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# Symbolic Algorithm for Inverting General k-Tridiagonal Interval Matrices

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Abstract. The k-tridiagonal matrices have received much attention in recent years. Many different algorithms have been proposed to improve the efficiency of k-tridiagonal matrix estimation. A novel method based on interval analysis has been identified to improve the efficiency of the calculation. This paper presents efficient and reliable computational algorithms for determining the determinant and inverse of general k-tridiagonal interval matrices built on generalized interval arithmetic. This study is based on the Doolittle LU factorization of the interval matrix. Finally, examples are presented to illustrate the algorithms.

## 1. Introduction

Tridiagonal matrices play an influential role in many areas of science and engineering. These areas include spline interpolation, parallel computing, signal processing, solving ordinary and partial differential equations using finite differences. In many of these areas, tridiagonal matrix inversion is a crucial procedure with various applications. k-tridiagonal matrices, a generalization of tridiagonal matrices, are widely used and frequently appear in various applications. For examples, Moawwad El-Mikkawy et al. [13–15] present breakdown-free algorithms for inverting general tridiagonal and k-tridiagonal matrices without imposing constraints. Moreover, they have developed a novel algorithm for inverting a non-singular k-tridiagonal matrix. Ji Teng Jia et al. [8, 9] developed a numerical algorithm for computing the determinants of a block k-tridiagonal matrix and a bordered k-tridiagonal matrix. The

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algorithm uses the fast block diagonalization method. Tanasescu A et al. [21, 22] proposed the singular value decomposition of a k-tridiagonal matrix that can be calculated in $O(n^3/k^2)$ and a technique for enhancing any existing SVD algorithm to make it suitable for this class of matrices. Da Fonseca CM et al. [3, 4] developed the spectral theory for k-tridiagonal matrices, which are the first type of matrices. Then they discussed the explosion of interest in them over the last two decades. Wei Y et al. [24] presented explicit formulae for determinants, inverses and eigenpairs of a periodic tridiagonal Toeplitz-like matrix with asymmetrically perturbed rows. Solary et al. [19] showed a symbolic algorithm for inverting a general k-heptadiagonal matrix and recursive relationships. This work is based on the LU factorization of the matrix. Fu Y et al. [5] studied the eigenvalues and eigenvectors of the tridiagonal Toeplitz matrix with opposite bordered rows. Alberto J et al. [1] studied the inverses of k-Toeplitz matrices in the context of resonator arrays with multiple receivers. Albuquerque H et al. [2] gave rational formulas for the determinant, the characteristic polynomial and the elements of the inverse of a tridiagonal k-Toeplitz matrix over any commutative unital ring. Kucuk AZ et al. [10] discussed recursive and combinational formulas for the permanents of general k-tridiagonal Toeplitz matrices. Takahira S et al. [20] presented bidiagonalization of $n$-by-$n$ $(k, k+1)$-tridiagonal matrices when $n \leq 2k$. Yalciner [23] proposed a k-tridiagonal matrix determinant based on LU factorization. In real-life, computations are inaccurate since uncertainty often exists. At most, it is possible to know the intervals of possible values. So, it is crucial to figure out how to handle the impact of unclear parameters on system properties. Interval analysis is a common way to deal with uncertain situations. It describes uncertain parameters as interval numbers. Then, the interval containing each potential solution must be computed. Ganesan et al. [6] presented a new set of arithmetic operations for interval numbers by which those discrepancies in general can be reduced to some extent. Kaucher [11] introduced interval analysis in extended interval space IR and dual as a significant monadic operator in interval calculations. Nirmala et al. [16] developed a new way to find the inverse of an interval matrix. This helps us solve systems of interval linear equations. Rohn [18] proposed theoretical and practical ways to figure out how to calculate the inverse interval matrix. After this inspiration and motivation, several authors, such as [7, 12, 17] have investigated uncertainty. The main goal of this study is to create effective computational algorithms based on generalized interval arithmetic. These algorithms are used to find the determinant and inverse of general k-tridiagonal interval matrices with interval Doolittle $LU$ factorization. This is explained with the help of two instructive numerical examples. The paper is organized as follows: Section 2 overviews generalized interval arithmetic. In Section 3, the main results and theorem are presented. Section 4 suggests algorithms for finding the determinant and inverse of the general k-tridiagonal interval matrix. Section 5 gives two numerical examples to show how the algorithm works.

## 2. Preliminary Notes

Let $\mathbb{D} = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{[u_1, u_2] : u_1, u_2 \in \mathbb{R}\}$ is the set of generalized intervals that are the proper and improper intervals, where $\overline{\mathbb{IR}} = \{\tilde{u} = [u_1, u_2] : u_1 > u_2 \text{ and } u_1, u_2 \in \mathbb{R}\}$ be the collection of all improper intervals on a real line $\mathbb{R}$. Be the collection of generalized intervals $\mathbb{D}$ is a group that maintains inclusion monotonicity while performing addition and multiplication operations over zero free intervals. The midpoint and width of an interval number $\tilde{u} = [u_1, u_2]$ is given by $m(\tilde{u}) = \left(\frac{u_1 + u_2}{2}\right)$ and $w(\tilde{u}) = \left(\frac{u_2 - u_1}{2}\right)$. Kaucher introduces the dual as a significant monadic operator [11] that expresses element to element symmetry between proper and improper intervals by reversing the end points numbers in the interval, intervals in $\mathbb{D}$. For $\tilde{u} = [u_1, u_2] \in \mathbb{D}$, its dual is given by $dual(\tilde{u}) = dual[u_1, u_2] = [u_2, u_1]$. An interval's opposite $\tilde{u} = [u_1, u_2]$ is opp $\{[u_1, u_2]\} = [-u_1, -u_2]$ which is the additive inverse of $[u_1, u_2]$ and $\left[\frac{1}{u_1}, \frac{1}{u_2}\right]$ is the multiplicative inverse of $[u_1, u_2]$, provided $0 \notin [u_1, u_2]$.

$$\begin{aligned}\text{That is, } \tilde{u} + (-\text{dual } \tilde{u}) &= \tilde{u} - \text{dual}(\tilde{u}) = [u_1, u_2] - \text{dual}([u_1, u_2]) \\ &= [u_1, u_2] - [u_2, u_1] = [u_1 - u_1, u_2 - u_2] = [0, 0] \text{ and}\end{aligned}$$

$$\begin{aligned}\tilde{u} \times \left(\frac{1}{\text{dual } \tilde{u}}\right) &= [u_1, u_2] \times \left(\frac{1}{\text{dual}([u_1, u_2])}\right) = [u_1, u_2] \times \frac{1}{[u_2, u_1]} \\ &= [u_1, u_2] \times \left[\frac{1}{u_1}, \frac{1}{u_2}\right] = [1, 1].\end{aligned}$$

2.1. **Arithmetic Operations on Interval Matrices.** If $\tilde{A}, \tilde{B} \in \mathbb{D}^{n\times n}$, $\tilde{\mathbf{x}} \in \mathbb{D}^n$ and $\tilde{\alpha} \in \mathbb{D}$, we propose a generalized interval arithmetic as,

(i). $\tilde{\alpha}\tilde{A} \approx \left(\tilde{\alpha}\tilde{a}_{ij}\right)$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$

(ii). $\tilde{A} + \tilde{B} \approx \left(\tilde{a}_{ij} + \tilde{b}_{ij}\right)$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$

(iii). $\tilde{A} - \tilde{B} \approx \begin{cases} \left(\tilde{a}_{ij} - \tilde{b}_{ij}\right)_{1\le i\le n,\ 1\le j\le n}, & \text{if } \tilde{A},\ \tilde{B} \text{ are not equivalent} \\ \tilde{A} - \text{dual}(\tilde{A}) \approx \tilde{O} = O, & \text{if } \tilde{A} \approx \tilde{B} \end{cases}$

(iv). $\tilde{A}\tilde{B} \approx \left(\sum_{k=1}^{n} \tilde{a}_{ik}\tilde{b}_{kj}\right)$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$

(v). $\tilde{A}\tilde{\mathbf{x}} \approx \left(\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{\mathbf{x}}\right)$ for $i = 1, 2, \cdots, n$

2.2. **Interval Arithmetic.** A new method of interval arithmetic on $\mathbb{IR}$ was proposed by Ganesan and Veeramani [6]. The set of generalized interval numbers is extended using these arithmetic procedures $\mathbb{D}$ by utilising the dual concept, For $\tilde{u} = [u_1, u_2]$, $\tilde{v} = [v_1, v_2] \in \mathbb{D}$ and for $* \in \{+, -, \cdot, \div\}$, we define $\tilde{u} * \tilde{v} = [m(\tilde{u}) * m(\tilde{v}) - j, m(\tilde{u}) * m(\tilde{v}) + j]$, where
$j = \min\{(m(\tilde{u}) * m(\tilde{v})) - \beta,\ \gamma - (m(\tilde{u}) * m(\tilde{v}))\}$, where the $\beta$ and $\gamma$ are the end points of the interval $\tilde{u} \odot \tilde{v}$ under the existing interval arithmetic. In particular,

(i) Addition: $\tilde{u} + \tilde{v} = [u_1, u_2] + [v_1, v_2] = [(m(\tilde{u}) + m(\tilde{v})) - j, (m(\tilde{u}) + m(\tilde{v})) + j]$, where $j = \left\{ \dfrac{(v_2 + u_2) - (v_1 + u_1)}{2} \right\}$.

(ii) Subtraction: $\tilde{u} - \tilde{v} = [u_1, u_2] - [v_1, v_2] = [(m(\tilde{u}) - m(\tilde{v})) - j, (m(\tilde{u}) - m(\tilde{v})) + j]$,
where $j = \left\{ \dfrac{(v_2 + u_2) - (v_1 + u_1)}{2} \right\}$.
Also if $\tilde{u} = \tilde{v}$, i.e. if $[u_1, u_2] = [v_1, v_2]$, then

$$\tilde{u} - \tilde{v} \quad = \quad \tilde{u} - \text{dual}(\tilde{u}) = [u_1, u_2] - [u_2, u_1] = [u_1 - u_1, u_2 - u_2] = [0, 0] \,.$$

(iii) Multiplication: $\tilde{u}.\tilde{v} = \tilde{u}\tilde{v} = [u_1, u_2]\,[v_1, v_2] = [(m(\tilde{u})m(\tilde{v})) - j, (m(\tilde{u})m(\tilde{v})) + j]$,
where $j = \min\{(m(\tilde{u})m(\tilde{v})) - \beta,\ \gamma - (m(\tilde{u})m(\tilde{v}))\}$,
$\beta = \min(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ and $\gamma = \max(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$.

(iv) Division: $1 \div \tilde{u} = \dfrac{1}{\tilde{u}} = \dfrac{1}{[u_1, u_2]} = \left[\dfrac{1}{m(\tilde{u})} - j, \dfrac{1}{m(\tilde{u})} + j\right]$, where
$j = \min\left\{ \dfrac{1}{u_2}\left(\dfrac{u_2 - u_1}{u_1 + u_2}\right), \dfrac{1}{u_1}\left(\dfrac{u_2 - u_1}{u_1 + u_2}\right) \right\}$ and
$m([u_1, u_2]) = \left(\dfrac{u_1 + u_2}{2}\right) \neq 0$.

Also if $\tilde{u} = \tilde{v}$, i.e. $[u_1, u_2] = [v_1, v_2]$, then

$$\frac{\tilde{u}}{\tilde{v}} = \frac{\tilde{u}}{\tilde{u}} = \frac{\tilde{u}}{\text{dual}(\tilde{u})} = [u_1, u_2] \,.\frac{1}{[u_2, u_1]} = [u_1, u_2] \,.\left[\frac{1}{u_1}, \frac{1}{u_2}\right] = [1, 1] \,.$$

From $(iii)$, it is clear that $\lambda\tilde{u} = \begin{cases} [\lambda u_1, \lambda u_2], & \text{for } \lambda \geq 0 \\ [\lambda u_2, \lambda u_1], & \text{for } \lambda < 0. \end{cases}$

It's worth noting that $\odot$ stands for existing interval arithmetic and $*$ stands for generalized interval arithmetic. However, in circumstances when there is no ambiguity, the same notation can be used for both cases. It is also to be noted that $\tilde{u} * \tilde{v} \subseteq \tilde{u} \odot \tilde{v}$, where $\odot \in \{\oplus, \ominus, \otimes, \oslash\}$ is the existing interval arithmetic.

**Note 2.1.** Without loss of generality, assume that for any interval number $\tilde{u} = [u_1, u_2]$ with $m(\tilde{u}) \neq 0$ and $0 \in \tilde{u}$, there exist $\tilde{v} = [m(\tilde{u}) - j, m(\tilde{u}) + j]$, where $0 < j < h$ and $h = \min\{|u_1|, |u_2|\}$, such that $\tilde{v} \approx \tilde{u}$ and $0 \notin \tilde{v}$. Hence, if $\dfrac{\tilde{a}}{\tilde{u}}$ with $m(\tilde{u}) \neq 0$ and $0 \in \tilde{u}$, then we replace $\dfrac{\tilde{a}}{\tilde{u}}$ by $\dfrac{\tilde{a}}{\tilde{v}}$ where $\tilde{v} \approx \tilde{u}$ and $0 \notin \tilde{v}$. In particular (for convenience) one may select $j$ in such a way that

$$j = \begin{cases} \dfrac{m(\tilde{u})}{2}, & \text{if} \quad m(\tilde{u}) > 0 \\ \dfrac{-m(\tilde{u})}{2}, & \text{if} \quad m(\tilde{u}) < 0 \end{cases}$$

Generalized interval arithmetic can be used to prove a lot of important things, like the distributive law for interval numbers.

## 3. Main Results

In this section, we provide some important results concerning the general k-tridiagonal interval matrix. A tridiagonal interval matrix is a matrix with three interval diagonals. The tridiagonal interval matrix has nonzero interval entries (midpoint of interval number not equal to zero) on the form's main diagonal, immediate sub diagonal and super diagonal.

$$\tilde{A} = \begin{bmatrix} [\underline{h}_1, \overline{h}_1] & [\underline{f}_1, \overline{f}_1] & \tilde{0} & \cdots & \cdots & \tilde{0} \\ [\underline{e}_1, \overline{e}_1] & [\underline{h}_2, \overline{h}_2] & [\underline{f}_2, \overline{f}_2] & \ddots & & \vdots \\ \tilde{0} & [\underline{e}_2, \overline{e}_2] & [\underline{h}_3, \overline{h}_3] & [\underline{f}_3, \overline{f}_3] & \tilde{0} & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \ddots & \tilde{0} \\ \vdots & & \ddots & \ddots & \ddots & [\underline{f}_{n-1}, \overline{f}_{n-1}] \\ \tilde{0} & \cdots & \cdots & \tilde{0} & [\underline{e}_{n-1}, \overline{e}_{n-1}] & [\underline{h}_n, \overline{h}_n] \end{bmatrix}. \tag{3.1}$$

Let $\mathbb{D}^{n\times n}$ be the collection of all $n \times n$ interval matrices.

The k-tridiagonal interval matrix $\tilde{A}_n^k$ is a more general tridiagonal interval matrix that can be expressed as follows:

$$\tilde{A}_n^k = \begin{bmatrix} [\underline{h}_1, \overline{h}_1] & [0,0] & \cdots & [0,0] & [\underline{f}_1, \overline{f}_1] & [0,0] & \cdots & [0,0] \\ [0,0] & [\underline{h}_2, \overline{h}_2] & [0,0] & \cdots & [0,0] & [\underline{f}_2, \overline{f}_2] & \ddots & \vdots \\ \cdots & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & [0,0] \\ [0,0] & \cdots & \ddots & [\underline{h}_{n-k}, \overline{h}_{n-k}] & \ddots & \cdots & \ddots & [\underline{f}_{n-k}, \overline{f}_{n-k}] \\ [\underline{e}_1, \overline{e}_1] & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\ [0,0] & [\underline{e}_2, \overline{e}_2] & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\ \vdots & \ddots & \ddots & [0,0] & \cdots & [0,0] & [\underline{h}_{n-1}, \overline{h}_{n-1}] & [0,0] \\ [0,0] & \cdots & [0,0] & [\underline{e}_{n-k}, \overline{e}_{n-k}] & [0,0] & \cdots & [0,0] & [\underline{h}_n, \overline{h}_n] \end{bmatrix} \tag{3.2}$$

where $1 \leq k < n$. For $k \geq n$, the interval matrix $\tilde{A}_n^k$ is a diagnal interval matrix, which has $k = 1$, gives a standard tridiagonal interval matrix in (3.1). The $3n - 2k$ memory locations can be used to store the nonzero interval numbers of the interval matrix $\tilde{A}_n^k$. Having this habit makes calculations easier. The midpoint of an k-tridiagonal interval matrix $\tilde{A}_n^k$ is defined as,

$$m(\tilde{A}_n^k) = \begin{bmatrix} m(\tilde{h}_1) & [0,0] & \cdots & [0,0] & m(\tilde{f}_1) & [0,0] & \cdots & [0,0] \\ [0,0] & m(\tilde{h}_2) & [0,0] & \cdots & [0,0] & m(\tilde{f}_2) & \ddots & \vdots \\ \cdots & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & [0,0] \\ [0,0] & \cdots & \ddots & m(\tilde{h}_{n-k}) & \ddots & \cdots & \ddots & m(\tilde{f}_{n-k}) \\ m(\tilde{e}_1) & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\ [0,0] & m(\tilde{e}_2) & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\ \vdots & \ddots & \ddots & [0,0] & \cdots & [0,0] & m(\tilde{h}_{n-1}) & [0,0] \\ [0,0] & \cdots & [0,0] & m(\tilde{e}_{n-k}) & [0,0] & \cdots & [0,0] & m(\tilde{h}_n) \end{bmatrix}.$$

The width of an k-tridiagonal interval matrix $\tilde{A}_n^k$ is defined as,

$$w(\tilde{A}_n^k) = \begin{bmatrix} w(\tilde{h}_1) & [0,0] & \cdots & [0,0] & w(\tilde{f}_1) & [0,0] & \cdots & [0,0] \\ [0,0] & w(\tilde{h}_2) & [0,0] & \cdots & [0,0] & w(\tilde{f}_2) & \ddots & \vdots \\ \cdots & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & [0,0] \\ [0,0] & \cdots & \ddots & w(\tilde{h}_{n-k}) & \ddots & \cdots & \ddots & w(\tilde{f}_{n-k}) \\ w(\tilde{e}_1) & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\ [0,0] & w(\tilde{e}_2) & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\ \vdots & \ddots & \ddots & [0,0] & \cdots & [0,0] & w(\tilde{h}_{n-1}) & [0,0] \\ [0,0] & \cdots & [0,0] & w(\tilde{e}_{n-k}) & [0,0] & \cdots & [0,0] & w(\tilde{h}_n) \end{bmatrix}$$

which is always nonnegative.

If $m(\tilde{A}_n^k) = m(\tilde{B}_n^k)$, then the interval matrices $\tilde{A}_n^k$ and $\tilde{B}_n^k$ are said to be equivalent and is denoted by $\tilde{A}_n^k \approx \tilde{B}_n^k$. In particular if $m(\tilde{A}_n^k) = m(\tilde{B}_n^k)$ and $w(\tilde{A}_n^k) = w(\tilde{B}_n^k)$, then $\tilde{A}_n^k = \tilde{B}_n^k$. If $m(\tilde{A}_n^k) = 0$ then $\tilde{A}_n^k$ is a *zero interval matrix*. In particular, if $m(\tilde{A}_n^k) = 0$ and $w(\tilde{A}_n^k) = 0$, then $\tilde{A}_n^k = \tilde{0}$. If $m(\tilde{A}_n^k) = 0$ and $w(\tilde{A}_n^k) \neq 0$, then $\tilde{A}_n^k \not\approx \tilde{0}$, if $\tilde{A}_n^k$ is said to be a *non-zero interval matrix*. If $m(\tilde{A}_n^k) = I$, then $\tilde{A}_n^k$ is an identity interval matrix. In specifically, if $m(\tilde{A}_n^k) = I$ and $w(\tilde{A}_n^k) = 0$, then $\tilde{A}_n^k = \tilde{I}$, if $m(\tilde{A}_n^k) = I$ and $w(\tilde{A}_n^k) \neq 0$, then $\tilde{A}_n^k \approx \tilde{I}$. Also $I$ denotes the identity matrix and the identity interval matrix is indicated by $\tilde{I}$. If 0 be the null matrix and $\tilde{0}$ be the matrix of null intervals.

**Theorem 3.1.** Let the k-tridiagonal interval matrix $\tilde{A}_n^k$ be as in (3.2), the LU factorization of $\tilde{A}_n^k$ can be expressed as,

$$\tilde{A}_n^k \approx \tilde{L}_n^k \tilde{U}_n^k$$

where

$$\tilde{L}_n^k = \begin{bmatrix} [1,1] & [0,0] & [0,0] & \cdots & \cdots & \cdots & \cdots & [0,0] \\ [0,0] & [1,1] & [0,0] & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & [0,0] & [1,1] & \ddots & \ddots & \ddots & \ddots & \vdots \\ [0,0] & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{[\underline{e}_1, \overline{e}_1]}{[\underline{k}_1, \overline{k}_1]} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & [0,0] \\ [0,0] & \frac{[\underline{e}_2, \overline{e}_2]}{[\underline{k}_2, \overline{k}_2]} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ [0,0] & \cdots & [0,0] & \frac{[\underline{e}_{n-k}, \overline{e}_{n-k}]}{[\underline{k}_{n-k}, \overline{k}_{n-k}]} & [0,0] & \cdots & \cdots & [1,1] \end{bmatrix},$$

$$\tilde{U}_n^k = \begin{bmatrix} [\underline{k}_1, \overline{k}_1] & [0,0] & \cdots & [0,0] & [\underline{f}_1, \overline{f}_1] & [0,0] & \cdots & [0,0] \\ [0,0] & [\underline{k}_2, \overline{k}_2] & [0,0] & \ddots & [0,0] & [\underline{f}_2, \overline{f}_2] & \ddots & \vdots \\ \vdots & [0,0] & [\underline{k}_3, \overline{k}_3] & \ddots & \ddots & \ddots & \ddots & [0,0] \\ [0,0] & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & [\underline{f}_{n-k}, \overline{f}_{n-k}] \\ [0,0] & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & [0,0] \\ [0,0] & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ [0,0] & \cdots & [0,0] & \cdots & [0,0] & \cdots & \cdots & [\underline{k}_n, \overline{k}_n] \end{bmatrix}.$$

with

$$\tilde{k}_i = \begin{cases} \tilde{h}_i & i = 1, 2, ..., k, \\ \tilde{h}_i - \tilde{s}_{i-k}\tilde{e}_{i-k} & i = k+1, k+2, \cdots, n. \end{cases} \tag{3.3}$$

where $\tilde{s}_i = \dfrac{\tilde{f}_i}{\tilde{k}_i}$ for $i = 1, 2, ..., n-k$.

In order to further discuss this article, we must consider the above results.

## 4. The Symbolic Inverse of a k-Tridiagonal Interval Matrix

In this section, we give two algorithms for finding the determinant and inverse of the general k-tridiagonal interval matrix $\tilde{A}_n^k$.

The following algorithm can be used to evaluate the value of $det(\tilde{A}_n^k)$ in the interval matrix $\tilde{A}_n^k$.

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**Algorithm 4.1. An algorithm for determining the determinant of the k-tridiagonal interval matrix.**

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**Step 1. Input:** $\tilde{e}_i, \tilde{h}_i, \tilde{f}_i$ and the order $n$.

**Step 2. For** $i = 1, 2, ..., k$ **do**

**Set:** $\tilde{k}_i = \tilde{h}_i$

**End do.**

**Setp 3. For** $i = 1, 2, ..., n-k$, **do**

**Set:** $\tilde{s}_i = \dfrac{\tilde{f}_i}{\tilde{k}_i}$ **If** $m(\tilde{k}_i) \neq 0$

**End do.**

**Step 4. For** $i = k+1, k+2, ..., n$ **do**

**Set:** $\tilde{k}_i = \tilde{h}_i - \tilde{s}_{i-k}\tilde{e}_{i-k}$

**End do.**

**Step 5. Compute and simplify:**

$\det(\tilde{A}_n^k) = \Pi_{i=1}^n \tilde{k}_i.$

**Step 6. Output:** The determinant of the tridiagonal interval matrix $(\tilde{A}_n^k)$.

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We can follow the procedure outlined below to find the inverse of a general k-tridiagonal interval matrix $(\tilde{A}_n^k)$.

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**Algorithm 4.2. Symbolic algorithm for inverting a k-tridiagonal interval matrix.**

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**Step 1.** Input: $\tilde{e}_i, \tilde{h}_i, \tilde{f}_i$, and the order $n$.

**For** $\tilde{e}_i, \tilde{f}_i, i = 1, 2, ..., n-k$. $\tilde{h}_i, i = 1, 2, ..., n$.

**Step 2. For** $i = 1, 2, ..., k$ **do**

**Set:** $\tilde{k}_i = \tilde{h}_i$. **If** $m(\tilde{k}_i) \neq 0$

**End do.**

**Step 3. For** $i = k+1, k+2, ..., n$ **do**

**Compute and simplify:**

$$\tilde{s}_{i-k} = \frac{\tilde{f}_{i-k}}{\tilde{k}_{i-k}}$$

$\tilde{k}_i = \tilde{h}_i - \tilde{e}_{i-k}\tilde{s}_{i-k}$ **if** $m(\tilde{k}_i) \neq 0$ **end if.**

$$\tilde{t}_{i-k} = \frac{\tilde{e}_{i-k}}{\tilde{k}_{i-k}}$$

**End do.**

**Step 4.** Use the determinant of the k-tridiagonal interval matrix algorithm (4.1) to check the non-singularity of the interval matrix in (3.2).

**Step 5.** **For** $i = n, n-1, ..., n-k+1$ **do**

**Compute and simplify:**

$$\tilde{\delta}_{i,i} = \frac{\tilde{1}}{\tilde{k}_i}$$

**End do.**

**Step 6.** **For** $i = n-k, n-k-1, ..., 1$ **do**

**Compute and simplify:**

$$\tilde{\delta}_{i,i} = \frac{\tilde{1}}{\tilde{k}_i} + \tilde{s}_i \tilde{t}_i \tilde{\delta}_{i+k,i+k}$$

**End do.**

**Step 7.** **For** $j = n, n-1, ..., 2$ **do**

**For** $i = j-k, j-2k, ..., 1$ **do**

**Compute and simplify:**

$$\tilde{\delta}_{i,j} = -\tilde{s}_i \tilde{\delta}_{i+k,j}$$

**End do.**

**Step 8.** **For** $i = n, n-1, ..., 2$ **do**

$j = i-k, i-2k, ..., 1$ **do**

**Compute and simplify:**

$$\tilde{\delta}_{i,j} = -\tilde{t}_j \tilde{\delta}_{i,j+k}$$

**End do.**

**Step 9.** Output: The inverse interval matrix $(\tilde{A}_n^k)^{-1} = \tilde{\Delta} = \tilde{\delta}_{i,j=1}^n$.

## 5. Numerical examples

In this section, we will examine the effectiveness of two numerical examples using the proposed algorithm.

**Example 5.1.** Let us consider the $k$-tridiagonal interval matrix $\tilde{A}_n^k$ with $n = 10, k = 4$.

$$\tilde{A}_{10}^4 = \begin{bmatrix} [-3.75, 1.75] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [-4.25,0.25] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [-4.25,0.25] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [-4.25,0.25] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] \\ [0.5,1.5] & [0,0] & [0,0] & [0,0] & [-4.25,0.25] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] \\ [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [-4.25,0.25] & [0,0] & [0,0] & [0,0] & [0.5,1.5] \\ [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [-4.25,0.25] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [-4.25,0.25] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [-4.25,0.25] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [-4.25,0.25] \end{bmatrix}.$$

**Solution:** Applying the symbolic algorithm (4.2) for inverting a k-tridiagonal interval matrix.

By using **steps 2-3,** we get:

$\tilde{k}_1 = [-3.75, 1.75], \tilde{k}_2 = [-4.25, 0.25], \tilde{k}_3 = [-4.25, 0.25], \tilde{k}_4 = [-4.25, 0.25], \tilde{k}_5 = [-4.083, 2.084],$

$\tilde{k}_6 = [-4.132, 1.132], \tilde{k}_7 = [-4.132, 1.132], \tilde{k}_8 = [-4.132, 1.132], \tilde{k}_9 = [-4.083, 2.084], \tilde{k}_{10} = [-4.139, 1.472].$

Using **step 4,** we yields:
$\det(\tilde{A}_{10}^4) = \Pi_{i=1}^{10} \tilde{k}_i = [-653997, 654068.2].$
Applying **steps 5-8,** we obtain $(\tilde{A}_n^k)^{-1}$

$$= \begin{bmatrix} [-5.156,-0.750] & [0,0] & [0,0] & [0,0] & [-3.756,-0.248] & [0,0] & [0,0] & [0,0] & [-1.928,-0.074] & [0,0] \\ [0,0] & [-1.047,-0.523] & [0,0] & [0,0] & [0,0] & [-0.924,-0.217] & [0,0] & [0,0] & [0,0] & [-0.474,-0.026] \\ [0,0] & [0,0] & [-0.836,-0.497] & [0,0] & [0,0] & [0,0] & [-0.562,-0.105] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [-0.836,-0.497] & [0,0] & [0,0] & [0,0] & [-0.562,-0.105] & [0,0] & [0,0] \\ [-3.756,-0.248] & [0,0] & [0,0] & [0,0] & [-3.260,-0.742] & [0,0] & [0,0] & [0,0] & [-1.778,-0.223] & [0,0] \\ [0,0] & [-0.924,-0.217] & [0,0] & [0,0] & [0,0] & [-1.363,-0.918] & [0,0] & [0,0] & [0,0] & [-0.889,-0.111] \\ [0,0] & [0,0] & [-0.562,-0.105] & [0,0] & [0,0] & [0,0] & [-0.889,-0.444] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [-0.562,-0.105] & [0,0] & [0,0] & [0,0] & [-0.889,-0.444] & [0,0] & [0,0] \\ [-1.928,-0.074] & [0,0] & [0,0] & [0,0] & [-1.778,-0.223] & [0,0] & [0,0] & [0,0] & [-1.333,-0.667] & [0,0] \\ [0,0] & [-0.347,-0.028] & [0,0] & [0,0] & [0,0] & [-0.632,-0.118] & [0,0] & [0,0] & [0,0] & [-1,-0.5] \end{bmatrix}.$$

**Example 5.2.** Let us consider the $k$-tridiagonal interval matrix $\tilde{A}_n^k$ with $n = 10, k = 6$.

$$\tilde{A}_{10}^6 = \begin{bmatrix} [1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0.3,1.7] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-1.5,-0.5] & [0,0] & [0,0] \\ [0,0] & [0,0] & [-3.75,1.75] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-0.25,4.25] & [0,0] \\ [0,0] & [0,0] & [0,0] & [2.8,3.2] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [2.58,5.42] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-2.5,-1.5] & [0,0] & [0,0] & [0,0] & [0,0] \\ [1.65,2.35] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [3.574,6.426] & [0,0] & [0,0] & [0,0] \\ [0,0] & [-3.75,1.75] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [2.8,3.2] & [0,0] & [0,0] \\ [0,0] & [0,0] & [2.8,3.2] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-1.5,-0.5] & [0,0] \\ [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [2.8,3.2] \end{bmatrix}.$$

**Solution:** Applying the symbolic algorithm (4.2) for inverting a k-tridiagonal interval matrix.

By using **steps 2-3,** we get:

$\tilde{k}_1 = [1.5, 2.5]$, $\tilde{k}_2 = [0.3, 1.7]$, $\tilde{k}_3 = [-3.75, 1.75]$, $\tilde{k}_4 = [2.8, 3.2]$, $\tilde{k}_5 = [0.5, 1.5]$, $\tilde{k}_6 = [-2.5, -1.5]$, $\tilde{k}_7 = [1.904, 6.096]$, $\tilde{k}_8 = [-2.186, 6.186]$, $\tilde{k}_9 = [-2.674, 12.674]$, $\tilde{k}_{10} = [-1.324, 1.988]$.

Using **step 4,** we yields:

$\det(\tilde{A}^6_{10}) = \Pi_{i=1}^{10} \tilde{k}_i = [-70840, 71158.73]$.

Applying **steps 5-8,** we obtain $(\tilde{A}^k_n)^{-1}$

$$= \begin{bmatrix} [0.422, 0.828] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-0.217, -0.033] & [0,0] & [0,0] & [0,0] \\ [0,0] & [-2.224, 5.224] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.098, 0.902] & [0,0] & [0,0] \\ [0,0] & [0,0] & [-1.909, 2.309] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-0.098, 0.898] & [0,0] \\ [0,0] & [0,0] & [0,0] & [1.076, 4.955] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-6.414, -1.622] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.667, 1.333] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-0.6, -0.4] & [0,0] & [0,0] & [0,0] & [0,0] \\ [-0.392, -0.108] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.164, 0.336] & [0,0] & [0,0] & [0,0] \\ [0,0] & [-1.648, 2.648] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.333, 0.667] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0.198, 1.002] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.133, 0.267] & [0,0] \\ [0,0] & [0,0] & [0,0] & [-3.077, -0.944] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [2.008, 4.016] \end{bmatrix}.$$

## 6. Conclusion

In this paper, we present two efficient algorithms for finding the determinant and inverse of k-tridiagonal interval matrices based on generalized interval arithmetic. These algorithms are based on interval Doolittle LU factorization and are efficient. Computational results are shown in numerical examples, illustrating the feasibility of the proposed algorithms.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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