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Sufficient Conditions for Convergence of Sequences of Henstock-Kurzweil Integrable Functions

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Abstract. The main aim of this paper is to present our approach of obtaining sufficient conditions for convergence of sequences of Henstock-Kurzweil integrable functions. Our approach involves the use of the concept of multiplier functions, where we define a class Φ of multipliers for the Henstock-Kurzweil integral. We consider a sequence (f_n) of Henstock-Kurzweil integrable functions on a non-degenerate interval $[a, b]$ and we assume that (f_n) converges point wise to a function f. Then we show that f is Henstock-Kurzweil integrable and its integral is equal to the limit of the sequence $(\int_a^b f_n)$ if there exists $\phi \in \Phi$ such that the defined functionals of the type $F(\phi, f_n)$ satisfy the imposed conditions. Beside the fact that the results regarding the convergence under the integral sign are always of great importance, the method introduced here can be imitated and used to obtain other results on the related areas.

1. Introduction

The Henstock-Kurzweil integral was originally introduced in [5] and [6]. It is a generalization of the Riemann integral. It is a very powerful technique of integration in a way that the space of all Henstock-Kurzweil integrable functions strictly contains the spaces of all Lebesgue and Riemann integrable functions. However, the space of all Henstock- Kurzweil integrable functions, unlike Lebesgue space, lacks the property of completeness. For further reading about the Henstock-Kurzweil integral reader may consult [4], [9], and [11]. In this paper, we investigate the sufficient conditions for the convergence of sequences of Henstock- Kurzweil integrable functions. Let (f_n) be a sequence of Henstock-Kurzweil integrable functions that converges point-wise to f on $[a, b]$. The properties and integrability of the function f was studied by many authors. One of the most well-known results for f was obtained by

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Charles Swartz in [8]. He showed that if the sequence (f_n) is uniformly Henstock- Kurzweil integrable then f is as well Henstock- Kurzweil integrable and $\int_a^b f = lim_{n\longrightarrow\infty} \int_a^b f_n$. In the following section, we obtain a similar result under a different set of conditions. For the convenience of the reader, we state some main definitions and results those will be needed later.

Definition 1.1. *A sequence of functions (F_n) is said to be equi-Lipschitz on an interval I if there exists $C > 0$ such that $|F_n(x) - F_n(y)| \leq C|x - y|$ for all $x, y \in I$ and for all n.*

Definition 1.2. *A sequence of functions (F_n) where $F_n(x) = \int_a^x f_n$ is said to be equi-absolutely continuous on an interval I if for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f_n \right| < \epsilon \tag{1.1}$$

for all n whenever $\{[x_{k-1}, x_k] : k = 1, ..., n\}$ is a collection of disjoint subintervals of I satisfying

$$\sum_{k=1}^{n} |x_k - x_{k-1}| < \delta. \tag{1.2}$$

Definition 1.3. *A sequence of functions (F_n) is said to be uniformly Cauchy on an interval I if for all $\epsilon > 0$ there exists N_ϵ such that $|F_n(x) - F_m(x)| < \epsilon$ for all $x \in I$ whenever $n, m \geq N_\epsilon$.*

A connection between a sequence of Lebesgue integrable functions and the sequence of their primitives (indefinite integrals) is given via the following theorem (see [4] and [7]).

Theorem 1.1. *Let f_n be a sequence of Lebesgue integrable functions that converges point-wise to a function f on $[a, b]$. If the sequence $F_n(x) = \int_a^x f_n$ is equi-absolutely continuous on $[a, b]$, then f is Lebesgue integrable and*

$$\lim_{n\longrightarrow\infty} \int_a^b f_n = \int_a^b f. \tag{1.3}$$

Another result that will be needed in the coming sections is the so-called Hake's theorem (see [2]), which can be stated as follows.

Theorem 1.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function, then f is Henstock- Kurzweil integrable on $[a, b]$ if and only if f is Henstock- Kurzweil integrable on $[a, c]$ for all $c \in [a, b)$ and*

$$\lim_{c\to b^-} \int_a^c f\,dt \quad \text{exists.} \tag{1.4}$$

In this case

$$\int_a^b f = \lim_{c\to b^-} \int_a^c f. \tag{1.5}$$

2. The Main Results

Definition 2.1. $\Phi([a,b]) =: \{\phi \in C^1([a,b])$ *such that* ϕ *is monotonic and* $\phi(b) = 0 \neq \phi'(b)$ $\}$.

In the above definition $C^1([a,b])$ denotes the class of all continuously differentiable functions on $[a,b]$. We note that $\Phi([a,b]) \subseteq BV([a,b])$ since Φ consists of monotonic functions. Therefore, the function ϕf is Henstock-Kurzwel integrable for all $\phi \in \Phi([a,b])$ and $f \in HK([a,b])$, (see [11]).

Lemma 2.1. *Let f_n be a sequence of HK-integrable functions that converges point-wise to a function f on $[a,b]$. If there exists $\phi \in \Phi$ such that $\int_a^x \phi f_n$ is equi-Lipschitz continuous on $[a,b]$, then*

i. $\phi f \in L^1([a,b])$, *and* $\int_a^b \phi f_n \longrightarrow \int_a^b \phi f$.

ii. $f \in L^1([a,r])$ *and* $\int_a^r f_n \longrightarrow \int_a^r f$ *for all* $r \in [a,b)$.

Proof. It is clear that ϕf_n converges point-wise to ϕf and the sequence $\int_a^r \phi f_n$ is equi-absolutely continuous on $[a,b]$ since it is equi-Lipschitz on $[a,b]$ where

$$\sum_{k=1}^{m} |\int_{x_{k-1}}^{x_k} \phi f_n| = \sum_{k=1}^{m} |\int_a^{x_k} \phi f_n - \int_a^{x_{k-1}} \phi f_n| \leq C \sum_{k=1}^{m} |x_k - x_{k-1}|.$$

Therefore, (i) follows immediately from Theorem 1.1. Now since ϕ is continuous on $[a,b]$ it attains its extreme values on any closed subinterval of $[a,b]$. Thus, $M_r = Min\{|\phi(t)| : t \in [a,r]\}$ exists for all $r \in [a,b]$. Moreover, since ϕ is a monotonic function and $\phi(b) = 0$, then $|\phi(x)|$ is a decreasing function. Also, since $\phi'(b) \neq 0$, then ϕ is a non-constant function in $[r,b]$ for all $r \in [a,b)$. Thus, there is $s \in [r,b]$ such that $\phi(s) \neq \phi(b) = 0$ and hence, $M_r > 0$ for all $r \in [a,b)$ since $|\phi(t)| \geq |\phi(r)| \geq |\phi(s)| > 0$ for all $t \in [a,r]$. Using this result, we get

$$\int_a^r |f| = \int_a^r |\frac{\phi f}{\phi}| \leq \frac{1}{M_r} \int_a^r |\phi f| \qquad \textit{for all } r \in [a,b), \textit{ and} \tag{2.1}$$

$$|\int_a^r f_n - f| = |\int_a^r \frac{\phi(f_n - f)}{\phi}| \leq \frac{1}{M_r} |\int_a^r \phi(f_n - f)| \qquad \textit{for all } r \in [a,b). \tag{2.2}$$

Therefore, we can obtain (ii) directly form (i), (2.1) and (2.2). □

Remark 2.1. *If ϕf is Lebesgue integrable on $[a,b]$ for some $f \in HK$ and $\phi \in \Phi$, then f can only attain a point of Lebesgue singularity at b (f oscillates very rapidly as approaching b). In fact, we would concentrate on the local case when f is Lebesgue integrable on any subinterval $[c,d] \subset [a,b]$ with $d \neq b$ (see [4]). For the special case $\phi = b - x$, the reader my refer to the results in [1].*

In the coming sections we set $F(r) = \int_a^r f$ and $F_n(r) = \int_a^r f_n$.

Lemma 2.2. *Let f_n be a sequence of HK-integrable functions that converges point-wise to a function f on $[a,b]$. If there exists $\phi \in \Phi$ such that the sequence $\frac{1}{\phi(r)} \int_r^b F_n d\phi(t)$ is uniformly Cauchy on $[a,b)$, then there are $C > 0$ and $s \in [a,b)$ such that*

$$|\frac{1}{\phi(r)} \int_r^b F_n d\phi(t)| \leq C \qquad \textit{for all } r \in (s,b) \textit{ and for all } n. \tag{2.3}$$

Proof. Since $\frac{1}{\phi(r)}\int_r^b F_n d\phi(t)$ is uniformly Cauchy on $[a,b)$, then there is N_1 such that

$$\left|\frac{1}{\phi(r)}\int_r^b (F_n-F_m)d\phi(t)\right|<1 \qquad \textit{for all } n,m\geq N_1. \tag{2.4}$$

Therefore,

$$\left|\int_r^b (F_n-F_m)d\phi(t)\right|<|\phi(r)| \qquad \textit{for all } n,m\geq N_1. \tag{2.5}$$

Also, since ϕ' and F_n are continuous we have that the function $\int_a^r F_n d\phi(t)$ is Lipschitz . Thus, for all n there is $K_n>0$ such that

$$\left|\int_r^b F_n d\phi(t)\right|\leq K_n(b-r). \tag{2.6}$$

Choosing $K=Max\{K_1,K_2,...,K_{N_1}\}$, we get

$$\left|\int_r^b F_n d\phi(t)\right|\leq K(b-r) \qquad \textit{for all } n\in\{1,2,...,N_1\}. \tag{2.7}$$

For the case $n>N_1$, we have

$$\begin{aligned}\left|\int_r^b F_n d\phi(t)\right| &\leq \left|\int_r^b (F_n-F_{N_1}d\phi(t)\right|+\left|\int_r^b F_{N_1}d\phi(t)\right|\\ &\leq |\phi(r)|+K(b-r) \qquad \textit{for all } n\geq N_1\end{aligned}$$

Combining the above results, we get

$$\left|\int_r^b F_n d\phi(t)\right|\leq |\phi(r)|+K(b-r) \qquad \textit{for all } n. \tag{2.8}$$

Now, using the fact that $\phi(b)=0\neq\phi'(b)$, we get

$$\lim_{r\to b^-}\frac{b-r}{-\phi(r)}=\frac{1}{\phi'(b)}. \tag{2.9}$$

Therefore, there exists $s\in(a,b)$ such that

$$\frac{(b-r)}{|\phi(r)|}\leq\frac{2}{|\phi'(b)|} \qquad \textit{for all } r\in(s,b). \tag{2.10}$$

Letting $M_s=Min\{|\phi(t)|:t\in[a,s]\}$ be as defined in Lemma 2.2, we obtain

$$\frac{(b-r)}{|\phi(r)|}\leq\frac{2}{|\phi'(b)|}+\frac{|b-a|}{M_s} \qquad \textit{for all } r\in[a,b), \tag{2.11}$$

and hence,

$$(b-r)\leq\left(\frac{2}{|\phi'(b)|}+\frac{|b-a|}{M_s}\right)|\phi(r)| \qquad \textit{for all } r\in[a,b) \tag{2.12}$$

Choosing $C=1+K\left(\frac{2}{|\phi'(b)|}+\frac{|b-a|}{M_s}\right)$ and and using it with (2.8) and (2.12), we get the desired result.

□

Theorem 2.1. *Let f_n be a sequence of HK-integrable functions that converges point-wise to a function f on $[a,b]$. If there exists $\phi\in\Phi$ such that:*

- *the sequence* $\frac{1}{\phi(r)}\int_r^b F_n d\phi(t)$ *is uniformly Cauchy on* $[a, b)$*, and*
- *the sequence* $\int_r^b \phi f_n dt$ *is equi-Lipschitz on* $[a, b]$*,*

then $f \in HK([a, b])$ *and*

$$\lim_{n\to\infty}\int_a^b f_n dt = \int_a^b f dt. \tag{2.13}$$

Proof. Similar to the argument above and since $\int_r^b \phi f_n dt$ is equi-Lipschitz, there exists $M_0 > 0$ such that

$$\left|\frac{1}{\phi(r)}\int_r^b \phi f_n dt\right| \le M_0\frac{|b-r|}{|\phi(r)|} \qquad \text{for all } r \in (a, b) \text{ and all } n. \tag{2.14}$$

Therefore, choosing s as in Lemma 2.2 and $C_2 = \frac{2M_0}{\phi'(b)}$, we get

$$\left|\frac{1}{\phi(r)}\int_r^b \phi f_n dt\right| \le C_2 \qquad \text{for all } r \in (s, b) \text{ and all } n. \tag{2.15}$$

Now integrating by parts and using the fact $F_n(a) = 0$, we get

$$\phi(r)F_n(r) = \int_a^r \phi f_n dt + \int_a^r F_n d\phi(t). \tag{2.16}$$

Also, since $\phi(b) = 0$, we have

$$\int_a^b \phi f_n dt + \int_a^b F_n d\phi(t) = 0, \tag{2.17}$$

which implies

$$\phi(r)F_n(r) = -\int_r^b \phi f_n dt - \int_r^b F_n d\phi(t) \tag{2.18}$$

and hence,

$$F_n(r) = -\frac{1}{\phi(r)}\int_r^b \phi f_n dt - \frac{1}{\phi(r)}\int_r^b F_n d\phi(t). \tag{2.19}$$

Applying Lemma 2.2 and (2.15), we get

$$|F_n(r)| \le C \tag{2.20}$$

for all n and all $r \in (s, b)$. Thus, passing the limit as $n \longrightarrow \infty$, we get

$$\left|\lim_{n\to\infty}\int_a^r f_n dt\right| \le C. \tag{2.21}$$

Applying lemma 2.1, we get

$$\left|\int_a^r f dt\right| \le C \qquad \text{for all } r \in (s, b). \tag{2.22}$$

Therefore,

$$\lim_{r\to b^-}|\phi(r)F(r)| \le \lim_{r\to b^-} C|\phi(r)| = 0. \tag{2.23}$$

using this result with the equation

$$\phi(r)F(r) = \int_a^r \phi f dt + \int_a^r F d\phi(t), \tag{2.24}$$

we get

$$0=\int_a^b \phi f dt+\int_a^b F d\phi(t) \tag{2.25}$$

Thus, in asimilar way to that for (2.19) we get

$$F(r)=-\frac{1}{\phi(r)}\int_r^b \phi f dt-\frac{1}{\phi(r)}\int_r^b F d\phi(t). \tag{2.26}$$

Now, we use (2.19), (2.26) and Lemma 2.2 to get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\phi(r)}\int_r^b F_n d\phi(t) &= \lim_{n\to\infty}-\frac{1}{\phi(r)}\int_r^b \phi f_n dt-F_n(r)\\ &= -\frac{1}{\phi(r)}\int_r^b \phi f dt-F(r)\\ &= \frac{1}{\phi(r)}\int_r^b F d\phi(t)\end{aligned}$$

Combining the above point-wise convergence with the given that $\left(\frac{1}{\phi(r)}\int_r^b F_n d\phi(t)\right)$ is uniformly Cauchy, we obtain that

$$\lim_{n\to\infty}\frac{1}{\phi(r)}\int_r^b F_n d\phi(t)=\frac{1}{\phi(r)}\int_r^b F d\phi(t) \qquad \textit{uniformly} \tag{2.27}$$

Now, since $\phi' F_n$ is continuous then by the Fundamental Theorem of Calculus $\int_a^r F_n d\phi(t)=\int_a^r \phi' F_n dt$ is differentiable and

$$\frac{d}{dr}\left(\int_a^r F_n d\phi(t)\right)=\phi' F_n$$

By L'Hopital rule, we have

$$\lim_{r\to b}\frac{1}{\phi(r)}\int_r^b F_n d\phi(t)=-\lim_{r\to b}\frac{\phi'(r)\cdot F_n(r)}{\phi'(r)}=-\lim_{r\to b}F_n(r) \tag{2.28}$$

Similarly,

$$\lim_{r\to b}\frac{1}{\phi(r)}\int_r^b F d\phi(t)=-\lim_{r\to b}F(r) \tag{2.29}$$

On the other hand, passing the limit in (2.20) as $r\longrightarrow b$ and using Theorem 1.2, we get

$$|F_n(b)|<C. \tag{2.30}$$

Therefore, by the completeness of $\mathbb{R}$

$$\lim_{n_k\to\infty}\int_a^b f_{n_k} dt=A \qquad \textit{for some } A\in\mathbb{R} \tag{2.31}$$

Also, in view of (2.27) we have (see [10] Theorem 7.11)

$$\lim_{r\to b}\lim_{n_k\to\infty}\frac{1}{\phi(r)}\int_r^b F_{n_k} d\phi(t)=\lim_{n_k\to\infty}\lim_{r\to b}\frac{1}{\phi(r)}\int_r^b F_{n_k} d\phi(t). \tag{2.32}$$

Now, using (2.29), (2.27), (2.32), (2.28), Theorem 1.2 and (2.31) respectively, we get

$$\begin{aligned}\lim_{r\to b}\int_a^r f\,dt &= -\lim_{r\to b}\frac{1}{\phi(r)}\int_r^b F\,d\phi(t)\\ &= -\lim_{r\to b}\lim_{n_k\to\infty}\frac{1}{\phi(r)}\int_r^b F_{n_k}\,d\phi(t)\\ &= -\lim_{n_k\to\infty}\lim_{r\to b}\frac{1}{\phi(r)}\int_r^b F_{n_k}\,d\phi(t)\\ &= -\lim_{n_k\to\infty}\lim_{r\to b}\int_a^r f_{n_k} = \lim_{n_k\to\infty}\int_a^b f_{n_k} = A.\end{aligned}$$

Thus, by Theorem 1.2 f is Henstock-Kurzwel integrable on $[a, b]$ and

$$\int_a^b f\,dt = \lim_{r\to b^-}\int_a^r f\,dt = A \tag{2.33}$$

Reversing the steps above and using the the result that f is Henstock-Kurzwel integrable, we obtain the convergence for the whole sequences as follows

$$\begin{aligned}\lim_{n\to\infty}\int_a^b f_n dt &= -\lim_{n\to\infty}\lim_{r\to b}\frac{1}{\phi(r)}\int_r^b F_n d\phi(t)\\ &= -\lim_{r\to b}\lim_{n\to\infty}\frac{1}{\phi(r)}\int_r^b F_n d\phi(t)\\ &= -\lim_{r\to b}\frac{1}{\phi(r)}\int_r^b F\,d\phi(t) = \int_a^b f\,dt\end{aligned}$$

This completes the proof. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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