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Bi-ideals and Weak Bi-ideals of Near Left Almost Rings

Thiti Gaketem*

Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics School of Science, University of Phayao, Phayao 56000, Thailand

**Corresponding author:* thiti.up.ac.th

Abstract. In this paper, we define bi-ideals and weak bi-ideals of nLA-ring. We investigate the properties of bi-ideals and weak bi-ideals of nLA-ring.

1. Introduction

M.A. Kazim and MD. Naseeruddin defined LA-semigroup as the following; a groupoid S is called a left almost semigroup, abbreviated as LA-semigroup if

$$(ab)c = (cb)a, \quad \forall\, a, b, c \in S$$

M.A. Kazim and MD. Naseeruddin [1, Proposition 2.1] asserted that, in every LA-semigroups G a *medial law* hold

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall\, a, b, c, d \in G.$$

Q. Mushtaq and M. Khan [3, p.322] asserted that, in every LA-semigroups G with left identity

$$(a \cdot b) \cdot (c \cdot d) = (d \cdot b) \cdot (c \cdot a), \quad \forall\, a, b, c, d \in G.$$

Further M. Khan, Faisal, and V. Amjid [2], asserted that, if a LA-semigroup G with left identity the following law holds

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall\, a, b, c \in G.$$

M. Sarwar (Kamran) [5] defined LA-group as the following; a groupoid G is called a left almost group, abbreviated as LA-group, if (i) there exists $e \in G$ such that $ea = a$ for all $a \in G$, (ii) for every $a \in G$ there exists $a' \in G$ such that, $a'a = e$, (iii) $(ab)c = (cb)a$ for every $a, b, c \in G$.

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Let $\langle G, \cdot \rangle$ be an LA-group and S be a non-empty subset of G and S is itself and LA-group under the binary operation induced by G, the S is called an LA-subgroup of G, then S is called an LA-subgroup of $\langle G, \cdot \rangle$.

S.M. Yusuf in [7, p.211] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations "+" and "$\cdot$" is called a left almost ring, if $\langle R, + \rangle$ is an LA-group, $\langle R, \cdot \rangle$ is an LA-semigroup and distributive laws of "$\cdot$" over "+" holds. T. Shah and I. Rehman [7, p.211] asserted that a commutative ring $\langle R, +, \cdot \rangle$, we can always obtain an LA-ring $\langle R, \oplus, \cdot \rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $\langle R, +, \cdot \rangle$ is said to be LA-integral domain if $a \cdot b = 0$, $a, b \in R$, then $a = 0$ or $b = 0$. Let $\langle R, +, \cdot \rangle$ be an LA-ring and S be a non-empty subset of R and S is itself and LA-ring under the binary operation induced by R, the S is called an LA-subring of R, then S is called an LA-subring of $\langle R, +, \cdot \rangle$. If S is an LA-subring of an LA-ring $\langle R, +, \cdot \rangle$, then S is called a left ideal of R if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.

By [4] a near-ring is a non-empty set N together with two binary operations "+" and "$\cdot$" such that $\langle N, + \rangle$ is a group (not necessarily abelian), $\langle N, \cdot \rangle$ is a semigroup and one sided distributive (left or right) of "$\cdot$" over "+" holds.

By [8] If a subgroup B of $\langle N, + \rangle$ is said to be a bi-ideal of N if $BNB \cap (BN) * B \subseteq B$. If N has a zero symmetric near-ring a subgroup B of $\langle N, + \rangle$ is a bi-ideal if and only if $BNB \subseteq B$.

By [9] A subgroup B of $\langle N, + \rangle$ is said to be a weak bi-ideal of N if $B^3 \subseteq B$. In this paper we will define bi-ideal of near-ring has a zero symmetric.

2. Near Left Almost Rings

T. Shah, F. Rehman and M. Raees [6, pp.1103-1111] introduces the concept of a near left almost ring (nLA-ring).

Definition 2.1. *[6]. A non-empty set N with two binary operation "+" and "$\cdot$" is called a* near left almost ring *(or simply an nLA-ring) if and only if*

1. *$\langle N, + \rangle$ is an LA-group.*
2. *$\langle N, \cdot \rangle$ is an LA-semigroup.*
3. *Left distributive property of $\cdot$ over $+$ holds, that is $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$.*

Definition 2.2. *[6]. An nLA-ring $\langle N, + \rangle$ with left identity 1, such that $1 \cdot a = a$ for all $a \in N$, is called an* nLA-ring with left identity.

Definition 2.3. *[6]. A non-empty subset S of an nLA-ring N is said to be an* nLA-subring *if and only if S is itself an nLA-ring under the same binary operations as in N.*

Definition 2.4. *[6]. An nLA-subring I of an nLA-ring N is called a* left ideal *of N if $NI \subseteq I$, and I is called a* right ideal *if for all $n, m \in N$ and $i \in I$ such that $(i+n)m - nm \in I$, and is called* two sided ideal *or simply ideal if it is both left and right ideal.*

Definition 2.5. *[6]. Let $\langle N, +, \cdot \rangle$ be an nLA-ring. A non-zero element a of N is called a left zero divisor if there exists $0 \neq b \in N$ such that $a \cdot b = 0$. Similarly a is a right zero divisor if $b \cdot a = 0$. If a is both a left and a right zero divisor, then a is called a zero divisor.*

Definition 2.6. *[6]. An nLA-ring $\langle D, +, \cdot \rangle$ with left identity 1, is called an* nLA-ring integral domain *if it has no left zero divisor.*

Definition 2.7. *[6]. An nLA-ring $\langle F, +, \cdot \rangle$ with left identity 1, is called a* near almost field *(n-almost field) if and only if each non-zero element of F has inverse under "$\cdot$"*

3. Bi-ideals and Weak Bi-ideals of Near Left Almost Rings

Next we defines of a bi-ideals and weak bi-ideals in nLA-ring is defines the same as a bi-ideal and weak bi-ideal in near-ring in [8] and [9].

Definition 3.1. *If a LA-subgroup B of $\langle N, +\rangle$ is said to be a* bi-ideal *of N if $(BN)B \cap (BN) * B \subseteq B$. If N has a zero symmetric nLA-ring a LA-subgroup B of $\langle N, +\rangle$ is a bi-ideal if and only if $(BN)B \subseteq B$.*

Lemma 3.1. *Let N be a zero symmetric nLA-ring. An LA-subgroup B of N is a bi-ideal if and only if $(BN)B \subseteq B$.*

Proof. For an LA-subgroup N of $\langle N, +\rangle$ if $(BN)B \subseteq B$ then B is a bi-ideal of N.

Conversely if B is a bi-ideal, we have $(BN)B \cap (BN) * B \subseteq B$. Since N is a zero symmetric nLA-ring, $NB \subseteq N * B$, we get

$$(BN)B = (BN)B \cap (BN)B \subseteq (BN)B \cap (BN) * B \subseteq B.$$

Thus $(BN)B \subseteq B$. □

Definition 3.2. *Let N be an nLA-ring. An LA-subgroup B of $\langle N, +\rangle$ is a bi-ideal if $(BN)B \subseteq B$.*

Theorem 3.1. *If B be a bi-ideal of a nLA-ring N and S is an nLA-subring of N. Then $B \cap S$ is a bi-ideal of S.*

Proof. Since B is a bi-ideal of N we have $(BN)B \subseteq B$. Assume that $C := B \cap S$. Then $(CS)C \subseteq (SS)S \subseteq S$, since S is a nLA-subring of N and $C \subseteq S$.

On the other hand $(CS)C \subseteq (BS)B \subseteq (BN)B \subseteq B$. Hence $(CS)C \subseteq B \cap S = C$. Therefore C is a bi-ideal of S. □

Theorem 3.2. *Let N be an nLA-ring and A, B be bi-ideals of an nLA-ring N. Then $A \cap B$ is a bi-ideal of N.*

Proof. Since A, B is bi-ideals of an nLA-ring N, we have $A \cap B$ is an LA-subgroup $\langle N, +\rangle$. Thus $[(A \cap B)N](A \cap B) \subseteq (AN)(A \cap B) = [(A \cap B)N]A \subseteq (AN)A \subseteq A$ and $[(A \cap B)N](A \cap B) \subseteq (BN)(A \cap B) = [(A \cap B)N]B \subseteq (BN)B \subseteq B$. It following that $A \cap B$ is a bi-ideal of N. □

Theorem 3.3. *The set of all bi-ideal of nLA-ring.*

Proof. Let $\{B_i\}_{i \in I}$ be a set of bi-ideal in N and $B := \cap_{i \in I} B_i$. Then $(BN)B \subseteq (\bigcap_{i \in I} B_i N) \bigcap_{i \in I} B_i \subseteq B_i$ for every $i \in I$. Thus B is a bi-ideal of N. □

Definition 3.3. *Let N be an nLA-ring. An element d of N is called* distributive *if $(n + n')d = nd + n'd$ for all $n, n' \in N$.*

Theorem 3.4. *Let N be an nLA-ring. If B is a bi-ideal of N then Bn and $n'B$ are bi-ideal of N where $n, n' \in N$ and n' is a distributive element in N.*

Proof. Since B is a bi-ideal we have Bn and $n'B$ are an LA-subgroup $\langle N, +\rangle$. Thus

$$((Bn)N)(Bn) \subseteq (BN)(Bn) = (BN)Bn \subseteq Bn.$$

Hence Bn is a bi-ideal of N.

Again

$$((n'B)N)(n'B) \subseteq ((n'B)N)B = (n'BN)B \subseteq n'B.$$

Thus $n'B$ are bi-ideal of N. □

Corollary 3.1. *If B is a bi-ideal of nLA-ring. For $b, c \in B$, if b is a distributive element in N, then bBc is a bi-ideal of N.*

Proof. Let B be a bi-ideal of nLA-ring and b is a distributive element in N. Then $b(n + n') = bn + dn'$ for all $n, n' \in N$. Since B is a bi-ideal we have bBc is an LA-subgroup $\langle N, +\rangle$ then $((bBc)N)(bBc) \subseteq (BN)B \subseteq B$. □

Definition 3.4. *An nLA-ring N is said to be B-*simple *if it has no proper bi-ideals.*

Theorem 3.5. *Let N be an nLA-ring with more than one element. If N is a near almost field. Then N is a B-simple.*

Proof. Let N be a near almost field then $\{0\}$ and N are the only bi-ideals of N. For if $0 \neq B$ is a bi-ideal of N, then for $0 \neq b \in B$ we get $N = Nb$ and $N = bN$.

Now $N = N^2 = (bN)(Nb) \subseteq bNb \subseteq B$, since B is a bi-ideal of N. Then $N = B$. Thus N is a B-simple. □

The following we defined weak bi-ideal and study properties it.

Definition 3.5. *An LA-subgroup B of $\langle N, +\rangle$ is said to be a* weak bi-ideal *of N if $B^3 \subseteq B$.*

Theorem 3.6. *Every bi-ideal of an nLA-ring is a weak bi-ideal.*

Proof. Since $B^3 = (BB)B \subseteq (BN)B \subseteq B$ we have every bi-ideal is a weak bi-ideal. $\square$

Theorem 3.7. *If B is a weak bi-ideal of a nLA-ring N and S is a nLA-subring of N. Then $B \cap S$ is a weak bi-ideal of N.*

Proof. Assume that $C := B \cap S$. Then

$$\begin{aligned} C^3 &= ((B \cap S)(B \cap S))(B \cap S) \\ &= ((B \cap S)(B \cap S))B \cap ((B \cap S)(B \cap S))S \\ &\subseteq (BB)B \cap SSS \\ &= B^3 \cap SSS \\ &\subseteq B^3 \cap SS \\ &\subseteq B^3 \cap S \\ &\subseteq B \cap S \\ &= C. \end{aligned}$$

Thus $C^3 \subseteq C$. Hence C is a weak bi-ideal of N. $\square$

Theorem 3.8. *Let N be an nLA-ring. If B is a weak bi-ideal of N then Bn and $n'B$ are bi-ideal of N where $n, n' \in N$ and n' is a distributive element in N*

Proof. Since B is a weak bi-ideal we have Bn and $n'B$ are an LA-subgroup $\langle N, + \rangle$. Thus

$$(Bn)^3 = (BnBn)Bn \subseteq (BB)Bn \subseteq B^3 n = Bn.$$

Hence Bn is a weak bi-ideal of N.

Again

$$(n'B)^3 = (n'Bn'B)n'B \subseteq (n'BB)B = n'B^3 \subseteq n'B.$$

Thus $n'B$ is a weak bi-ideal of N. $\square$

Corollary 3.2. *If B is a weak bi-ideal of nLA-ring. For $b, c \in B$, if b is a distributive element in N, then bBc is a weak bi-ideal of N.*

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References

[1] M.A. Kazim, Md. Naseeruddin, On almost semigroup, Portugaliae Math. 36 (1977), 41-47. http://eudml.org/doc/115301.

[2] M. Khan, Faisal, V. Amjid, On some classes of Abel-Grassmann's groupoids, (2010). http://arxiv.org/abs/1010.5965.

[3] Q. Mushtaq, M. Khan, M-system in LA-semigroups, Southeast Asian Bull. Math. 33 (2009), 321-327.

[4] G. Pilz, Near rings, North-Holland, Amsterdam, 1977.

[5] M. Sarwar (Kamran), Conditions for LA-semigroup to resemble associative structures, PhD thesis, Quaid-i-Azam University, 1993.

[6] T. Shah, F. ur Rehman, M. Raees, On near left almost rings, Int. Math. Forum, 6 (2011), 1103-1111.

[7] T. Shah, I. Rehman, On LA-rings of finitely nonzero function, Int. J. Contemp. Math. Sci. 5 (2010), 209-222.

[8] T.T. Chelvam, N. Ganesan, On bi-ideal of near-ring, Indian J. Pure Appl. Math. 18 (1987), 1002-1005.

[9] Y.U. Cho, T.T. Chelvam, S. Jayalakshmi, Weak bi-ideal of near-ring, J. Korea Soc. Math. 14 (2007), 153-159.