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Asymptotic Behavior of Some Parabolic Equations and Application in Image Restoration

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Abstract. In this paper, we consider some nonlinear parabolic problem involving the well known $p-$laplacian and some operator having exponential growth with respect to the gradient. We start by dealing the asymptotic behavior for some evolution equation then we give some numerical results with an application in image processing.

1. Introduction

Image processing has always been a challenging problem, this field has become "hot". In recent years, image processing has been a very active field of computer application and research [9].
The most active topics in this field is image restoration because it allow to recovery lost information from the observed degraded image data.
In [4, 6, 19, 24, 25] the authors have studied the partial differential equation (PDE) and fractional partial differential equation (FPDE) methods in image processing and proved the fundamental tools for image diffusion and restoration.
In 1987, the Perona Malik is the first attempts to derive a model from an image within a PDE framework in [21].
Then, by using Perona Malik the authors were concluded a nonlinear diffusion model (anisotropic model).

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In [1], the authors have studied a diffusion model, this model is a combination of fast growth with respect to low gradient and slow growth when the gradient is large which used for restoration in image processing.
Then, in [14] the researchers showed some class of nonlinear parabolic inequalities in Orlicz spaces.
The authors presented a novel model for image denoising and compared the results with the model of Perona-Malik and the method of the total variation (see [18]).
And, in [3] the authors proposed a novel parabolic equations for image restoration and enhacement.They proved the existence of solution, established a nonnegative weak solution obtained as limit of approximation and give some application in image processing.
Also, in [15] we find some optimal control problem for the Perona-Malik equation. The authors obtained existence results and approximation of an optimal control problem.
In [8], authors studied the minimisations problem numerically with anistropic diffusion and obtained the results in image restoration.

Let Ω be a regular open bounded subset of $\mathbb{R}^N$ with $N \geq 2$. And let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$.
We consider the following nonlinear parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} + A(u) = f & \text{in } Q \\ a_{\alpha,\beta}(x,t,\nabla u).\eta = 0, & \text{On } \partial Q = \partial\Omega \times (0,T) \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

$$A(u) = -\mathrm{div}\left(a_{\alpha,\beta}(x,t,\nabla u)\right), \tag{1.2}$$

with $a_{\alpha,\beta}(x,t,\nabla u)$ satisfying the following expression

$$a_{\alpha,\beta}(x,t,\nabla u) = \exp(\alpha|\nabla u|)\nabla u\chi_{\{|\nabla u|\leq\beta\}} + \frac{\nabla u}{|\nabla u|}\log^{\gamma}(1+|\nabla u|)\chi_{\{|\nabla u|>\beta\}} + \kappa|\nabla u|^{p-2}\nabla u. \tag{1.3}$$

For instance, if we take $\beta = \gamma = 0$ and $p = 1$ in problem (1.1), so we obtain the curvature-driven diffusion (see [22]).
If $\alpha = 0$, $\beta = 1$, $\gamma = 0$ and $\kappa = 0$ is treated in [7].

And, if we make $\kappa = 0$ and $\beta \geq 0$ then, the problem (1.1) has been recently studied in ([1], [20]) and successfully used in image processing.
Actually, nonlinear partial differential equations of type (1.1) can be considered as Perona–Malik equations see [21].
In this work, we will study the problem (1.1) when $\beta = +\infty$, $\kappa > 0$ and $p \to +\infty$. More precisely ,

we will show that the problem

$$(P_p)\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(a_\alpha\left(x,t,\nabla u\right)\right) = f & \text{in } Q \\ a_\alpha\left(x,t,\nabla u\right).\eta = 0, & \text{On } \ \partial Q = \partial\Omega \times (0,T) \\ u\left(x,0\right) = u_0 & \text{in } \ \Omega, \end{cases} \tag{1.4}$$

with

$$a_\alpha\left(x,t,\nabla u\right) = \exp\left(\alpha\left|\nabla u\right|\right)\nabla u + \kappa\left|\nabla u\right|^{p-2}\nabla u, \quad \text{for } p > 2, \tag{1.5}$$

admits at least one solution $u_p \in W^{1,x}L_{A_\alpha}(Q)$ where $A_\alpha(t) = t^2\exp(\alpha t)$. Next, we study the asymptotic behavior of the solution u_p as $p \to +\infty$, we show that the limit of u_p satisfies some parabolic obstacle problem.

In our work, we use an application illustrating that the problem can be used for denoising filter in image processing.

For recent works which involving the partial differential equations with nonstandard growth and with applications in image processing, the reader can refereed to [8], [17], [18] and [13].

This work is organized as follows: in the next section, we present somme lemmas and spaces; in section 3, we obtain main results; in section 4, we get some numerical results.

2. Preliminaries

In this section, we shall give some corollaries and definitions which will be used throughout this work.

2.1. $N-$ **Functions.** Let $A:\mathbb{R}^+ \to \mathbb{R}^+$ be an $N-$ function, i.e. A is continuous and convex with $A > 0$, for $t > 0$, $\frac{A(t)}{t} \to 0$ as $t \to 0$ and $\frac{A(t)}{t} \to +\infty$ as $t \to +\infty$.

Equivalently, A admits the representation: $A(t) = \int\limits_0^t a(s)ds$, where $a:\mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The $N-$ function $\bar{A}$ conjugate to A is defined by $\bar{A}(t) = \int\limits_0^t \bar{a}(s)ds$, where $\bar{a}:\mathbb{R}^+ \to \mathbb{R}^+$is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ ([2]).

The N-function is said to satisfy the Δ_2 condition ($\exists\ k > 0 : A(t) \leq kA(t), \forall t \geq 0$), so for some $k > 0$ we obtain,

$$A(2t) \leq kA(t),\ \forall t \geq 0, \tag{2.1}$$

when (2.1) holds only for some $t > 0$ then, A is said to satisfy the Δ_2 condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

2.2. **The Orlicz spaces.** Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz space $L_A(\Omega)$ is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that:

$$\int_\Omega A\left(\frac{u(x)}{\lambda}\right)dx < +\infty \quad \text{for some } \lambda > 0. \tag{2.2}$$

$L_A(\Omega)$ is Banach space under the norm

$$\|u\|_{A,\Omega} = \inf\left\{\lambda > 0, \int_\Omega A\left(\frac{u(x)}{\lambda}\right) dx \leq 1\right\}. \tag{2.3}$$

The closure in $L_A(\Omega)$ of the set of bounded measurable functions with compact support in Ω is denoted by $E_A(\Omega)$. The equality $E_A(\Omega) = L_A(\Omega)$ holds if only if A satisfies Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_A(\Omega)$ can be identified with $L_A(\Omega)$ by means of the pairing $\int_\Omega uv dx$, and the dual norm of $L_A(\Omega)$ is equivalent to $\|.\|_{A,\Omega}$.

The space $L_A(\Omega)$ is reflexive if and only if A and $\bar{A}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. **The Orlicz-Sobolev Spaces.** Now, we turn to the Orlicz-Sobolev space, $W^1L_A(\Omega)$ (respectively $W^1E_A(\Omega)$) is the space of all functions u and its distributional derivatives up to order 1 lie in L_A (respectively $E_A(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,A} = \sum_{|k|\leq 1} \left\|D^k u\right\|_A. \tag{2.4}$$

Thus, $W^1L_A(\Omega)$ and $W^1E_A(\Omega)$ can be identified with sub-spaces of product of $N+1$ copies of L_A. Denoting this product by ΠL_A, we will use the weak topologies $\sigma(\Pi L_A, \Pi E_{\bar{A}})$ and $\sigma(\Pi L_A, \Pi L_{\bar{A}})$.

We say that u_n converges to u for the modular convergence in $W^1L_A(\Omega)$ if for some $\lambda > 0$

$$\int_\Omega A\left(\frac{D^k u_n - D^k u}{\lambda}\right) dx \to 0 \text{ for all } |k| \leq 1. \tag{2.5}$$

This implies convergence for $\sigma(\Pi L_A, \Pi L_{\bar{A}})$.

If A satisfies Δ_2 condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

2.4. **Duality in Orlicz-Sobolev space.** Let $W^{-1}L_A(\Omega)$ denote the space of distributions on Ω which can be written as sums of derivatives of order < 1 of functions in $L_{\bar{A}}$. It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property then the space $D(\bar{\Omega})$ is dense in $W^1L_A(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_A, \Pi L_{\bar{A}})$ ([12]). Consequently, the action of a distribution in $W^{-1}L_{\bar{A}}(\Omega)$ on an element of $W^1L_A(\Omega)$ is well defined.

2.5. **Inhomogeneous Orlicz-Sobolev spaces.** Let Ω be an abounded open subset of $\mathbb{R}^N$, $T > 0$, and set $Q = \Omega \times (0, T)$. Let A be an $N-$function. For each $k \in \mathbb{N}^N$, denote by D_x^k the distributional derivatives on Q of order k with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces of order 1 are defined as follows

$$W^{1,x}L_A(Q) = \left\{u \in L_A(Q) : D_x^k u \in L_A(Q),\ \forall |k| \leq 1\right\}, \tag{2.6}$$

and

$$W^{1,x}E_A(Q) = \left\{u \in E_A\left(Q\right) : D_x^k \in E_A\left(Q\right),\ \forall \left|k\right| \le 1\right\}. \tag{2.7}$$

The latest space is a subset of the first one. They are Banach spaces under the norm

$$\|u\| = \sum_{|k|=1} \left\|D_x^k u\right\|_{A,Q}. \tag{2.8}$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product spaces $\Pi L_A\left(Q\right)$ which has $N+1$ copies. We shall also consider the weak topologies $\sigma(\Pi L_A, \Pi E_{\bar{A}})$ and $\sigma(\Pi L_A, \Pi L_{\bar{A}})$. If $u \in W^{1,x}L_A\left(Q\right)$ then the function $t \to u(t) = u(.,t)$ is defined on $(0,T)$ with values in $W^1L_A(\Omega)$. If, further, $u \in W^{1,x}E_A(Q)$ then $u(t)$ is a $W^1E_A(\Omega)$ valued and is strongly measurable. Furthermore, the following continuous imbedding holds: $W^{1,x}E_A(Q) \subset L^1(0,T;W^1E_A(\Omega))$. The space $W^{1,x}L_A\left(Q\right)$ is not in general separable, if $u \in W^{1,x}L_A\left(Q\right)$, we cannot conclude that the function $u(t)$ is measurable from $(0,T)$ into $W^1L_A(\Omega)$. However, the scalar function $t \to \left\|D_x^k u(t)\right\|_{A,\Omega}$, is in $L^1(0,T)$ for all $|k| \le 1$.

2.6. **Duality in inhomogeneous Orlicz-Sobolev spaces.** We denote by $F = W^{-1,x}L_{\bar{A}}\left(Q\right)$ the space

$$F = \left\{ f = \sum_{|k|=1} D_x^k f_k : f_k \in L_{\bar{A}}\left(Q\right) \right\}. \tag{2.9}$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|k|=1} \|f_k\|_{\bar{A},Q}, \tag{2.10}$$

where the inf is taken on all possible decomposition $f = \sum\limits_{|k|=1} D_x^k f_k : f_k \in L_{\bar{A}}(Q)$. The space $F_0 = W^{-1,x}L_{\bar{A}}\left(Q\right)$ is then given by

$$F_0 = \left\{ f = \sum_{|k|=1} D_x^k f_k : f_k \in E_{\bar{A}}\left(Q\right) \right\}. \tag{2.11}$$

The following corollary will be useful in the proof of our existence theorem.

Corollary 2.1. *([10]). Let A be an $N-$function. Let (u_n) be a sequence of $W^{1,x}L_A\left(Q\right)$ such that*

$$u_n \to u \quad \text{weakly} \quad \text{in} \quad W^{1,x}L_A\left(Q\right) \quad \text{for} \ \ \sigma(\Pi L_A, \Pi E_{\bar{A}}) \tag{2.12}$$

and $\frac{\partial u_n}{\partial t}$ is bounded in $W^{-1,x}L_{\bar{A}}\left(Q\right) + M(Q)$, where $M(Q)$ is the space of measures defined on Q. Then $u_n \to u$ strongly in $L^1_{loc}(Q)$.

3. The Main Results of the Existence

Theorem 3.1. *Let $f \in L^{\infty}(Q)$, and u_0 that $|\nabla u_0| \leq 1$. Then the problem*

$$(P_p)\begin{cases} u_p \in W^{1,x}L_{A_\alpha}(Q) \\ \left\langle \frac{\partial u_p}{\partial t}, v\right\rangle + \int\limits_Q \nabla u_p \exp(\alpha\,|\nabla u_p|)\nabla v dxdt + \int\limits_Q |\nabla u_p|^{p-2}\,\nabla u_p \nabla v dxdt = \int\limits_Q f v dxdt, \\ \text{for } v \in W^{1,x}L_{A_\alpha}(Q)\cap L^2(Q) \quad \text{such that } \frac{\partial v}{\partial t} \in W^{-1,x}L_{\bar{A}_\alpha}(Q) + L^2(Q), \end{cases} \tag{3.1}$$

admits at least one solution $u_p \in W^{1,x}L_{A_\alpha}(Q)$ such that $u_p \to u$ for the modular convergence where u is solution of the following parabolic inequality:

$$(P)\begin{cases} |\nabla u| \leq 1 \\ \langle\langle \frac{\partial v}{\partial t}, u - v\rangle\rangle + \int\limits_Q a(x,\nabla u)\,(\nabla u - \nabla v)\,dxdt \leq \int\limits_Q \langle f, u - v\rangle\,dxdt + \frac{1}{2}\int\limits_\Omega (u_0 - v(x,0))^2\,dx \\ \text{for } v \in W^{1,x}L_{A_\alpha}(Q)\cap L^2(Q) \quad \text{such that } \frac{\partial v}{\partial t} \in W^{-1,x}L_{\bar{A}_\alpha}(Q) + L^2(Q) \text{ and } |\nabla v| \leq 1. \end{cases}$$

Remark 3.1. *Since $\left\{v \in W^{1,x}L_{A_\alpha}(Q)\cap L^2(Q) \ : \frac{\partial v}{\partial t} \in W^{-1,x}L_{\bar{A}_\alpha}(Q) + L^2(Q)\right\} \subset C\left(([0,T]), L^2(\Omega)\right)$ (see, [11]) the least term of problem (P) is well defined.*

Proof. **Step 1**

Let us consider the following approximate problem:

$$(P_{p,n})\begin{cases} \frac{\partial u_p^n}{\partial t} - \operatorname{div}(\nabla u_p^n \exp(\alpha\left|\nabla u_p^n\right|)) - \operatorname{div}(\left|\nabla u_p^n\right|^{p-2}\nabla u_p^n) + \frac{1}{n}\left(u_p^n - M\right)\exp\left(\alpha\left|u_p^n - M\right|\right) = f \quad \text{in } Q \\ u_p^n(x,0) = u_0 \quad \text{in } \Omega \\ \exp(\alpha\left|\nabla u_p^n\right|)\frac{\partial u_p^n}{\partial n} + \left|\nabla u_p^n\right|^{p-2}\frac{\partial u_p^n}{\partial n} = 0 \quad on \quad \partial\Omega \times (0,T), \end{cases} \tag{3.2}$$

where $M = \max(\|f\|_\infty, \|u_0\|_\infty)$. As it is done in [11], one can is seen easily that the problem $(P_{p,n})$ admits at least one solution $u_p^n \in W^{1,x}L_{A_\alpha}(Q)$ furthermore

$$\left\|u_p^n\right\|_\infty \leq \max\left\{\|f\|_\infty, \|u_0\|_\infty\right\}. \tag{3.3}$$

By choosing $u_p^n - M$ as test function in $(P_{p,n})$, we obtain $\int\limits_Q \left|\nabla u_p^n\right|^2 \exp(\alpha\left|\nabla u_p^n\right|)dxdt \leq C$, and via (3.3) it follows that

$$\left\|u_p^n\right\|_{1,A_\alpha} \leq M',$$

and thanks to $(P_{p,n})$, we deduce that $\frac{\partial u_p^n}{\partial t}$ is bounded in $W^{-1,x}L_{\bar{A}_\alpha}(Q) + L^\infty(Q)$. Thanks to corollary (2.1) we have $u_p^n \to u_p$ in $L^1(Q)$ as $n \to +\infty$ almost everywhere convergence in Q.

Arguing as in [10] and [5], we pass to the limit in $(P_{p,n})$ to obtain

$$(P_p)\begin{cases} \frac{\partial u_p}{\partial t} - \operatorname{div}(\nabla u_p \exp(\alpha\,|\nabla u_p|)) - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = f \quad \text{in } Q \\ u(x,0) = u_0 \quad \text{in } \Omega \\ \exp(\alpha\,|\nabla u_p|)\frac{\partial u_p}{\partial n} + |\nabla u|^{p-2}\frac{\partial u_p}{\partial n} = 0 \quad on \quad \partial\Omega \times (0,T), \end{cases} \tag{3.4}$$

with $u_p \in W^{1,x}L_{A_\alpha}(Q)\cap L^\infty(Q)$ and $\|u_p\|_\infty \leq M''$.

Step 2: **A priori estimates**

Choosing $v = u_p$ as test function in (P_p) we obtain:

$$\left\langle\left\langle \frac{\partial u_p}{\partial t}, u_p \right\rangle\right\rangle + \int_Q |\nabla u_p|^2 \exp(\alpha |\nabla u_p|) dxdt + \int_Q |\nabla u_p|^p \, dxdt = \int_Q f u_p dxdt,$$

which gives by using Young's inequality

$$\frac{1}{2}\int_\Omega u_p^2(x,t)dx - \frac{1}{2}\int_\Omega u_0^2 dx + \int_Q |\nabla u_p|^p \, dxdt + \int_Q |\nabla u_p|^2 \exp(\alpha |\nabla u_p|) dxdt \leq \frac{1}{2}\int_Q f^2 dxdt + \frac{1}{2}\int_Q u_p^2 dxdt$$

$$\frac{1}{2}\int_\Omega u_p^2(x,t)dx + \frac{1}{2}\int_\Omega u_0^2 dx + \int_Q |\nabla u_p|^p \, dxdt + \int_Q |\nabla u_p|^2 \exp(\alpha |\nabla u_p|) dxdt \leq c + \frac{1}{2}\int_0^t \int_\Omega u_p^2 dxdt,$$

by Gronwall's lemma (see [23]), we get

$$\int_\Omega u_p^2(x,t)dx + \int_Q |\nabla u_p|^2 \exp(\alpha |\nabla u_p|) dxdt + \int_Q |\nabla u_p|^p \, dxdt \leq C.$$

Consequently, since u_p is bounded in $W^{1,x}L_{A_\alpha}(Q) \cap L^2(Q)$ so there exist some $u \in W^{1,x}L_{A_\alpha}(Q) \cap L^2(Q)$ such that (for a subsequence still denoted by u_p)

$$u_p \to u \text{ weakly in } W^{1,x}L_{A_\alpha}(Q) \cap L^2(Q).$$

Step 3:

To obtain $|\nabla u| \leq 1$, we will use the estimate

$$\int_Q |\nabla u_p|^p \, dxdt \leq C. \tag{3.5}$$

Let $q < p$, we have

$$\int_Q |\nabla u_p|^q \, dxdt = \int_{|\nabla u_p| \leq 1} |\nabla u_p|^q \, dxdt + \int_{|\nabla u_p| > 1} |\nabla u_p|^q \, dxdt$$

$$\leq meas(Q) + \int_{|\nabla u_p| > 1} |\nabla u_p|^q \, dxdt \leq meas(Q) + C,$$

which gives

$$\int_Q |\nabla u_p|^q \, dxdt \leq M,$$

by letting $p \to \infty$ for q fixed, we obtain

$$\int_Q |\nabla u|^q \, dxdt \leq M.$$

Now, let $k > 1$, we get

$$\int_{|\nabla u| \geq k} |\nabla u|^q \, dxdt \leq M \Longrightarrow meas\{|\nabla u| \geq k\} \leq \frac{M}{k^q} \Longrightarrow meas\{|\nabla u| \geq 1\} = 0,$$

which gives

$$|\nabla u| \leq 1.$$

Step 4: **Modular convergence of** $u_p \to u$ **in** $W^{1,x}L_{A_\alpha}(Q)$:

Let $w_\mu = u_\mu + e^{-\mu t}u_0$, where u_μ is the mollifier function defined in [16] with respect to time of u and the function w_μ have the following properties:

$$\frac{\partial w_\mu}{\partial t} = \mu\left(u - w_\mu\right); \quad w_\mu\left(0\right) = u_0, \tag{3.6}$$

with

$$u_\mu = \mu \int_{-\infty}^{t} u(x,s)\chi_{(0,T)} \exp(\mu\left(s-t\right))ds, \tag{3.7}$$

$$\nabla w_\mu = \mu \int_{-\infty}^{t} \nabla u(x,s)\chi_{(0,T)} \exp(\mu\left(s-t\right))ds + \exp\left(-\mu t\right)\nabla u_0. \tag{3.8}$$

By using $|\nabla u| < 1$ and $|\nabla u_0| < 1$, we get:

$$|\nabla w_\mu| \leq \mu \int_{0}^{t} \exp(\mu\left(s-t\right))ds + \exp\left(-\mu t\right) = \left[\exp(\mu\left(s-t\right))\right]_0^t + \exp\left(-\mu t\right) = 1.$$

This implies that

$$|\nabla w_\mu| \leq 1.$$

Now, we proof that $u_p \to u$ in $W^{1,x}L_{A_\alpha}(Q)$, for the modular convergence as $p \to +\infty$.
For this, we will denote by $\varepsilon\left(p,\mu,\theta\right)$ function with all quantities such that

$$\lim_{\mu\to+\infty}\lim_{\theta\to 1}\lim_{p\to+\infty}\varepsilon\left(p,\mu,\theta\right) = 0, \tag{3.9}$$

and we will respect the order of the parameters p, θ, μ. Similarly, we will write $\varepsilon\left(p\right)$, or $\varepsilon\left(p,\mu\right)$ that the limits are made only on the specified parameters. Firstly take $v_p = u_p - \theta w_\mu$ for $0 < \theta < 1$ as test function in (P_p), which belong to $W^{1,x}L_{A_\alpha}(Q)$, we get

$$\left\langle \frac{\partial u_p}{\partial t}, v_p \right\rangle + \int_Q \nabla u_p \exp\left(\alpha\left|\nabla u_p\right|\right)\nabla\left(u_p - \theta w_\mu\right)dxdt \tag{3.10}$$

$$+\int_Q \left|\nabla u_p\right|^{p-2}\nabla u_p \nabla\left(u_p - \theta w_\mu\right)dxdt = \int_Q f\left(u_p - \theta w_\mu\right)dxdt.$$

On the other hand, by using the monotonicity of the $p-$Laplacien, we deduce that;

$$\int_Q \left|\nabla u_p\right|^{p-2}\nabla u_p \nabla\left(u_p - \theta w_\mu\right)dxdt \geq \theta^{p-1}\int_Q \left|\nabla w_\mu\right|^{p-2}\nabla w_\mu\left(\nabla u_p - \theta\nabla w_\mu\right)dxdt, \tag{3.11}$$

by using the Holder's inequality

$$\left|I_1\left(p,\mu,\theta\right)\right| = \left|\int_Q \left|\nabla w_\mu\right|^{p-2}\nabla w_\mu\left(\nabla u_p - \theta\nabla w_\mu\right)dxdt\right| \leq \left(\int_Q \left|\nabla w_\mu\right|^p dxdt\right)^{\frac{1}{p'}}\left(\int_Q \left|\nabla u_p - \theta\nabla w_\mu\right|^p dxdt\right)^{\frac{1}{p}},$$

which implies

$$\left(\int_Q |\nabla u_p - \theta \nabla w_\mu|^p \, dxdt\right)^{\frac{1}{p}} \leq \left(2^p \left(\int_Q (|\nabla u_p|^p + \theta^p \, |w_\mu|^p) dxdt\right)\right)^{\frac{1}{p}} \leq 2M^{\frac{1}{p}}.$$

And finally, we obtain

$$I_1 \left(p, \mu, \theta\right) \leq \varepsilon \left(p\right).$$

On the other hand

$$\left\langle\left\langle \frac{\partial u_p}{\partial t}, z_p \right\rangle\right\rangle = \left\langle\left\langle \frac{\partial u_p}{\partial t}, u_p - \theta w_\mu \right\rangle\right\rangle = \left\langle\left\langle \frac{\partial u_p}{\partial t} - \theta \frac{\partial w_\mu}{\partial t}, u_p - \theta w_\mu \right\rangle\right\rangle + \theta \left\langle\left\langle \frac{\partial w_\mu}{\partial t}, u_p - \theta w_\mu \right\rangle\right\rangle$$
$$= J_1 + \theta J_2.$$

With

$$J_1 = \left\langle\left\langle \frac{\partial u_p}{\partial t} - \theta \frac{\partial w_\mu}{\partial t}, u_p - \theta w_\mu \right\rangle\right\rangle = \int_\Omega \left(u_p - \theta w_\mu\right)^2 dx - \int_\Omega \left(u_0 - \theta w_\mu\right)^2 dx,$$

we deduce that

$$J_1 \geq - \int_\Omega \left(u_0 - \theta w_\mu\right)^2 dx \geq -\left(1-\theta\right)^2 \int_\Omega u_0^2 dx + \varepsilon\left(\mu\right) = \varepsilon\left(\mu, \theta\right).$$

For what concerns J_2, we deduce that

$$J_2 = \left\langle\left\langle \frac{\partial w_\mu}{\partial t}, u_p - \theta w_\mu \right\rangle\right\rangle = \mu \int_Q \left(u - w_\mu\right)\left(u_p - \theta w_\mu\right) dxdt$$

$$\lim_{\theta \to 1} \lim_{p \to +\infty} J_2 = \lim_{\theta \to 1} \lim_{p \to +\infty} \left\langle \frac{\partial w_\mu}{\partial t}, u_p - \theta w_\mu \right\rangle \geq 0.$$

Finally, we get

$$\lim_{\mu \to +\infty} \lim_{\theta \to 1} \lim_{p \to +\infty} \int_Q a\left(x, \nabla u_p\right)\left(\nabla u_p - \theta \nabla w_\mu\right) dxdt \leq 0. \tag{3.12}$$

Since

$$\int_Q \left[a\left(x, \nabla u_p\right) - a\left(x, \nabla u\right)\right]\left[\nabla u_p - \nabla u\right] dxdt - \int_Q a\left(x, \nabla u_p\right)\left(\nabla u_p - \theta \nabla w_\mu\right) dxdt$$
$$= -\int_Q a\left(x, \nabla u_p\right) \nabla u dxdt - \int_Q a\left(x, \nabla u\right)\left[\nabla u_p - \nabla u\right] dxdt + \theta \int_Q a\left(x, \nabla u_p\right) \nabla w_\mu dxdt.$$

Since $a\left(x, \nabla u_p\right)$ is bounded in $\left(L_{\bar{A}_\alpha}\left(Q\right)\right)^N$, we have $a\left(x, \nabla u_p\right) \to h$ weakly for $\sigma(\Pi L_{\bar{A}_\alpha}, \Pi E_{A_\alpha})$ consequently

$$\int_Q \left[a\left(x, \nabla u_p\right) - a\left(x, \nabla u\right)\right]\left[\nabla u_p - \nabla u\right] dxdt - \int_Q a\left(x, \nabla u_p\right)\left(\nabla u_p - \theta \nabla w_\mu\right) dxdt$$

$$= -\int_Q h \nabla u dxdt + \theta \int_Q h \nabla w_\mu dxdt + \varepsilon\left(p\right)$$

$$= -\int_Q h\nabla u dxdt + \int_Q h\nabla w_\mu dxdt + \varepsilon(p,\theta) = \varepsilon(p,\theta,\mu).$$

Which gives

$$\int_Q [a(x,\nabla u_p) - a(x,\nabla u)][\nabla u_p - \nabla u]\, dxdt - \int_Q a(x,\nabla u_p)(\nabla u_p - \theta\nabla w_\mu)\, dxdt = \varepsilon(p,\theta,\mu),$$

by using (3.12), we obtain

$$\int_Q [a(x,\nabla u_p) - a(x,\nabla u)][\nabla u_p - \nabla u]\, dxdt \to 0 \text{ as } p \to +\infty.$$

By strict monotonicity of $a(.,.)$, we obtain that $\nabla u_p \to \nabla u$ a.e in Q.

Finally $a(x,\nabla u_p) \to h = a(x,\nabla u)$, weakly for $\sigma(\Pi L_{\bar{A}_\alpha}, \Pi E_{A_\alpha})$, consequently

$$\int_Q [a(x,\nabla u_p) - a(x,\nabla u)][\nabla u_p - \nabla u]\, dxdt$$

$$= \int_Q a(x,\nabla u_p)\nabla u_p dxdt - \int_Q a(x,\nabla u_p)\nabla u dxdt - \int_Q a(x,\nabla u)[\nabla u_p - \nabla u]\, dxdt,$$

because $\nabla u_p \to \nabla u$ weakly in $\left(L_{\bar{A}_\alpha}(Q)\right)^N$, we get

$$\lim_{p\to+\infty}\int_Q [a(x,\nabla u_p) - a(x,\nabla u)][\nabla u_p - \nabla u]\, dxdt$$

$$= \lim_{p\to+\infty}\int_Q a(x,\nabla u_p)\nabla u_p dxdt - \int_Q a(x,\nabla u)\nabla u dxdt = 0.$$

Since $a(x,\nabla u_p)\nabla u_p = A_\alpha(|\nabla u_p|)$ we get

$$\lim_{p\to+\infty}\int_Q A_\alpha(|\nabla u_p|)\, dxdt = \int_Q A_\alpha(|\nabla u|)\, dxdt.$$

Thanks to fact that

$$A_\alpha\left(\frac{|\nabla u_p - \nabla u|}{2}\right) \le \frac{1}{2}\left(A_\alpha(|\nabla u_p|) + A_\alpha(|\nabla u|)\right).$$

By using Vitali's theorem, we obtain

$$\int_Q A_\alpha\left(\frac{|\nabla u_p - \nabla u|}{2}\right) dxdt \to 0 \text{ as } p \to +\infty.$$

Which shows that ∇u_p converges to ∇u for the modular convergence in $L_{A_\alpha}(Q)$.

Step 5: The passage to the limit

Let us consider $v \in W^{1,x}L_{A_\alpha}(Q) \cap L^2(Q) = W^{1,x}L_{A_\alpha}(Q)$ such that $|\nabla v| < 1$, $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\bar{A}_\alpha}(Q) + L^2(Q)$ and $0 < \theta < 1$. Using $u_p - \theta v$ as test function in (P_n), the fact that

$$\left\langle\left\langle \frac{\partial u_p}{\partial t}, u_p - \theta v \right\rangle\right\rangle + \int_Q \nabla u_p \exp(\alpha|\nabla u_p|)\nabla(u_p - \theta v)\, dxdt \tag{3.13}$$

$$+\int_Q |\nabla u_p|^{p-2}\nabla u_p \nabla (u_p-\theta v)\,dxdt \le \int_Q f\,(u_p-\theta v)\,dxdt,$$

we have

$$\left\langle\left\langle \frac{\partial u_p}{\partial t}-\theta\frac{\partial v}{\partial t}, u_p-\theta v\right\rangle\right\rangle+\theta\left\langle\left\langle \frac{\partial v}{\partial t}, u_p-\theta v\right\rangle\right\rangle$$

$$+\int_Q \nabla u_p \exp(\alpha\,|\nabla u_p|)\nabla(u_p-\theta v)\,dxdt+\int_Q |\nabla u_p|^{p-2}\nabla u_p\nabla(u_p-\theta v)\,dxdt\le\int_Q f\,(u_p-\theta v)\,dxdt,$$

since

$$\int_Q |\nabla u_p|^{p-2}\nabla u_p\nabla(u_p-\theta v)\,dxdt\ge\theta^{p-1}\int_Q |\nabla v|^{p-2}\nabla v\nabla(u_p-\theta v)\,dxdt,$$

which gives

$$\theta\left\langle\left\langle \frac{\partial v}{\partial t}, u_p-\theta v\right\rangle\right\rangle+\int_Q a(x,\nabla u_p)\,(\nabla u_p-\theta\nabla v)\,dxdt+\theta^{p-1}\int_Q |\nabla v|^{p-2}\nabla v\nabla(u_p-\theta v)\,dxdt$$

$$\le\int_Q \langle f, u_p-\theta v\rangle\,dxdt+\int_\Omega (u_0-\theta v)^2\,dx.$$

Since $a\,(x,\nabla u)$ belongs to $\left(L_{\bar{A}_\alpha}(Q)\right)^N$, and using Fatou's lemma in the first term of the last side gives

$$\liminf_{p\to+\infty}\int_Q a(x,\nabla u_p)\,(\nabla u_p-\theta\nabla v)\,dxdt\ge\int_Q a(x,\nabla u)\,(\nabla u-\theta\nabla v)\,dxdt,$$

then, we can easily pass to the limit as $\theta\to 1$ and p tend to infinity, we obtain

$$\left\langle\left\langle \frac{\partial v}{\partial t}, u-v\right\rangle\right\rangle+\int_Q a(x,\nabla u)\,(\nabla u-\nabla v)\,dxdt\le\int_Q\langle f,u-v\rangle\,dxdt+\frac{1}{2}\int_\Omega (u_0-v(x,0))^2\,dx.$$

Which completes the proof. □

4. Numerical results

We consider the following model problem:

$$(P1)\begin{cases}\frac{\partial u_p}{\partial t}-\operatorname{div}(\nabla u_p\exp(\alpha\,|\nabla u_p|))-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=f \quad \text{in } Q\\ u(x,0)=u_0 \ \ \text{in } \Omega\\ \exp(\alpha\,|\nabla u_p|)\frac{\partial u_p}{\partial n}+|\nabla u|^{p-2}\frac{\partial u_p}{\partial n}=0 \ \ in \ \ \partial\Omega\times(0,T),\end{cases}\tag{4.1}$$

where u_0 represents the input image. We apply finite differences method to this problem. We denote respectively by h and Δt the spatial and time steps sizes.

In what follows, we take $h=1$ and we define for every field $p=(p_1,p_2)\in\mathbb{R}^2$, the discrete divergence approximation:

$$\operatorname{div}_{i,j}(p)=\begin{cases}p_1(i,j)-p_1(i-1,j) & \text{if } 1<i<n\\ p_1(i,j) & \text{if } \ i=1\\ p_1(i-1,j) & \text{if } \ i=n\end{cases}+\begin{cases}p_2(i,j)-p_2(i,j-1) & \text{if } 1<j<n\\ p_2(i,j) & \text{if } \ j=1\\ -p_2(i,j-1) & \text{if } \ j=m,\end{cases}\tag{4.2}$$

where n and m is an integer greater than 2.

One can write the following scheme:

$$u^{k+1}(i,j) = u^k(i,j) + \Delta t\left[\left(\operatorname{div}\left(d_\alpha(x,\nabla u) + \operatorname{div}(q(x,\nabla u))\right)^k(i,j)\right], \qquad 1 \le k \le N \tag{4.3}$$

where $d_\alpha(x,\nabla u) = \nabla u_p \exp(\alpha\,|\nabla u_p|)$, $q(x,\nabla u) = |\nabla u|^{p-2}\nabla u$, $u(t_k, x_i, y_j) = u^k(i,j)$, $x_i = ih$, $y_j = jh$, $t_k = k\Delta t$, and $\Delta t = \frac{T}{N}$.

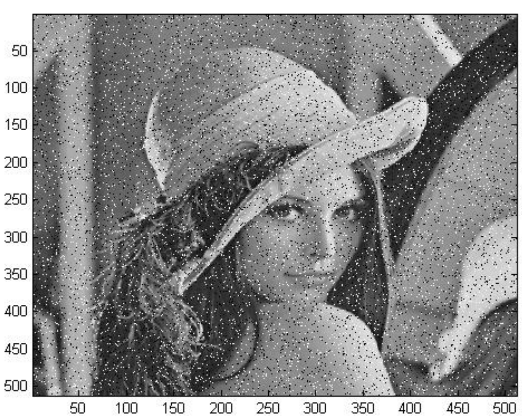
In our numerical tests we take $\Delta t = \frac{T}{N} = 0.1$, and we compute the PNSR (Peak Signal to Noise Ratio) quotient of every image.

In Figs. 1-3, we give some examples by taking $\alpha = 0.25$, σ is the standard deviation of the distribution which performs an edge-preserving average filter on the image and with different values of p.

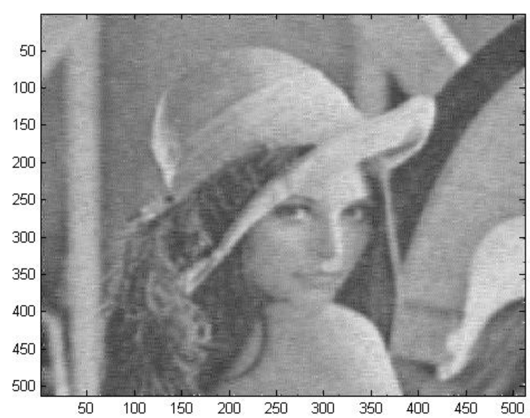
We give in Fig. 4, tests with different values of α, with $p = 40$.

Noisy image with $salt\&pepper = 0.008$

$p = 40$, $PSNR = 24.2487$

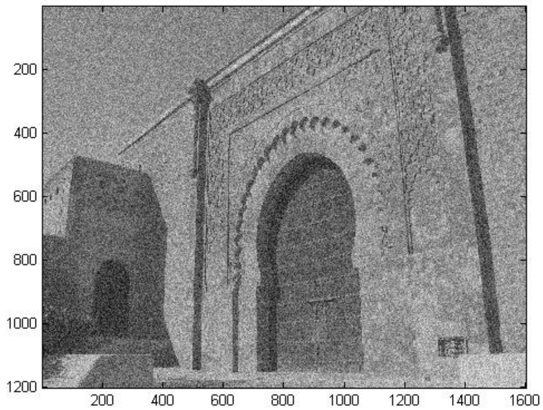


Noisy image with $salt\&pepper = 0.08$

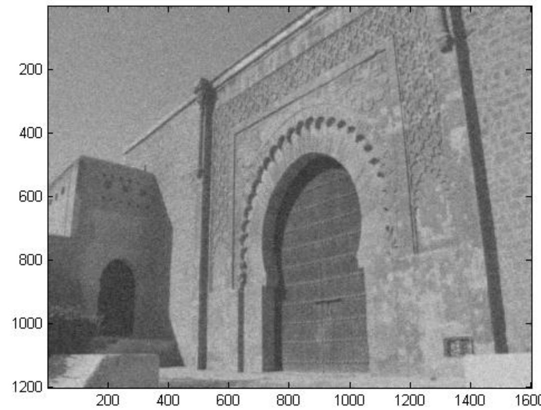


$p = 40$, $PSNR = 16.0574$

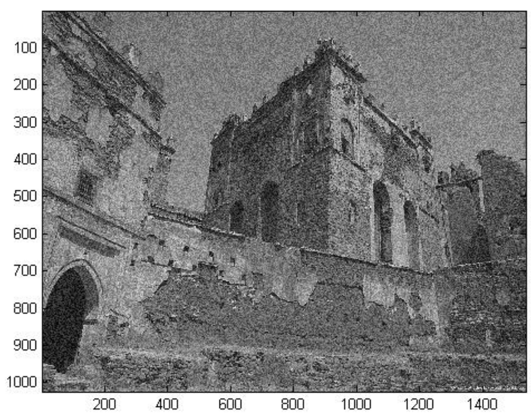
fig 1.



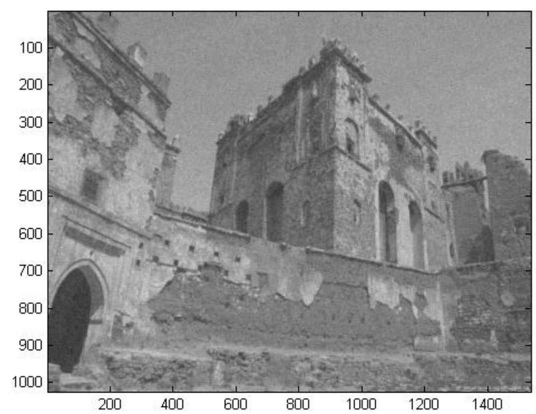
Noisy image with $\sigma = 1$



$p = 900E900, PSNR = 15.2328$

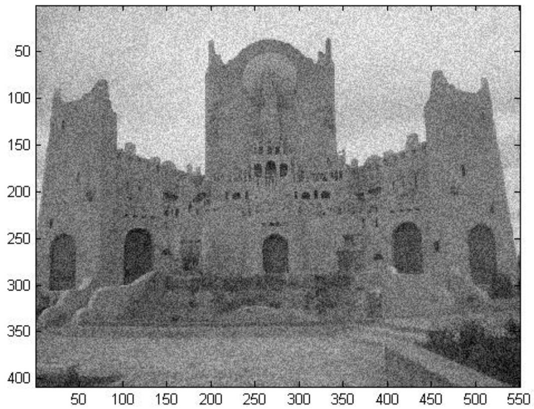


Noisy image with $\sigma = 1$

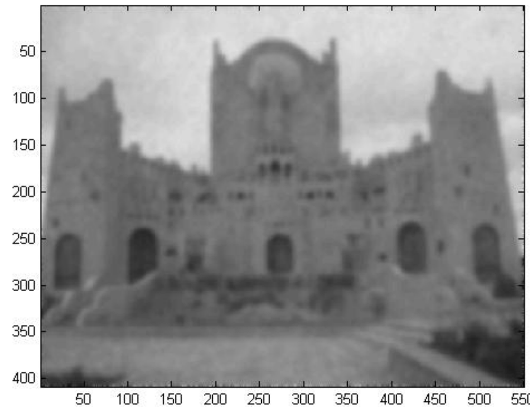


$p = 900E900, PSNR = 16.4051$

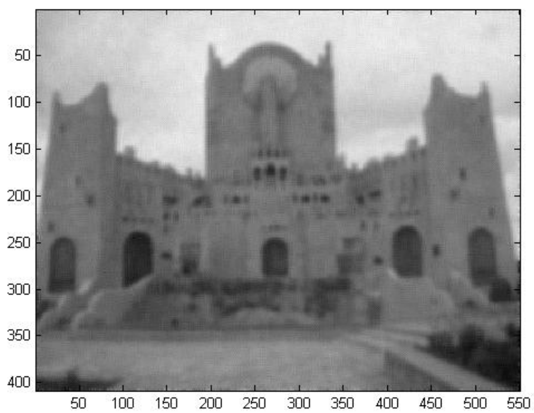
fig 2.



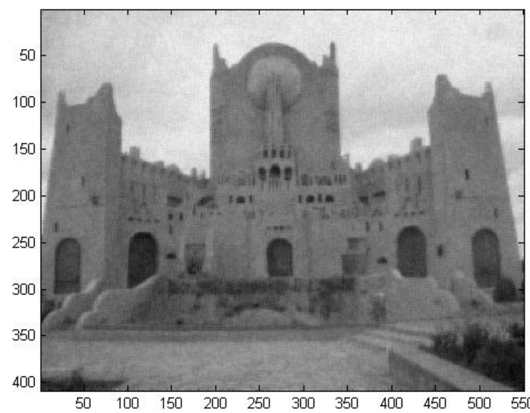
Noisy image with $\sigma = 0.9$



$p = 3, PSNR = 19.4112$

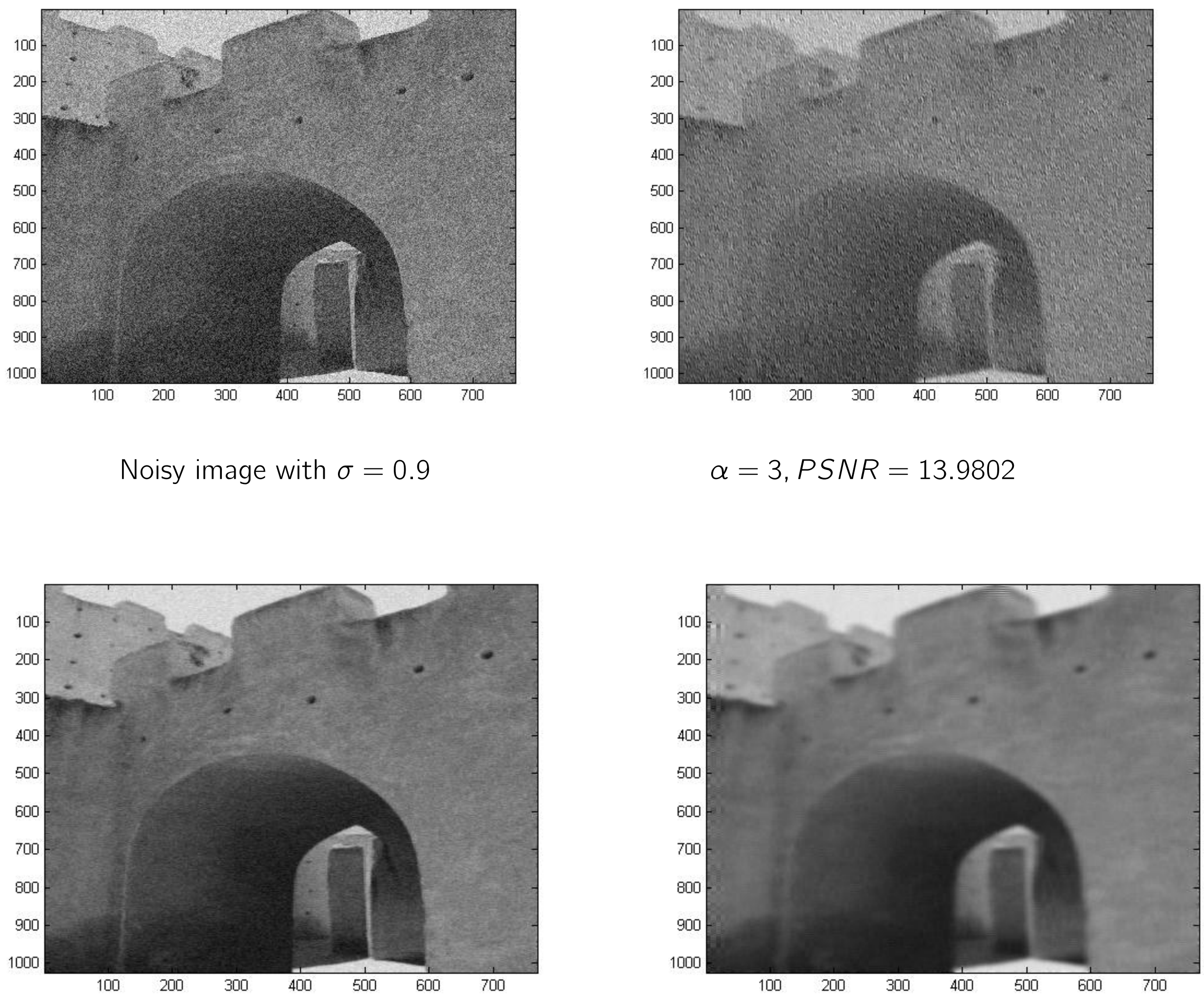


p=300, PSNR=25.1073



$p = 900E900, PSNR = 25.1176$

fig 3.



Noisy image with $\sigma = 0.9$

$\alpha = 3, PSNR = 13.9802$

$\alpha = 0.25$, PSNR=20.1509

$\alpha = 10E - 15, PSNR = 18.5800$

fig 4.

In Numerical tests, we show the better value of a α which gives a good restored image is equal to 0.25, so we should not take α close to 0 and no more than 0.25.

5. Conclusion

In this article, we presente a parabolic model for image denoising and restoration, with their theoretical results and numerical results.

This model preserve the contours of image more than other models.

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