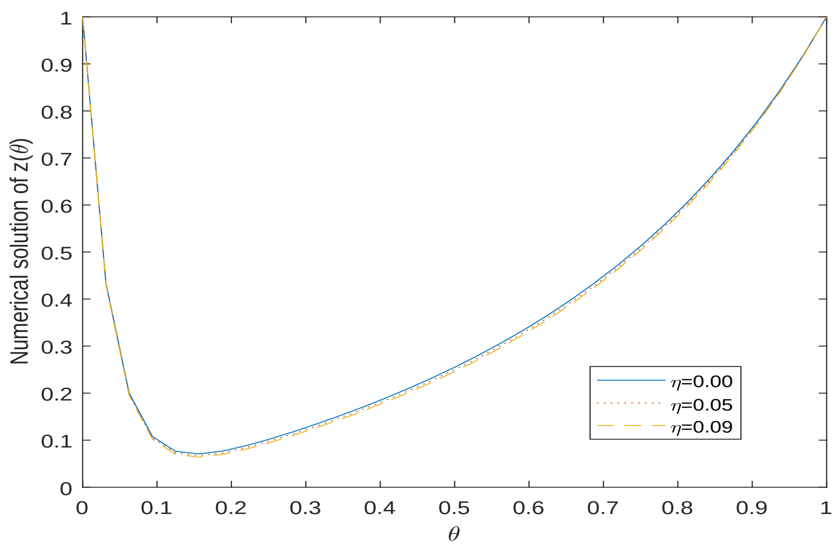

Figure 4. Layer profile of Example 5.2 with η , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

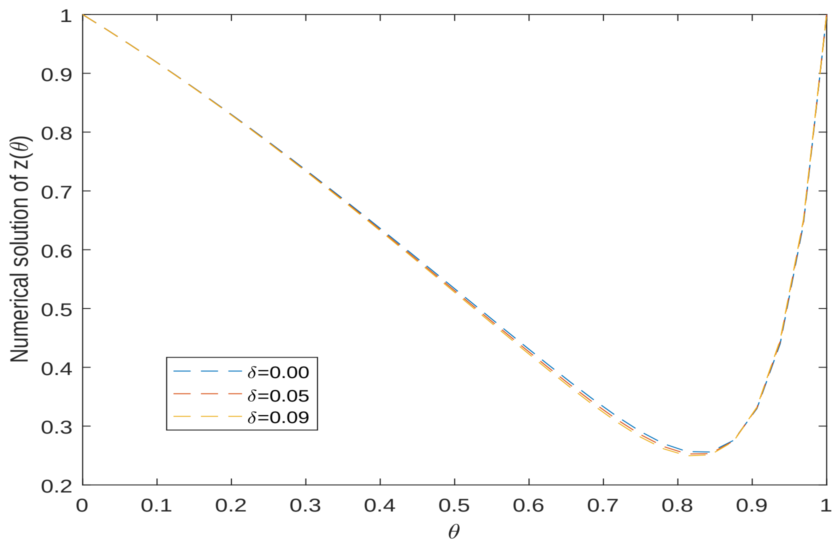

Figure 5. Layer profile of Example 5.3 with δ , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

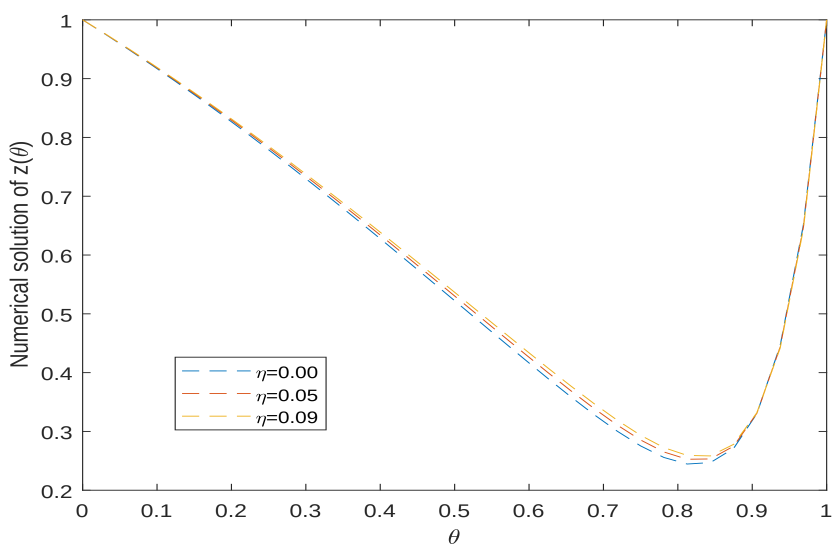

Figure 6. Layer profile of Example 5.3 with η , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

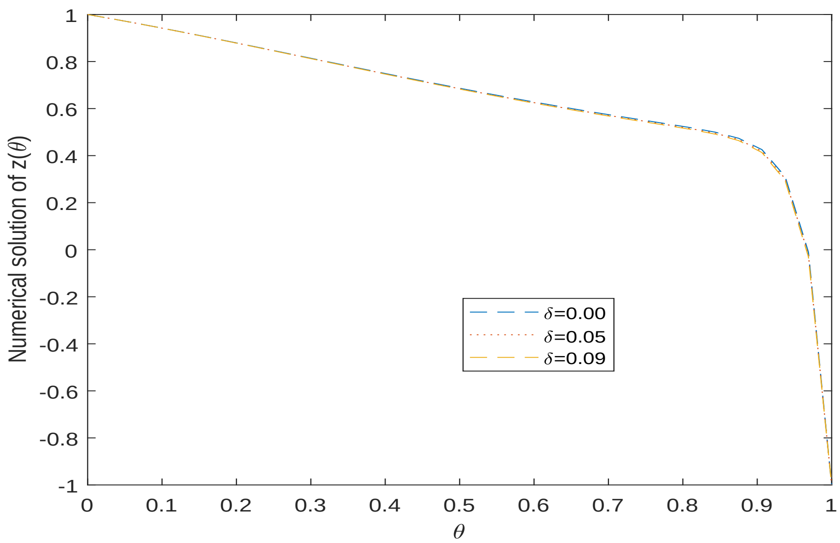

Figure 7. Layer profile of Example 5.4 with δ , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

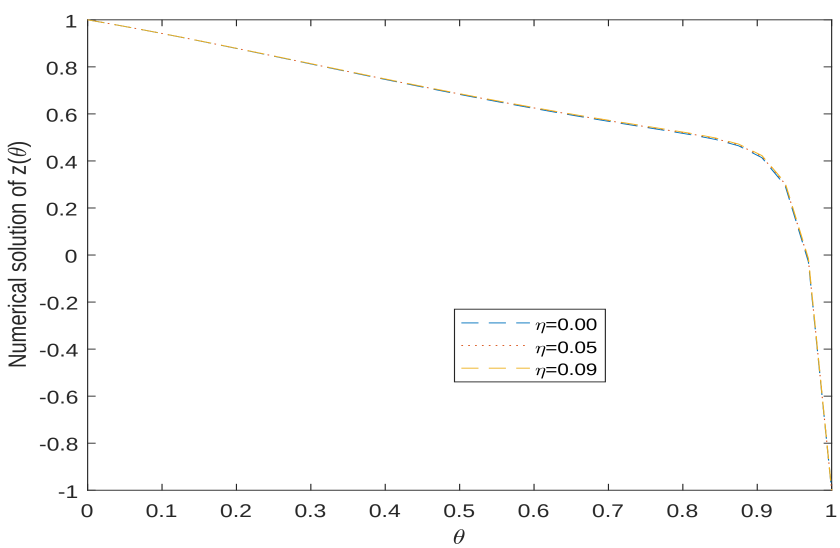

Figure 8. Layer profile of Example 5.4 with η , $N = 2^5$, $\varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

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