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Functional Impulsive Fractional Differential Equations Involving the Caputo-Hadamard Derivative and Integral Boundary Conditions

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for functional impulsive fractional differential equations and integral boundary conditions. Our results are based on some fixed point theorems. Finally, we provide an example to illustrate the validity of our main results.

1. Introduction

In this paper, we discuss the existence and uniqueness of solutions to a boundary value problem (BVP for short) for functional impulsive fractional differential equation, in the following form:

$$ {}_{H}^{C}D^{r}y(t) = f(t, y_t), \quad t \in J = [a, T], t \neq t_k, \quad k = 1, .., m, \tag{1.1}$$

$$ \Delta y \mid_{t=t_k} = I_k(y(t_k^-)), \quad t = t_k, \quad k = 1, .., m, \tag{1.2}$$

$$ \Delta y' \mid_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad t = t_k, \quad k = 1, .., m, \tag{1.3}$$

$$ y(t) = \phi(t), \quad t \in [a-\tau, a], \quad y'(T) = \int_a^T h(s, y(s))ds. \tag{1.4}$$

where ${}_{H}^{C}D^{r}$ is the Caputo-Hadamard fractional derivative of order $1 < r \leq 2$, $a > 0$, $f : J \times C([a-\tau, a], \mathbb{R}) \to \mathbb{R}$, $h : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $\phi \in C([a-\tau, a], \mathbb{R})$ and $I_k, \overline{I_k} \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, ..., m$, $a = t_0 < t_1 < ... < t_m < t_{m+1} = T$. For any continuous functions y defined on

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$J' = J \backslash \{t_1, ..., t_m\}$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ and $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, $y(t_k^+)$, $y(t_k^-)$ represent the right and the left limits of $y(t)$ at $t = t_k$, we denote by y_t the element of $C_\tau = C([a-\tau, a], \mathbb{R})$, defined by $y_t(\theta) = y(t+\theta)$,$\theta \in [a-\tau, a]$, hence $y_t(\cdot)$ represents the history of the state from time $t - \tau$ up to the present time t.

In the last few decades, the analysis of impulsive boundary value problems has developed. It has also been extremely useful in developing various applied mathematical models of real-world processes in engineering and applied sciences. Tian and Bai [16] discussed some existence results of impulsive boundary value problems involving Caputo's type fractional derivatives. Results of existence and uniqueness have been developed using fixed-point theorem. Recently, it has been noted that much of the works on this subject are focused on the fractional differential equations of Riemann-Liouville and Caputo types with different conditions such as impulses, time delays, boundary value conditions [1, 6–9, 14, 20, 21].

The Hadamard fractional derivative, introduced in 1892, [10] is another type of fractional derivative that appears in the literature alongside the Riemann-Liouville and Caputo derivatives. It differs from the previous ones in that it contains an arbitrary logarithm function, further details can be found in [3–5]. Next, Jarad et al. proposed a Caputo-type modification of the Hadamard fractional derivative in [15] by the Caputo Hadamard fractional derivative and implemented the fundamental fractional calculus theorem in the Caputo-Hadamard. Recently, some researchers have focused on impulsive differential equations with Hadamard and Caputo-Hadamard derivatives (see [11–13, 17, 18] and the references therein).

The rest of the paper is organized as follows. In Section 2, we introduce some notions preliminary and properties on the fractional culcules. In Section 3, we give a supporting lemma describing the solutions of the considered problem and discuss the main findings. Finally, we give an example to illustrate the obtained results.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts that will be used in the remainder of this paper.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

Also C_τ is endowed with the norm

$$\|\phi\|_{C_\tau} = \sup\{\|\phi(\theta)\| : a - \tau \leq \theta \leq a\}.$$

Let $L^1(J, \mathbb{R})$ as the Banach space of Lebesgue integrable functions $y : J \longrightarrow \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_a^T |y(t)| dt.$$

The space $AC(J,\mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$ that are absolutely continuous. Let $\delta = t\frac{d}{dt}$, and then we set

$$AC_\delta^n(J,\mathbb{R}) = \{y : J \longrightarrow \mathbb{R},\ \delta^{n-1}y(t) \in AC(J,\mathbb{R})\}.$$

Definition 2.1. *[19] The Hadamard derivative of fractional order r for a C^{n-1} function $y : [a,T] \to \mathbb{R}$ is defined by*

$$^{H}D^{r}y(t) = \frac{1}{\Gamma(n-r)}(t\frac{d}{dt})^{n}\int_a^t \left(\log\frac{t}{s}\right)^{n-r-1} y(s)\frac{ds}{s}, n-1 < r < n, n = [r]+1.$$

Definition 2.2. *[19] The Hadamard fractional integral of order r for a continuous function y is defined as a function*

$$^{H}I^{r}y(t) = \frac{1}{\Gamma(r)}\int_a^t \left(\log\frac{t}{s}\right)^{r-1} y(s)\frac{ds}{s},\ r > 0,$$

provided the integral exists.

Definition 2.3. *[19] For an $n-$times differentiable function $y : [a,T] \to \mathbb{R}$ the Caputo type Hadamard derivative of fractional order r is defined as*

$$^{C}_{H}D^{r}y(t) = \frac{1}{\Gamma(n-r)}\int_a^t \left(\log\frac{t}{s}\right)^{n-r-1} \delta^{n}y(s)\frac{ds}{s},\ \ n-1 < r < n,\ n = [r]+1,$$

where $\delta = t\frac{d}{dt}$ and $[r]$ denotes the integer part of the real number r and $\log(.) = \log_e(.)$.

Lemma 2.1. *[2] Let $r \in \mathbb{R}^+$ and $n = [r] = 1$. If $y(t) \in AC_\delta^n(J,\mathbb{R})$ then Caputo-Hadamard fractional differential equation*

$$^{C}D_a^{r}y(t) = 0$$

has a solution

$$y(t) = \sum_{k=0}^{n-1} c_k \left(\log\frac{t}{a}\right)^{k},$$

and the following formula holds:

$$^{H}I_a^{r}(^{C}_{H}D_a^{r}y)(t) = y(t) + \sum_{k=0}^{n-1} c_k \left(\log\frac{t}{a}\right)^{k},$$

where $c_k \in \mathbb{R},\ k = 0,1,2,...,n-1$.

3. Existence of Solutions

In this section, we will establish the existence and uniqueness of solutions for (1.1)-(1.4).

$$AC'(J,\mathbb{R}) = \left\{ \begin{array}{l} y : J \to \mathbb{R},\ y \in AC_\delta^2((t_k,t_{k+1}],\mathbb{R}) \text{ and there exist } y(t_k^+) \text{ and } y(t_k^-),\ \ k = 1,...,m, \\ \text{with, } y(t_k^-) = y(t_k) \end{array} \right\},$$

with the norm

$$\|y\|_{AC'} = \sup\{\|y(t)\| : a \le t \le T\}.$$

Let B be set defined by

$$B = \{y : (a-\tau, T] \to \mathbb{R} \ \backslash \ y \in AC'(J, \mathbb{R}) \cap C_\tau\},$$

is endowed with the norm

$$\|y\|_B = \sup\{\|y(t)\| : t \in [a-\tau, T]\}.$$

Definition 3.1. *A function $y \in B$ is said to be a solution of the problem (1.1)-(1.4) if y satisfies the equation ${}^C_H D^r y(t) = f(t, y_t)$ on J' and the conditions (1.2)-(1.4).*

We need the following auxiliary lemma to prove the existence and uniqueness of solutions to the problem(1.1)-(1.4).

Lemma 3.1. *Let $1 < r \leq 2$. Assume that $\sigma, \varrho \in AC^2(J, \mathbb{R})$, then the following BVP :*

$${}^H_C D^r y(t) = \sigma(t), \quad t \in J = [a, T], t \neq t_k, \tag{3.1}$$

$$\Delta y \mid_{t=t_k} = I_k(y(t_k^-)), \quad t = t_k, \quad k = 1, .., m, \tag{3.2}$$

$$\Delta y' \mid_{t=t_k} = \overline{I}_k(y(t_k^-)) \ , \quad t = t_k, \quad k = 1, .., m, \tag{3.3}$$

$$y(a) = \overline{y}, \quad y'(T) = \int_a^T \varrho(s) ds, \tag{3.4}$$

has the following integral equation:

$$y(t) = \begin{cases} \overline{y} + c_2 \left(\log \frac{t}{a}\right) + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s}, & if \ t \in [a, t_1] \\ \overline{y} + c_2 \left(\log \frac{t}{a}\right) + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ +\frac{1}{\Gamma(r)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ +\sum_{i=1}^k \frac{\left(\log \frac{t}{t_i}\right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \sum_{i=1}^k I_i(y(t_i^-)) \\ +\sum_{i=1}^k t_i \left(\log \frac{t}{t_i}\right) \overline{I}_i(y(t_i^-)), \ if \ t \in (t_k, t_{k+1}], \ k = 1, ..., m, \end{cases}$$

where

$$c_2 = T \int_a^T \varrho(s) ds - \left[\frac{1}{\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{T}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \sum_{i=1}^m \frac{1}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \sum_{i=1}^m (t_i) \overline{I}_i(y(t_i^-))\right]$$

proof: Let y be the solution of (3.1)-(3.4). For $t \in [a, t_1]$. Using Lemma 2.1, for some constants $c_1, c_2 \in \mathbb{R}$, we have

$$y(t) = \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + c_1 + c_2 \left(\log \frac{t}{a}\right)$$

Acccording the a condition $y(a) = \overline{y}$, we deduce that $c_1 = \overline{y}$ and thus

$$y(t) = \overline{y} + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + c_2 \left(\log \frac{t}{a}\right),$$

$$y'(t) = \frac{1}{t\Gamma(r-1)} \int_a^t \left(\log \frac{t}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \frac{c_2}{t}.$$

If $t \in (t_1, t_2]$, then we have

$$y(t) = \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + d_1 + d_2 \left(\log \frac{t}{t_1}\right),$$

and

$$y'(t) = \frac{1}{t\Gamma(r-1)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \frac{d_2}{t}.$$

Using the impulses conditions $\Delta y \mid_{t=t_1} = y(t_1^+) - y(t_1^-) = I_1(y(t_1^-))$ and $\Delta y' \mid_{t=t_1} = y'(t_1^+) - y'(t_1^-) = \overline{I}_1(y(t_1^-))$, we obtain

$$d_1 = I_1(y(t_1^-)) + \overline{y} + c_2 \left(\log \frac{t_1}{a}\right) + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \sigma(s) \frac{ds}{s}.$$

$$d_2 = t_1 \overline{I}_1(y(t_1^-)) + c_2 + \frac{1}{\Gamma(r-1)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \sigma(s) \frac{ds}{s}.$$

Thus, for $t \in (t_1, t_2]$ we have

$$y(t) = \overline{y} + \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + \frac{\left(\log \frac{t}{t_1}\right)}{\Gamma(r-1)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \sigma(s) \frac{ds}{s}$$
$$+ t_1 \left(\log \frac{t}{t_1}\right) \overline{I}_1(y(t_1^-)) + I_1(y(t_1^-)) + c_2 \left(\log \frac{t}{a}\right).$$

Continuinq in the same manner, we obtain for $t \in (t_m, T]$,

$$y(t) = \overline{y} + \frac{1}{\Gamma(r)} \int_{t_m}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + \frac{1}{\Gamma(r)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + \sum_{i=1}^m I_i(y(t_i^-))$$
$$+ \sum_{i=1}^m \frac{\left(\log \frac{t}{t_i}\right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \sum_{i=1}^m t_i \left(\log \frac{t}{t_i}\right) \overline{I}_i(y(t_i^-)) + c_2 \left(\log \frac{t}{a}\right),$$

and

$$y'(t) = \frac{c_2}{t} + \frac{1}{t\Gamma(r-1)} \int_{t_m}^t \left(\log \frac{t}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \sum_{i=1}^m \frac{1}{t\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s) \frac{ds}{s} + \sum_{i=1}^m \left(\frac{t_i}{t}\right) \overline{I}_i(y(t_i^-)).$$

By the application of the boundary condition $y'(T)=\int_a^T \varrho(s)ds$, we have

$$y'(T)=\frac{c_2}{T}+\frac{1}{T\Gamma(r-1)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-2}\sigma(s)\frac{ds}{s}+\sum_{i=1}^{m}\frac{1}{T\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\sigma(s)\frac{ds}{s}$$
$$+\sum_{i=1}^{m}\left(\frac{t_i}{T}\right)\overline{I}_i(y(t_i^-)).$$

We obtain the reguired value of the constant c_2, where

$c_2=T\int_a^T\varrho(s)ds-\left[\frac{1}{\Gamma(r-1)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-2}\sigma(s)\frac{ds}{s}+\sum_{i=1}^{m}\frac{1}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\sigma(s)\frac{ds}{s}+\sum_{i=1}^{m}(t_i)\overline{I}_i(y(t_i^- t))\right].$

This completes the proof.

Our first result is based on the uniqueness of solutions for problem (1.1)-(1.4) and relies on the Banach fixed point theorem.

Theorem 3.1. *Assume that :*

(H1) *There exists a constant $L_1>0$ such that*

$$|f(t,u)-f(t,v)|\leq L_1\|u-v\|_{C_\tau},\quad \textit{for each } t\in J \textit{ and } u,v\in C_\tau.$$

(H2) *There exists a constants $L_2>0$ such that*

$$|h(t,x)-h(t,y)|\leq L_2|x-y|,\quad \textit{for each } t\in J \textit{ and } x,y\in\mathbb{R}.$$

(H3) *For each $k=1,2,...,m$, there exist l, $l^*>0$ such that*

$$|I_k(x)-I_k(y)|\leq l|x-y|,\ |\overline{I}_k(x)-\overline{I}_k(y)|\leq l^*|x-y|,\quad \textit{for each}\quad x,y\in\mathbb{R}.$$

If the condition

$$\left[L_1\left(\frac{m+1}{\Gamma(r+1)}+\frac{1+2m}{\Gamma(r)}\right)\left(\log\frac{T}{a}\right)^r+L_2T(T-a)\left(\log\frac{T}{a}\right)+ml+2ml^*T\left(\log\frac{T}{a}\right)\right]<1, \tag{3.5}$$

then the boundary value problem (1.1)-(1.4) has a unique solution on $[a-\tau,T]$.

proof : Transform the problem (1.1)-(1.4) into a fixed point problem. Consider the operator $F:B\to B$ defined by:

$$(Fy)(t)=\begin{cases}\phi(t), & \text{if}\quad t\in(a-\tau,a]\\ \phi(a)+T\left(\log\frac{t}{a}\right)\int_a^T h(s,y(s))ds-\left(\log\frac{t}{a}\right)[\frac{1}{\Gamma(r-1)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-2}f(s,y_s)\frac{ds}{s} & \\ +\frac{1}{\Gamma(r-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}f(s,y_s)\frac{ds}{s}+\sum_{i=1}^{m}t_i\overline{I}_i(y(t_i^-))] & \\ +\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}f(s,y_s)\frac{ds}{s}+\frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}f(s,y_s)\frac{ds}{s} & \\ +\sum_{i=1}^{k}\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}f(s,y_s)\frac{ds}{s}+\sum_{i=1}^{k}I_i(y(t_i^-)) & \\ +\sum_{i=1}^{k}t_i\left(\log\frac{t}{t_i}\right)\overline{I}_i(y(t_i^-)), & \text{if}\ t\in[t_k,t_{k+1}],\ k=1,...,m.\end{cases} \tag{3.6}$$

Clearly, the fixed point of the operator F are solutions of problem (1.1)-(1.4). Let $x, y \in B$, if $t \in [a-\tau, a]$, we have

$$|(Fx)(t) - (Fy)(t)| = |\phi(t) - \phi(t)| = 0$$

If $t \in [a, T]$, by (H1)-(H3), we have:

$$\begin{aligned}
& |(Fx)(t) - (Fy)(t)| \\
\leq\ & T|\left(\log\frac{t}{a}\right)|\int_a^T |h(s,x(s)) - h(s,y(s))|ds + \frac{|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\int_{t_m}^T \left(\log\frac{T}{s}\right)^{r-2}|f(s,x_s) - f(s,y_s)|\frac{ds}{s} \\
& + \frac{|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\sum_{i=1}^m \int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,x_s) - f(s,y_s)|\frac{ds}{s} \\
& + |\left(\log\frac{t}{a}\right)|\sum_{i=1}^m t_i|\bar{I}_i(x(t_i^-)) - \bar{I}_i(y(t_i^-))| + \frac{1}{\Gamma(r)}\int_{t_k}^t \left(\log\frac{t}{s}\right)^{r-1}|f(s,x_s) - f(s,y_s)|\frac{ds}{s} \\
& + \frac{1}{\Gamma(r)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|f(s,x_s) - f(s,y_s)|\frac{ds}{s} \\
& + \sum_{i=1}^k \frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,x_s) - f(s,y_s)|\frac{ds}{s} \\
& + \sum_{i=1}^k |I_i(x(t_i^-)) - I_i(y(t_i^-))| + \sum_{i=1}^k t_i\left(\log\frac{t}{t_i}\right)|\bar{I}_i(x(t_i^-)) - \bar{I}_i(y(t_i^-))| \\
\leq\ & T\left(\log\frac{t}{a}\right)\int_a^T L_2|x(s) - y(s)|ds + \frac{|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}L_1\|x_s - y_s\|_{C_\tau}\frac{ds}{s} \\
& + \frac{|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}L_1\|x_s - y_s\|_{C_\tau}\frac{ds}{s} + |\left(\log\frac{t}{a}\right)|\sum_{i=1}^m t_i l^*|x(t_i^-) - y(t_i^-)| \\
& + \frac{1}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}L_1\|x_s - y_s\|_{C_\tau}\frac{ds}{s} + \frac{1}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}L_1\|x_s - y_s\|_{C_\tau}\frac{ds}{s} \\
& + \sum_{i=1}^k\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}L_1\|x_s - y_s\|_{C_\tau}\frac{ds}{s} + \sum_{i=1}^k l|x(t_i^-) - y(t_i^-)| \\
& + \sum_{i=1}^k t_i\left(\log\frac{t}{t_i}\right)l^*|x(t_i^-) - y(t_i^-)| \\
\leq\ & T\left(\log\frac{T}{a}\right)(T-a)L_2\|x-y\| + \left(\log\frac{T}{a}\right)[\frac{L_1\|x-y\|}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}\frac{ds}{s} \\
& + \frac{L_1\|x-y\|}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\frac{ds}{s} + mTl^*\|x-y\|] \\
& + \frac{L_1\|x-y\|}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}\frac{ds}{s} + \frac{L_1\|x-y\|}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\frac{ds}{s} \\
& + \sum_{i=1}^k\frac{L_1\|x-y\|\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\frac{ds}{s} + ml\|x-y\| + mT\left(\log\frac{T}{a}\right)l^*\|x-y\| \\
\leq\ & \left[L_2(T-a)T\left(\log\frac{T}{a}\right) + L_1\left(\frac{m+1}{\Gamma(r+1)} + \frac{1+2m}{\Gamma(r)}\right)\left(\log\frac{T}{a}\right)^r + ml + 2mT\left(\log\frac{T}{a}\right)l^*\right]\|x-y\|
\end{aligned}$$

Thus, we have

$$\|Fx - Fy\| \leq \left[L_1\left(\frac{m+1}{\Gamma(r+1)} + \frac{1+2m}{\Gamma(r)}\right)\left(\log\frac{T}{a}\right)^r + L_2T(T-a)\left(\log\frac{T}{a}\right) + ml + 2ml^*T\left(\log\frac{T}{a}\right)\right]\|x-y\|.$$

Consequently by (3.5), F is a contraction, as a conseguence of Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1)-(1.4). This completes the proof.

Our second result deals with the existence of solutions for problem (1.1)-(1.4) by applying on Scheafer fixed point theorem.

Theorem 3.2. *Assume that the following conditions hold :*

(H4) *The function $f : J \times C_\tau \to \mathbb{R}$ is continuous.*

(H5) *The function $h : \mathbb{R} \to \mathbb{R}$ is continuous.*

(H6) *The functions I_k, $\bar{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous.*

(H7) *There exists a constant $N > 0$ such that $|f(t, y)| \leq N$, for each $t \in J$ and $y \in C_\tau$.*

(H8) *There exists a constant $N^* > 0$ such that $|h(t, x)| \leq N^*$ for each $x \in \mathbb{R}$.*

(H9) *There exist two constants $N_1 > 0$, $N_2 > 0$ such that $|I_k(x)| \leq N_1$, $|\bar{I}_k(x)| \leq N_2$ for each , $x \in \mathbb{R}$, $k = 1, \ldots, m$,*

then the boundary value problem (1.1)-(1.4) has at least one solution on $[a - \tau, T]$,

Proof: We shall use Scheafer fixed point theorem to prove that F has a fixed point, defined by 3.6. The proof will be given in several steps.

Step 1: *F is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in B. If $t \in [a, T]$, we have

$$
\begin{aligned}
|F(y_n)(t) - F(y)(t)| \leq\; & T\left(\log\frac{t}{a}\right)\int_a^T |h(s, y_n(s)) - h(s, y(s))|ds \\
& + \frac{(\log\frac{t}{a})}{\Gamma(r-1)}\int_{t_m}^T \left(\log\frac{T}{s}\right)^{r-2} |f(s, y_{ns}) - f(s, y_s))|\frac{ds}{s} \\
& + \frac{(\log\frac{t}{a})}{\Gamma(r-1)}\sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(\log\frac{t_i}{s}\right)^{r-2} |f(s, y_{ns}) - f(s, y_s))|\frac{ds}{s} \\
& + \left(\log\frac{t}{a}\right)\sum_{i=1}^m t_i|\bar{I}(y_n(t_i^-)) - \bar{I}_i(y(t_i^-))| \\
& + \frac{1}{\Gamma(r)}\int_{t_k}^t \left(\log\frac{t}{s}\right)^{r-1} |f(s, y_{ns}) - f(s, y_s)|\frac{ds}{s} \\
& + \frac{1}{\Gamma(r)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(\log\frac{t_i}{s}\right)^{r-1} |f(s, y_{ns}) - f(s, y_s)|\frac{ds}{s} \\
& + \sum_{i=1}^k \frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i} \left(\log\frac{t_i}{s}\right)^{r-2} |f(s, y_{ns}) - f(s, y_s)|\frac{ds}{s} \\
& + \sum_{i=1}^k |I_i(y_n(t_i^-)) - I_i(y(t_i^-))| + \sum_{i=1}^k t_i\left(\log\frac{t}{t_i}\right)|\bar{I}(y_n(t_i^-)) - \bar{I}_i(y(t_i^-))|
\end{aligned}
$$

Since $f,\ h$ and $I_k, \bar{I}_k,\quad k=1,\dots,m$, are continuous functions, we have

$$\|F(y_n)-F(y)\|_\infty \to 0 \text{ as } n\to\infty.$$

Step 2: *F maps bounded sets into bounded sets in B.*

Indeed, it is enough to show that for any $\eta^*>0$, there exists a positive constant L such that for each $y\in D_{\eta^*}=\{y\in B:\|y\|\le\eta^*\}$, we have $\|F(y)\|\le L$. By (H7), (H8) and (H9), for each $t\in J$, we can obtain

$$\begin{aligned}
|F(y)(t)| &\le |\varphi(a)|+T\left(\log\frac{t}{a}\right)\int_a^T|h(s,y(s))|ds+\frac{|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}|f(s,y_s)|\frac{ds}{s}\\
&\quad+\frac{|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y_s)|\frac{ds}{s}+|\left(\log\frac{t}{a}\right)|\sum_{i=1}^m t_i|\bar{I}_i(y(t_i^-))|\\
&\quad+\frac{1}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}|f(s,y_s)|\frac{ds}{s}+\frac{1}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|f(s,y_s)|\frac{ds}{s}\\
&\quad+\sum_{i=1}^k\frac{|\left(\log\frac{t}{t_i}\right)|}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y_s)|\frac{ds}{s}+\sum_{i=1}^k|I_i(y(t_i^-))|\\
&\quad+\sum_{i=1}^k t_i|\left(\log\frac{t}{t_i}\right)||\bar{I}_i(y(t_k^-))|\\
&\le |\varphi(a)|+N^*T\left(\log\frac{t}{a}\right)\int_a^T ds+\frac{N|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}\frac{ds}{s}\\
&\quad+\frac{N|\left(\log\frac{t}{a}\right)|}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\frac{ds}{s}+m\left(\log\frac{T}{a}\right)TN_2\\
&\quad+\frac{N}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}\frac{ds}{s}+\frac{N}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\frac{ds}{s}\\
&\quad+\sum_{i=1}^k\frac{N|\left(\log\frac{t}{t_i}\right)|}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\frac{ds}{s}+mN_1+mT(\log\frac{T}{a})N_2\\
&\le \|\varphi\|+(T-a)T\left(\log\frac{T}{a}\right)N^*+N\frac{(1+2m)\left(\log\frac{T}{a}\right)^r}{\Gamma(r)}+N\frac{(1+m)\left(\log\frac{T}{a}\right)^r}{\Gamma(r+1)}\\
&\quad+mN_1+2mT\left(\log\frac{T}{a}\right)N_2.
\end{aligned}$$

Therefore

$$\|Fy\| \le \|\varphi\|+N\left[\frac{1+m}{\Gamma(r+1)}+\frac{1+2m}{\Gamma(r)}\right]\left(\log\frac{T}{a}\right)^r+mN_1+[(T-a)N^*+2mN_2]\,T\left(\log\frac{T}{a}\right):=L.$$

Step 3: *F maps bounded sets into equicontinuous sets of B .*

Let $\tau_1,\tau_2\in J,\tau_1<\tau_2$, D_{η^*} be a bounded set of B as in Step 2, and let $y\in D_{\eta^*}$. Then

$$
\begin{aligned}
& |F(y)(\tau_2)-F(y)(\tau_1)| \\
\leq\ & T\left(\log\frac{\tau_2}{\tau_1}\right)\int_a^T|h(s,y(s))|ds+\frac{(\log\frac{\tau_2}{\tau_1})}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}|f(s,y_s)|\frac{ds}{s} \\
& +\frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(r-1)}\sum_{i=i}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y_s)|\frac{ds}{s}+\left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^m t_i|\overline{I}_i(y(t_i^-))| \\
& +\frac{1}{\Gamma(r)}\int_{t_k}^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{r-1}-\left(\log\frac{\tau_1}{s}\right)^{r-1}\right]|f(s,y_s)|\frac{ds}{s}+\frac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{r-1}|f(s,y_s)|\frac{ds}{s} \\
& +\sum_{i=1}^k\frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y_s)|\frac{ds}{s}+\left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^k t_i|\overline{I}_i(y(t_i^-))|.
\end{aligned}
$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $F : B \to B$ is completly continuous.

Step 4: *A priori bounds.*

Now it remains to show that the set $\varepsilon = \{y \in B \to B : y = \lambda F(y)\ for\ some\ 0 < \lambda < 1\}$ is bounded. Let $y \in \varepsilon$, then $y = \lambda F(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$
\begin{aligned}
(Fy)(t) \ = \ & \lambda\varphi(a)+\lambda T\left(\log\frac{t}{a}\right)\int_a^T h(s,y(s))ds-\frac{\lambda\left(\log\frac{t}{a}\right)}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}f(s,y_s)\frac{ds}{s} \\
& -\frac{\lambda\left(\log\frac{t}{a}\right)}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}f(s,y_s)\frac{ds}{s}-\lambda\left(\log\frac{t}{a}\right)\sum_{i=1}^m t_i\overline{I}_i(y(t_i^-)) \\
& +\frac{\lambda}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}f(s,y_s)\frac{ds}{s}+\frac{\lambda}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}f(s,y_s)\frac{ds}{s} \\
& +\lambda\sum_{i=1}^k\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}f(s,y_s)\frac{ds}{s}+\lambda\sum_{i=1}^k I_i(y(t_i^-)) \\
& +\lambda\sum_{i=1}^k t_i\left(\log\frac{t}{t_i}\right)\overline{I}_i(y(t_i^-))
\end{aligned}
$$

For each $t \in J$, by (H7)-(H9), we have

$$
\|Fy\| \ \leq\ \|\varphi\|+T(T-a)\left(\log\frac{T}{a}\right)N^*+N\left[\frac{1+m}{\Gamma(r+1)}+\frac{1+2m}{\Gamma(r)}\right]\left(\log\frac{T}{a}\right)^r+mN_1+2mT\left(\log\frac{T}{a}\right)N_2.
$$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1)-(1.4).

By applying the of Leray-Schauder nonlinear alternative type.

Theorem 3.3. *Assume that (H4)-(H6) and the following conditions hold :*

(H10) *There exist* $\phi_f \in C(J, \mathbb{R}^+)$ *and* $\psi : [0, \infty) \to [0, \infty)$ *continuous and non-decreasing such that*

$$|f(t, u)| \leq \phi_f(t)\psi(|u|), \ \ for \ \ all \ \ t \in J, \ \ u \in C_\tau.$$

(H11) *There exist* $\phi_h \in L(J, \mathbb{R}^+)$ *and* $\psi^* : [0, \infty) \to [0, \infty)$ *continuous and non-decreasing such that*

$$|h(t, v)| \leq \phi_h(t)\psi^*(|v|), \ \ for \ \ all \ \ t \in J, \ \ v \in \mathbb{R}.$$

(H12) *There exist* $\bar{\psi}^*$, $\bar{\psi}^{**} : [0, \infty) \to [0, \infty)$ *continuous and non-decreasing such that*

$$|I_k(v)| \leq \bar{\psi}^*(|v|), \quad |\bar{I}_k(v)| \leq \bar{\psi}^{**}(|v|), \ \ for \ \ all \ \ v \in \mathbb{R}, \ \ k = 1, \ldots, m.$$

(H13) *There exists a number* $\overline{M} > 0$ *such that*

$$\frac{\overline{M}}{\|\varphi\| + T\left(\log\frac{T}{a}\right)\psi^*(\overline{M})\|\phi_h\|_{L^1} + \phi\psi(\overline{M})\left(\frac{1+m}{\Gamma(r+1)} + \frac{1+2m}{\Gamma(r)}\right)\left(\log\frac{T}{a}\right)^r + m\bar{\psi}^*(\overline{M}) + 2mT\left(\log\frac{T}{a}\right)\bar{\psi}^{**}(\overline{M})} \geq 1,$$

where $\phi = \sup\{\phi_f(t) : t \in J\}$, *then the problem(1.1)-(1.4) has at least one solution on* $[a - \tau, T]$.

Proof: Consider the operator F defined as in 3.6. It can be easily shown that F is continuous and completely continuous.
For $\lambda \in [0, 1]$ and each $t \in J$, let $y(t) = \lambda(Fy)(t)$. Then from(H10)-(H12), we have

$$\begin{aligned}
|(Fy)(t)| \leq\ & |\varphi(a)| + T\left(\log\frac{t}{a}\right)\int_a^T |h(s, y(s))|ds + \frac{\left(\log\frac{t}{a}\right)}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}|f(s, y_s)|\frac{ds}{s} \\
&+ \frac{\left(\log\frac{t}{a}\right)}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s, y_s)|\frac{ds}{s} + \left(\log\frac{t}{a}\right)\sum_{i=1}^m t_i|\bar{I}_i(y(t_i^-))| \\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}|f(s, y_s)|\frac{ds}{s} + \frac{1}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|f(s, y_s)|\frac{ds}{s} \\
&+ \sum_{i=1}^k\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s, y_s)|\frac{ds}{s} + \sum_{i=1}^k|I_i(y(t_i^-))| + \sum_{i=1}^k t_i\left(\log\frac{t}{t_i}\right)|\bar{I}_i(y(t_i^-))| \\
\leq\ & |\varphi(a)| + T\left(\log\frac{t}{a}\right)\int_a^T\phi_h(s)\psi^*(|y(s)|)ds + \frac{\left(\log\frac{t}{a}\right)}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}\phi_f(s)\psi(|y_s|)\frac{ds}{s} \\
&+ \frac{\left(\log\frac{t}{a}\right)}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\phi_f(s)\psi(|y_s|)\frac{ds}{s} + \left(\log\frac{t}{a}\right)\sum_{i=1}^m t_i\bar{\psi}^{**}(|y(t_i^-)|) \\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}\phi_f(s)\psi(|y_s|)\frac{ds}{s} + \frac{1}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\phi_f(s)\psi(|y_s|)\frac{ds}{s} \\
&+ \sum_{i=1}^k\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\phi_f(s)\psi(|y_s|)\frac{ds}{s} + \sum_{i=1}^k\bar{\psi}^*(|y(t_i^-)|) + \sum_{i=1}^k t_i\left(\log\frac{t}{t_i}\right)\bar{\psi}^{**}(|y(t_i^-)|) \\
\leq\ & \|\varphi\| + T\left(\log\frac{T}{a}\right)\psi^*(\|y\|)\int_a^T\phi_h(s)ds + \frac{\left(\log\frac{T}{a}\right)^r}{\Gamma(r)}\phi\psi(\|y\|) + \frac{m\left(\log\frac{T}{a}\right)^r}{\Gamma(r)}\phi\psi(\|y\|)
\end{aligned}$$

$$
\begin{aligned}
&+mT\left(\log\frac{T}{a}\right)\bar{\psi}^{**}(\|y\|)+\frac{\left(\log\frac{T}{a}\right)^r}{\Gamma(r+1)}\phi\psi(\|y\|)+\frac{m\left(\log\frac{T}{a}\right)^r}{\Gamma(r+1)}\phi\psi(\|y\|)\\
&+\frac{m\left(\log\frac{T}{a}\right)^r}{\Gamma(r)}\phi\psi(\|y\|)+m\bar{\psi}^{*}(\|y\|)+mT\left(\log\frac{t}{a}\right)\bar{\psi}^{**}(\|y\|)\\
&\le\|\varphi\|+T\left(\log\frac{T}{a}\right)\psi^{*}(\|y\|)\|\phi_h\|_{L^1}+\phi\psi(\|y\|)\left(\frac{(1+m)}{\Gamma(r+1)}+\frac{(1+2m)}{\Gamma(r)}\right)\left(\log\frac{T}{a}\right)^r\\
&+m\bar{\psi}^{*}(\|y\|)+2mT\left(\log\frac{T}{a}\right)\bar{\psi}^{**}(\|y\|)
\end{aligned}
$$

Thus

$$
\frac{\|y\|}{\|\varphi\|+T\left(\log\frac{T}{a}\right)\psi^{*}(\|y\|)\|\phi_h\|_{L^1}+\phi\psi(\|y\|)\left(\frac{1+m}{\Gamma(r+1)}+\frac{1+2m}{\Gamma(r)}\right)\left(\log\frac{T}{a}\right)^r+m\bar{\psi}^{*}(\|y\|)+2mT\left(\log\frac{T}{a}\right)\bar{\psi}^{**}(\|y\|)}\le 1.
$$

Then by condition (H13), there exists $\overline{M}$ such that $\|y\|\neq\overline{M}$. Let

$$
U=\{y\in B:\|y\|\le\overline{M}\}.
$$

The operator $F:\overline{U}\to B$ is continuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y=\lambda F(y)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that F has a fixed point $y\in\overline{U}$ which is a solution of the problem (1.1)-(1.4).
This completes the proof.

4. Example

Let consider the following problem:

$$
{}^{C}_{H}D^{\frac{3}{2}}y(t)=\frac{e^t}{(e^t+5)^2}\frac{|y_t|}{(1+|y_t|)},\qquad t\in[1,2],\,t\neq\frac{4}{3},\tag{4.1}
$$

$$
\Delta y(\frac{4}{3})=\frac{|y(\frac{4}{3}^{-})|}{15+|y(\frac{4}{3}^{-})|},\tag{4.2}
$$

$$
\Delta y'(\frac{4}{3})=\frac{|y(\frac{4}{3}^{-})|}{17+|y(\frac{4}{3}^{-})|},\tag{4.3}
$$

$$
y(t)=\phi(t),\ \ t\in[1-\tau,1],\ \ y'(2)=\int_1^2\frac{|y(s)|}{13+|y(s)|}ds.\tag{4.4}
$$

Set

$$
f(t,y_t)=\frac{e^t}{(e^t+5)^2}\frac{|y_t|}{(1+|y_t|)},\qquad (t,y)\in J\times C([1-\tau,1],\mathbb{R}),
$$

$$
h(t,y(t))=\int_1^2\frac{|y(s)|}{13+|y(s)|}ds,\qquad (t,y)\in J\times\mathbb{R},
$$

$$
I(y)=\frac{|y|}{15+|y|},\qquad \bar{I}(y)=\frac{|y|}{17+|y|},\qquad y\in\mathbb{R}.
$$

Hence the hypotheses (H1)-(H3) holds, with $L_1 = \frac{1}{36}, L_2 = \frac{1}{13},\ l = \frac{1}{15},\ \ l^* = \frac{1}{17}$.
We shall check that condition (3.5). With $r = \frac{3}{2}, m = 1, t_1 = \frac{4}{3}, T = 2, a = 1$.
Further

$$\left[L_2 T(T-a)\left(\log \frac{T}{a}\right) + L_1\left(\frac{m+1}{\Gamma(r+1)} + \frac{1+2m}{\Gamma(r)}\right)\left(\log \frac{T}{a}\right)^r + ml + 2ml^* T\left(\log \frac{T}{a}\right)\right]$$
$$= \left[\frac{2}{13}(\log 2) + \frac{1}{36}\left(\frac{2}{\Gamma\left(\frac{5}{2}\right)} + \frac{3}{\Gamma(\frac{3}{2})}\right)(\log 2)^{\frac{3}{2}} + ml + \frac{4}{17}(\log 2)\right]$$
$$= 0.414779517 < 1.$$

Note that $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi},\ \Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$.
Then all hypotheses of Theorem (3.1) are fulfilled, and consequently the boundary value problem (4.1)-(4.4) has a unique solution on $[1, 2]$.

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