Int. J. Anal. Appl. (2023), 21:35

International Journal of Analysis and Applications

On the Stability and Controllability of Degenerate Differential Systems in Hilbert Spaces

Norelhouda Beghersa[1,*], **Fares Yazid**[2]

[1] *Norelhouda Beghersa, Department of Mathematics, Faculty of Mathematics and Computer Sciences, University of Sciences and Technology Mohamed Boudiaf of Oran USTO-MB, El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algeria*

[2] *Laboratory of pure and applied Mathematics, Amar Teledji Laghouat University, 03000, Laghouat, Algeria*

[*] *Corresponding author:* norelhouda.beghersa@univ-usto.dz

Abstract. We apply the famous theorem of Lyapunov for some degenerate differential systems taken the form $Ax'(t) + Bx(t) = \Phi(t, x(t))$, where $t \in \mathbb{R}_+$ and A, B are linear bounded operators in Hilbert spaces, Φ is a given function. The obtained results are used to study the stabilizability and controllability of certain implicit controlled systems.

1. Introduction

The fondamental challenges of control theory and the most variety problems of this area in mathematical science attracted the attention of many researchers, that's why they often study linear differential systems (continuous or discrete, see [4]), of the form:

$$x'(t) = Px(t) + \Phi(t, x(t)), \tag{1.1}$$

or

$$x_{n+1} = Px_n + \Phi(t, x(t)), \quad n = 0, 1, 2, 3, ..., \tag{1.2}$$

where P is a linear operator, or matrix in an appropriate (finite or infinite dimensional) Hilbert spaces. The major interest on different physical and mechanical problems was described by another more

Received: Feb. 28, 2023.

2020 *Mathematics Subject Classification.* 15A22, 93D15, 93D05, 47D03.

Key words and phrases. Operator pencil; Stabilizability; Controllability; Semigroup.

https://doi.org/10.28924/2291-8639-21-2023-35
ISSN: 2291-8639

general form than 1.1 and 1.2 which has been studied extensively in 1970 by many mathematicians as Rutkas [1] and Gantmacher [6]. In [5] Favini and Vlasenko have recently ensured the existence of weak and strict solutions by using the operator method of Grisvard when the differential system is non-stationary.

In the present paper, we consider the following differential system of the form:

$$Ax'(t) + Bx(t) = \Phi(t, x(t)), \quad t \geqslant 0, \tag{1.3}$$

with initial condition

$$x(0) = x_0,$$

where A and B are linear bounded operators acting in the same Hilbert space $\mathcal{H}$ and $\Phi : [0, \infty[\times\mathcal{H}$ into $\mathcal{H}$ is a continuous function. We call the system 1.3 with a non invertible operator A degenerate or implicit stationary differential system but if A is the identity operator then, the equation 1.3 is said to be explicit. We denote by $\lambda A + B$ the operator pencil obtained by substituting the exponential function $e^{\lambda t}v$ for all $v \in \mathcal{H}$ into 1.3 when $\Phi \equiv 0$. Some real and practical applications of 1.3 can be found in [2], [7] and the references therein.

The organization of this paper is as follows: in section 2, we present an important result which is the generalization of the Lyapunov theorem obtained by using some properties of the spectral theory and an appropriate conformal transformation to the corresponding operator pencil. In section 3, we apply the achieved results to study the stabilizability and controllability of certain implicit differential control system described as follows:

$$Ax'(t) + Bx(t) = -Cu(t), \quad 0 \leqslant t \leqslant T, \quad x(0) = x_0; \tag{1.4}$$

here $x(t)$ and the control $u(t)$ (Bochner integrable function) lie in the Hilbert spaces $\mathcal{H}$ and U. Also, T is a time and the operator C is bounded on $\mathcal{H}$.

2. Stability of a stationary implicit differential system

In this section, we investigate the homogeneous case of the system 1.3 as follows:

$$Ax'(t) + Bx(t) = 0, \quad \text{for all} \quad t \geq 0. \tag{2.1}$$

In the sequel, we need the following definitions.

Definition 2.1. *The system 2.1 is said to be exponentially stable, if there exist two constants M, $\alpha > 0$ such that for any solution $x(t)$ the bellow hypothesis is verified:*

$$\|x(t)\| \leq Me^{-\alpha t}, \quad \forall t \geq 0. \tag{2.2}$$

Definition 2.2. *The equation 2.1 is called well-posed, if it satisfies the following conditions:*

- *for any solution $x(0)$ such that $x(0) = x_0$, then $x(t) = 0$, for each $t \geq 0$;*

- *it generates an evolution semi-group of bounded operators $\Gamma(t) : x_0 \to x(t)$, for all $t \geq 0$, where the operators $\Gamma(t)$ are defined on the set of admissible initial vectors denoted by $D_* = \{x_0\}$.*

Definition 2.3. *let D_* be a subspace of the Hilbert space $\mathcal{H}$ and A_0 is the restriction of the operator A in D_*. The operator A_0^{-1} exists if the system 2.1 is well-posed.*

Definition 2.4. *[1] The comlex parameter λ is said to be a regular point of the pencil $\lambda A + B$, if the operator $(\lambda A + B)^{-1}$ exists and it is bounded. We denote by $\rho(A, B)$ the set of all regular points and its complement by $\sigma(A, B) = \mathbb{C} \backslash \rho(A, B)$ which is also called the spectrum of the operator pencil $\lambda A + B$.*
The set of all eigen values is denoted by $\sigma_p(A, B)$, such that

$$\sigma_p(A, B) = \{\lambda \in \mathbb{C} \backslash \exists v \neq 0, \quad (\lambda A + B)v = 0\}. \tag{2.3}$$

We recall that our main objective is to generalize the Lyapunov theorem were laid in the monograph [2] concerning the classical linear differential systems for the operator pencil $\lambda A + B$, we have thereby obtained the next result.

Theorem 2.1. *Assume that A and B are bounded linear operators acting in the same Hilbert space $\mathcal{H}$. If the spectrum of the pencil $\lambda A + B$ lies inside the disk of radius r, then for any uniformly positive operator S, there exists an operator $W \gg 0$ such that:*

$$Re(W(B - rA)(B + rA)^{-1}) = -S. \tag{2.4}$$

Proof. Suppose that $\sigma(A, B) \subset \{\lambda \in \mathbb{C} : Re(\lambda) < r\}$. Then, r is a regular point and the operator $T = (B - rA)(B + rA)^{-1}$ is bounded. Now, we define the conformal transformation ζ given by:

$$\zeta = \chi(\lambda) = \frac{\lambda + r}{\lambda - r}. \tag{2.5}$$

Using 2.5 to compute $T - \zeta I$ then, we obtain:

$$\begin{aligned} T - \zeta I &= (B - rA)(B + rA)^{-1} - \frac{\lambda + r}{\lambda - r}(B + rA)(B + rA)^{-1}, \\ &= \frac{-2r}{\lambda - r}(\lambda A + B)(B + rA)^{-1}. \end{aligned}$$

Indeed, the operator $T - \zeta I$ is invertible if the inverse of the pencil $\lambda A + B$ exists. Hence, $\rho(T) = \chi(\rho(A, B))$ and we conclude that $\sigma(T) = \chi(\sigma(A, B))$. Therefore, $\sigma(T)$ lies inside the disk of radius r.
According to Lyapunov theorem [2], we have:

$$\forall S \gg 0, \quad \exists W \gg 0 / \quad Re(WT) = \frac{WT + T^*W}{2}. \tag{2.6}$$

So,

$$\begin{aligned} Re(WT) &= \frac{1}{2}\{W(B-rA)(B+rA)^{-1}+(B^*+rA^*)^{-1}(B^*-rA^*)W\}, \\ &= (B^*+rA^*)^{-1}[B^*WB-r^2A^*WA](B+rA)^{-1}, \\ &= -S \end{aligned}$$

which is equivalent to:

$$B^*WB-r^2A^*WA=G, \quad \text{where} \quad G=-(B^*+rA^*)S(B+rA). \tag{2.7}$$

To complete the proof, we use the fact that $G \gg 0$, it means:

$$G=G^*\text{and} <Gx,x>\geq c\|x\|^2, \quad \forall c>0. \tag{2.8}$$

□

Theorem 2.2. *Suppose that the equation 2.7 is satisfied for any pair of non negative uniform operators (W,G) then, r is not an eigen value of the operator pencil $\lambda A+B$.*

Proof. For $r\in\sigma_p(A,B)$, there exists an eigen-vector $v\neq 0$, such that $(rA+B)v=0$ so, $Bv=-rA$. Now, we compute the scalar product $<Gv,v>$ then, we obtain:

$$\begin{aligned} <Gv,v> &= <WBv,Bv>-r^2<WAv,Av>, \\ &= r^2<WAv,Av>-r^2<WAv,Av>, \\ &= 0. \end{aligned}$$

We use the fact that G is a uniformly positive operator, which implies a contraduction with our main hypothesis[1]. Thus, Theorem 2.2 is proved. □

Proposition 2.1. *In finite dimentional spaces (i.e., $dim(\mathcal{H})<\infty$), if:*

$$\sigma(A,B)=\sigma_p(A,B)\subset\{\lambda : Re(\lambda)<\omega^2\},$$

then, the system 2.1 is exponentially stable under the condition $\alpha\leq r\leq\omega$.

Proof. We assume that, the system 2.1 is not exponentially stable. We use the method of elementary divisors (see for example [6]) and we suppose that the matrix pencil $\lambda A+B$ is regular in order to prove our proposition. We have:

$$\lambda A+B\sim\lambda\widetilde{A}+\widetilde{B}=\{N^{\mu_1},N^{\mu_2},...,N^{\mu_s};J+\lambda I\},$$

[1] $G\gg 0 \Longleftrightarrow G=G^*, \quad \text{and} \quad <Gx,x>\geq c\|x\|^2>0.$

[2] it presents the strict Lyapunov exponent as in [3]

where the first diagonal blocks correspond to the infinite elementary divisors. Now, we pose $x(t) = Qz(t)$ where $det(Q) \neq 0$. So, the system 2.1 is equivalent to the following form:

$$\begin{cases} Az'(t) + Bz(t) = 0, \\ \widetilde{A} = AQ, \quad \widetilde{B} = BQ, \quad \lambda\widetilde{A} + \widetilde{B} = (\lambda A + B)Q. \end{cases} \tag{2.9}$$

Also, we can write the system 2.1 as follows:

$$\begin{cases} N^{\mu_k}\dfrac{dz_k}{dt} = 0, \qquad k = 1, 2, ..., s \\ \dfrac{d\widetilde{z}_k}{dt} + \Im\widetilde{z} = 0, \quad \text{where} \quad z = (z_1, z_2, ..., z_s, \widetilde{z})^t. \end{cases} \tag{2.10}$$

As, $\sigma(A, B) = \sigma(\widetilde{A}, \widetilde{B}) = \sigma(I, J) = \sigma(-J) \subset \{\lambda : Re(\lambda) < r\}$ then we obtain:

$$\|e^{-J.t}\| \leq M_\omega e^{-\omega.t},$$

and

$$\|\widetilde{z}(t)\| = \|e^{-J.t}\widetilde{z}(t_0)\| \leq M_\omega e^{-\omega.t}\|\widetilde{z}(t_0)\|.$$

Therefore,

$$\|x(t)\| = \|Qz(t)\| \leq \|Q\| M_\omega e^{-\omega.t}\|z(t)\|.$$

Hence, the system 2.1 is exponentially stable for all $r \leq \omega$, which implies a contradiction with our main hypothesis thus, this proposition is proved. □

Remark 2.1. *We can easly show that the necessary and sufficient conditions on the stability (exponential stability) of the implicit differential system 2.1 can be obtained by using Theorem* $1-3$ *in [4].*

Let us return now to the problem 2.1 when the Hilbert space $\mathcal{H}$ has finite dimension, in this case we often use theory of elementery divisors for the matrix pencil $\lambda A + B$ (see [6]) and we have thereby obtained the next result.

Theorem 2.3. *Consider the problem 2.1 in finite dimentional Hilbert space $\mathcal{H}$, the following assertions are equivalent:*

(1) *The system 2.1 is exponentially stable,*
(2) $\sigma_p(A, B) = \sigma(A, B) \subset \{\lambda \in \mathbb{C}, \quad Re(\lambda) < r\}$,
(3) *There exists a uniform positive definite matrix W, such that:*

$$B^*WB - r^2A^*WA \gg 0. \tag{2.11}$$

According to Theorem 2.2 in the reference [3] and Theorem 2.2 of this paper, we can show that (1) is equivalent to the relation (2). From Proposition 2 and Theorem 2.1, we obtain: (2) $\Leftrightarrow$ (3).

3. Basic preliminaries and results on the controlled degenerate systems

Now, we give some important definitions about the exact controllability and stabilizability for some types of systems governed by 1.4, we also establish some results more general than the classical explicit case under a given assumption.

Definition 3.1. *The equation 1.4 is said to be exactly controllable if for all $x_0, x_1 \in \mathcal{H}$, there exists a time T and control $u \in L_2((0,T),U)$ such that:*

$$x(T, u(.), x_0) = x_1.$$

Particulary, if $x_1 = 0$, here we talk about the notion of exact null controllability.

Definition 3.2. *[7] The implicit controlled system 1.4 is said to be exponentially stabilizable by means of a direct feedback $u(t) = Kx(t)$, if the given system:*

$$Ax'(t) + (B + CK)x(t) = 0, \quad 0 \leq t \leq T \tag{3.1}$$

is exponentially stable[3] (A, B, C, K suppose to be linear bounded operators in the Hilbert space $\mathcal{H}$).

Proposition 3.1. *[8] The system 1.4 is exactly null controlable in the class L_2 if and only if:*

$$\exists \sigma > 0, \forall x \in \mathcal{H}, \int_0^T \|C^*(A_0^*)^{-1}\Gamma^*(t)x\|^2 dt \geqslant \sigma^2 \|\Gamma^*(T)x\|^2. \tag{3.2}$$

Before studying the cotrollability of a system such 1.4, we can observe that the controllability problem has always a strong relation with the exponential stabilizability also, it can be reduced to the controllability problem of an explicit differential system (if A is an invertible operator). To be more precise we propose the next assumption then, we expand it to the system 1.4.

Assumption 3.1. *Let $\widetilde{B}$ be the infinitesimal generator of a C_0-group $\widetilde{\Gamma}(t)$ of bounded operators in the Hilbert space $\mathcal{H}$. $\widetilde{B}$ satisfies the assumption if for some time $T > 0$, there exists $\sigma > 0$, such that:*

$$\int_0^T \|\widetilde{B}^* \widetilde{\Gamma}^*(-t)x\|^2 \geqslant \sigma \|x\|^2, \forall x \in \mathcal{H}. \tag{3.3}$$

Theorem 3.1. *If the operator $A_0^{-1}B$ complies with the previous assumption then, the problem of control considered by the equation 1.4 is stabilizable.*
Additionally, given $\alpha > 0$ there exists a linear bounded operator K where the group $\Gamma(t)$ generated by $A_0^{-1}(B + CK)$ verifies:

$$\|\Gamma(t)\| \leqslant M_\alpha e^{-\alpha t}, \quad \forall t \geq 0, \quad \alpha > 0, \quad \textit{and} \quad M_\alpha \geqslant 1. \tag{3.4}$$

[3] It means: $\forall \alpha > 0, \quad \|e^{A_0^{-1}(B+CK)}\| \leqslant M_\alpha e^{-\alpha t}, \quad M_\alpha \geqslant 1.$

Proof. Let $D_{T,\alpha}$ be a bounded linear operator defined as follows:

$$D_{T,\alpha}x = \int_0^T e^{-2\alpha t}\Gamma(-t)A_0^{-1}CC^*(A_0^*)^{-1}\Gamma^*(-t)x dt. \tag{3.5}$$

Obviuosly, $D^*_{T,\alpha} = D_{T,\alpha}$, using the assumption 3.1, we can affirme that the inverse of the operator $D_{T,\alpha}$ exists and it is bounded. Now, we consider another implicit control system represented by the abstract form:

$$Ax'(t) = -Bx(t) + \alpha x(t) - Cu(t), \tag{3.6}$$

where the group $\Gamma_\alpha(t)$ has the infinitesimal generator $A_0^{-1}(B - \alpha I)$.
The equation 3.6 is also equivalent to

$$Ax'(t) = -Bx(t) + \alpha x(t) - CKx(t). \tag{3.7}$$

We define the linear feedback $u(t)$ as follows:

$$u(t) = Kx(t) = -C^*(A_0^*)^{-1}D_{T,\alpha}^{-1}x(t). \tag{3.8}$$

Also, we have the group $\Gamma^*_{\alpha,K}(t)$ generated by the operator $A_0^{-1}(-B + \alpha I) + A_0^{-1}CC^*(A_0^*)^{-1}D_{T,\alpha}^{-1}$ is a solution of the system 3.7.
To complete the proof, it is necessary to compute the scalar product so, for each $x_* \in D(B^*(A_0^*)^{-1})$, we have:

$$\begin{aligned}\frac{d}{dt}(< \Gamma^*_{\alpha,K}(t)x_*, D_{T,\alpha}\Gamma^*_{\alpha,K}(t)x_* >) = -\|e^{-\alpha T}C^*(A_0^*)^{-1}\Gamma^*(-T)\Gamma^*_{\alpha,K}(t)x_*\|^2 \\ -\|C^*(A_0^*)^{-1}\Gamma_{\alpha,K}(t)x_*\|^2,\end{aligned} \tag{3.9}$$

for more details of computation, see the reference [8].
Passing now to the denseness of the group $\Gamma_{\alpha,K}$, we can show that:

$$\forall\alpha > 0, \exists K : \|\Gamma_K(t)\| = \|e^{A_0^{-1}(B+CK)}\| \leqslant Me^{-\alpha t}, \forall t \geqslant 0. \tag{3.10}$$

Hence, the system is exponentially stabilizable. It follows immediately from the definition of the bounded operator K such that:

$$K = -C^*(A_0^*)^{-1}D_{T,\alpha}^{-1}.$$

□

Remark 3.1. *If we replace B by $B + CK$ in Theorem 2.3 then, we obtain another important result on the stabilization of implicit controlled systems ($dim(\mathcal{H}) < \infty$, $r = 0$).*

Corollary 3.1. *The following expressions are equivalent:*

- *The equation 2.1 is exponentially stabilizable,*
- $\sigma(A, B + CK) = \sigma_p(A, B + CK) \subset \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}$,
- *There exists a positive definite matrix $W \gg 0$, such that:*

$$B^*WB + K^*C^*WB \gg 0. \tag{3.11}$$

Remark 3.2. *According to reference [9], we can show that if all the matrices of the system are real and A^{-1} exists then, the spectrum is formed by real or complex conjugate numbers.*

Finally, we provide a simple example to illustrate and explain the last point of our paper.

Example 3.1. *Consider the system 1.4 as follows:*

$$\begin{cases} 2x'(t) - y'(t) = 3x(t) - y(t) + u_1(t) \\ 2y'(t) = -x(t) + 2y(t) + u_2(t), \end{cases}$$

with the direct feedback $u(t) = I_2(x(t), y(t))^t$.
We have,

$$E_c(\lambda) = det(A^{-1}(B + CK) - \lambda I_2) = \lambda^2 - \frac{13}{4}\lambda + \frac{11}{4},$$

then, we obtain:

$$\sigma(A, B + CK) = \sigma(I, A^{-1}(B + CK)) = \{\frac{13}{8} + \frac{1}{8}i\sqrt{7}; \quad \frac{13}{8} - \frac{1}{8}i\sqrt{7}\}.$$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A.G. Rutkas, Spectral Methods for Studying Degenerate Differential-Operator Equations. I, J. Math. Sci. 144 (2007), 4246-4263. https://doi.org/10.1007/s10958-007-0267-2.

[2] J.L. Doletski, M.G. Krein, Stability of Solutions of Differential Equations in Banach Space, Amer. Math. Soc. 2002.

[3] M. Benabdallah, M. Hariri, On The Stability of The Quasi-Linear Implicit Equations in Hilbert Spaces, Khayyam J. Math. 5 (2019), 105-112. https://doi.org/10.22034/kjm.2019.81222.

[4] M. Benabdallakh, A.G. Rutkas, A.A. Solov'ev, On the Stability of Degenerate Difference Systems in Banach Spaces, J. Soviet Math. 57 (1991), 3435-3439. https://doi.org/10.1007/bf01880215.

[5] A. Favini, L. Vlasenko, Degenerate Non-Stationary Differential Equations With Delay in Banach Spaces, J. Differ. Equ. 192 (2003), 93-110. https://doi.org/10.1016/s0022-0396(03)00090-1.

[6] F.R. Gantmacher, The Theory of Matrices, Vol. 1 and Vol. 2, Chelsea Publishing Company, New York, 1959.

[7] L. Baghdadi, R. Rabah, A Note on the Stabilization of Linear Systems in Hilbert Spaces, Demonstr. Math. 21 (1988), 631-641. https://doi.org/10.1515/dema-1988-0307.

[8] M. Slemrod, A Note on Complete Controllability and Stabilizability for Linear Control Systems in Hilbert Space, SIAM J. Control. 12 (1974), 500-508. https://doi.org/10.1137/0312038.

[9] W.M. Wonham, Linear Multivariable Control: A Geometric Approach, 3rd edition., Springer, New York, 1985.