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Composition Operators on $\mathcal{N}_K(p,q)$-Type Spaces on the Unit Ball

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Abstract. We describe the boundedness and compactness of the composition operators C_φ acting in $\mathcal{N}_K(p,q)$ on the open unit ball $\mathbb{B}$.

1. Introduction

For the unit ball $\mathbb{B}$ of $\mathbb{C}^n$, $\mathcal{HOI}(\mathbb{B})$ denotes the class of all holomorphic functions on $\mathbb{B}$ while $H^\infty = H^\infty(\mathbb{B})$ denotes the class of all functions that are holomorphic $u \in \mathcal{HOI}(\mathbb{B})$ equipped with the norm $\|u\|_\infty = \sup_{\zeta\in\mathbb{B}} |u(\zeta)|$. For any $d > 0$, the weighted Banach space $H_d^\infty = H_d^\infty(\mathbb{B})$ consists of all functions $u \in \mathcal{HOI}(\mathbb{B})$ such that

$$\|u\|_d^\infty := \sup_{\zeta\in\mathbb{B}} (1-|\zeta|)^d |u(\zeta)| < \infty.$$

The space $H_{d,0}^\infty = H_{d,0}^\infty(\mathbb{B})$ indicate the closed subspace of H_d^∞ such that

$$\lim_{|\zeta|\to 1} |u(\zeta)|(1-|\zeta|)^d = 0.$$

For further details about the properties of H_d^∞ spaces see [10]).

For $\zeta \in \mathbb{B}$, we let dV be the Lebesgue measure on $\mathbb{B}$ with

$$V(\mathbb{B}) = \int_B dV(\zeta) = 1.$$

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In addition, we let $d\omega$ be the surface measure on $\mathbb{S}$, normalized so that $\omega(\mathbb{S}) \equiv 1$. If u is a nonnegative Lebesgue measurable function on $\mathbb{B}$, then the measures V and ω are related by

$$\int_{\mathbb{B}} u(\zeta) dV(\zeta) = 2n \int_0^1 t^{2n-1} dt \int_{\mathbb{S}} u(t\zeta) d\omega(\zeta).$$

Moreover, the formulas for integration on $\mathbb{S}$ (see, [11]) as:

$$\int_{\mathbb{S}} u d\omega = \int_{\mathbb{S}} d\omega(\zeta) \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}\zeta) d\theta, \quad \text{for all } 0 \leq \theta \leq 2\pi.$$

For any $\psi \in Aut(\mathbb{B}), u \in L^1(\mathbb{B})$, the Möbius invariant on $\mathbb{B}$ (see e.g., [5]) such that

$$\int_{\mathbb{B}} u(\zeta) d\lambda(\zeta) = \int_{\mathbb{B}} u \circ \psi(\zeta) \frac{dV}{(1-|\zeta|^2)^{n+1}}.$$

The inner product of $\zeta = (\zeta_1, \dots, \zeta_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ in $\mathbb{C}^n$, is given by

$$\langle \zeta, \eta \rangle = \sum_{i=1}^{n} \zeta_i \overline{\eta}_i.$$

For any $\zeta \in \mathbb{B}$, we define the complex gradient and the radial derivative of the function $u \in \mathcal{HOl}(\mathbb{B})$ respectively as follows:

$$\begin{aligned} \nabla u(\zeta) &= \left(\frac{\partial u}{\partial \zeta_1}(\zeta), \cdots, \frac{\partial u}{\partial \zeta_n}(\zeta) \right), \\ Ru(\zeta) &= \langle \nabla u(\zeta), \overline{\zeta} \rangle = \sum_{i=1}^{n} \zeta_i \frac{\partial u}{\partial \zeta_1}(\zeta). \end{aligned}$$

We know the Bloch space $\mathcal{B}^d = \mathcal{B}^d(\mathbb{B})$ is the Banach space of functions $u \in \mathcal{HOl}(\mathbb{B})$ such that $Ru \in H_d^\infty$ which has the norm

$$\|u\|_{\mathcal{B}^d} := |f(0)| + \|Ru\|_d^\infty.$$

The involution automorphisms Ψ_b (the Möbius transformation of $\mathbb{B}$) is define for $\zeta \in \mathbb{B}$ and $b \in \mathbb{B} - \{0\}$ as

$$\Psi_b(\zeta) = \frac{b - \frac{\langle \zeta, b \rangle b}{|b|^2} - \sqrt{1-|b|^2} \left(\zeta - \frac{\langle \zeta, b \rangle b}{|b|^2} \right)}{1 - \langle \zeta, b \rangle},$$

where $\Psi_0(\zeta) = -\zeta$, $\Psi_b(0) = b$, $\Psi_b(b) = 0$ and $\Psi_b = \Psi_b^{-1}$. It is well known that for any $\zeta \in \mathbb{B}$

$$1 - |\Psi_b(\zeta)|^2 = \frac{(1-|b|^2)(1-|\zeta|^2)}{|1 - \langle b, \zeta \rangle|^2}.$$

The Bergman metric and the Bergman metric ball on $\mathbb{B}$, for $\zeta, \eta \in \mathbb{B}$ and $M > 0$ as follows:

$$\begin{aligned} \beta(\zeta, \eta) &= \frac{1}{2} \log \frac{1 + |\Psi_\zeta(\eta)|}{1 - |\Psi_\zeta(\eta)|}, \\ D(\zeta, M) &= \{\eta \in \mathbb{B} : \beta(\zeta, \eta) < M\}. \end{aligned}$$

Let $\mathcal{RC}^+$ denote the set of all right-continuous nondecreasing functions $K \neq 0$ and $K : [0, \infty) \to [0, \infty)$. For $K \in \mathcal{RC}^+$ and $p, q > 0$, the weighted Banach type spaces $\mathcal{N}_K(p, q) = \mathcal{N}_K(p, q)(\mathbb{B})$ consists of functions $u \in \mathcal{HOI}(\mathbb{B})$ such that

$$\mathcal{N}_K(p, q) := \{u \in H(\mathbb{B}) : \|u\|^p_{\mathcal{N}_K(p,q)} < \infty\},$$

where

$$\|u\|^p_{\mathcal{N}_K(p,q)} = \sup_{b\in\mathbb{B}} \int_{\mathbb{B}} |u(\zeta)|^p (1 - |\zeta|^2)^q K\big((1 - |\Psi_b(\zeta)|^2)^n\big) dV(\zeta).$$

This space was introduced first by Bakhit and Aljuaid in [1] who study several fundamental properties of $\mathcal{N}_K(p, q)$-type spaces and its closed subspaces $\mathcal{N}_{K,0}(p, q)$, which are Banach spaces of functions that are analytic and their norms determined by a weighted function $K \in \mathcal{RC}^+$, together with a Möbius transformation. Also in [1] the authors show that the norm of $\mathcal{N}_K(p, q)$-type space is equivalent to the norm

$$\|u\|^p_{\mathcal{N}_K(p,q)} = \sup_{b\in\mathbb{B}} \int_{\mathbb{B}} |u(\zeta)|^p (1 - |\zeta|^2)^q K\big(G(b, \zeta)\big) dV(\zeta) < \infty,$$

where $G(b, \zeta) = \log \frac{1}{|\Psi_b(\zeta)|}$. We set the integral $J_{K,q}(t)$ with $q > n$ as:

$$J_{K,q}(t) = \int_0^1 \frac{t^{2n-1}}{(1 - t^2)^{n+1-q}} K\big((1 - t^2)^n\big) dt. \tag{1.1}$$

Throughout the paper, we suppose that $J_{K,q}(t) < \infty$, then $\mathcal{N}_K(p, q)$ contain all the polynomials, otherwise $\mathcal{N}_K(p, q)$ consists only of zero functions.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two function spaces on $\mathbb{B}$ and consider φ be a holomorphic self-map of $\mathbb{B}$. We define the composition operator $C_\varphi : \mathcal{X} \to \mathcal{Y}$ by

$$C_\varphi(u)(\zeta) = u \circ \varphi, \ \ \forall\, u \in \mathcal{X}.$$

Recall that, for any two normed linear spaces X and Y, the linear operator $T : X \longrightarrow Y$ is said to be bounded if there exists $C > 0$ such that $\|Tu\|_Y \leq C\|u\|_X, \forall u \in X$. Furthermore, a linear operator $T : X \longrightarrow Y$ is said to be compact if it maps every bounded set in X to a relatively compact set in Y (i.e., a set whose closure is compact) (see e.g., [12]).

Studying the composition operators acting in different spaces is a quite classical topic since they arise in different problems; see the excellent monographs [2], [3] and [4]. Some of the earlier study on this topic is reflected in [9] descriptions of bounded and compact composition operators on $F(p, q, s)$ spaces were provided [8].

This paper is organized as follows: in Section 2 we shortly give the preliminaries and background information. In Section 3 we establish proving our main results respectively.

We use the notation $a \lesssim b$ in what follows to mean that there is a constant $C > 0$ with $a \leq Cb$. and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$.

2. Preliminaries

For $0 < t < \infty$, we use the auxiliary function $\phi_K(t) = \sup_{s\in(0,1]} \frac{K(st)}{K(s)}$ (see e.g., [6], [7]). The following constraints on $\phi_K(t)$ play a significant role in the study of any class of $\mathcal{N}_K(p,q)$ spaces:

$$J_K(t) = \int_0^1 \phi_K(t)\frac{dt}{t} < \infty, \tag{2.1}$$

and

$$\int_1^\infty \phi_K(t)\frac{dt}{t^2} < \infty, \tag{2.2}$$

and more generally,

$$\int_1^\infty \phi_K(t)\frac{dt}{t^{1+\varrho}} < \infty, \ \varrho > 0. \tag{2.3}$$

In the case that K satisfies condition (2.1), then $K(2t) \lesssim K(t)\ \forall\ 0 \le 2t \le 1$. If we started with the property that $K(t) = K(1)$ for $t \ge 1$, then $K(2t) \approx K(t)$ for $t > 0$ (see, [6]).

The following results will have an important role in the subsequent. The following lemma was proven in [1].

Lemma 2.1. *Let $K \in \mathcal{RC}^+$, $p \ge 1$ and $q > 0$ then*

- $\mathcal{N}_K(p,q) \subseteq H^\infty_{q/p}(\mathbb{B})$.
- $\mathcal{N}_K(p,q) = H^\infty_{q/p}(\mathbb{B})$ *if*

$$I_K(t) = \int_0^1 \frac{t^{2n-1}}{(1-t^2)^{n+1}} K\big((1-t^2)^n\big)dt < \infty. \tag{2.4}$$

We can find the subsequent result in [11].

Lemma 2.2. *Let $\delta \in (0,1]$ then there is a sequence $\{n_i\} \in \mathbb{B}$ such that*

- $\lim_{i\to\infty} |n_i| = 1$.
- $\mathbb{B} = \bigcup_{i=1}^\infty D(m_i,\delta)$.
- *Let $N > 0$ be an integer, then $\zeta \in \bigcap_{k=1}^{N+1} D(m_{i_k}, 4\delta)$ and $m_{i_k} \in D(\zeta, 4\delta)$ for each $\zeta \in \mathbb{B}, 1 \le k \le N+1$.*

Lemma 2.3. *For any $K \in \mathcal{RC}^+, \delta > 0$, let $p, q > 0$ and $\zeta, b \in \mathbb{B}$. Then there is a positive constant C, such that*

$$|u(\zeta)|^p \ \le\ \frac{(1-|z|^2)^{-q-n-1}}{K\big((1-|\Psi_b(\zeta)|^2)^n\big)} \int_{D(z,2\delta)} |u(w)|^p(1-|w|^2)^q K(1-|\Psi_b(w)|^2)dV(w),$$

for all $\zeta \in D(z,\delta)$ and $u \in \mathcal{HOI}(\mathbb{B})$.

Proof. By the result in Lemma 2.24 in [5], we obtain

$$|u(\zeta)|^p \ \le\ \frac{1}{(1-|\zeta|^2)^{n+1}} \int_{D(\zeta,\delta)} |u(w)|^p dV(w),$$

for all $\zeta \in \mathbb{B}$ and $u \in \mathcal{HOI}(\mathbb{B})$.

Now let $\zeta \in D(z,\delta)$ and $w \in D(\zeta,\delta)$, then obtain $\beta(w,z) \le \beta(w,\zeta) < 2\delta$. Thus, $D(\zeta,\delta) \subset D(z,2\delta)$. From some results in [5], we obtain

$$1-|\zeta|^2 \asymp 1-|z|^2 \asymp 1-|w|^2,$$

$$|1-\langle b,w\rangle| \asymp |1-\langle b,z\rangle|.$$

Thus,

$$|u(\zeta)|^p \le \frac{(1-|z|^2)^{-q-n-1}}{K\big((1-|\Psi_b(\zeta)|^2)^n\big)} \int_{D(z,2\delta)} |u(w)|^p(1-|w|^2)^q K\big((1-|\Psi_b(w)|^2)^n\big)\, dV(w).$$

□

Lemma 2.4. *Let ϕ be a holomorphic self-map of $\mathbb{B}$ and $b \in \mathbb{B}$. If u is a nonnegative Lebesgue measurable function on $\mathbb{B}$, then*

$$\int_{\mathbb{B}} u(\zeta)\, d\lambda_{K,q,\varphi}(\zeta) = \int_{\mathbb{B}} u(\varphi(\zeta))\,(1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta),$$

where

$$\lambda_{K,q,\varphi} = \int_{\varphi^{-1}(E)} (1-|\zeta|^2)^q\, K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta),$$

for any Borel measurable set $E \subseteq \mathbb{B}$.

Proof. Let u be a nonnegative simple Lebesgue measurable function. Assume that

$$u(\zeta) = \sum_{i=1}^{n} b_i\, \varkappa_{E_i},$$

where E_i is the measurable set on $\mathbb{B}$. Then,

$$\begin{aligned}
\int_{\mathbb{B}} u(\zeta) d\lambda_{K,q,\varphi}(\zeta) &= \sum_{i=1}^{n} b_i \lambda_{K,q,\varphi}(E_i) = \sum_{i=1}^{n} b_i \int_{E_i} d\lambda_{K,q,\varphi}(\zeta) \\
&= \sum_{i=1}^{n} b_i \int_{\phi^{-1}(E_i)} (1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big) dV(\zeta) \\
&= \int_{\mathbb{B}} \Big(\sum_{i=1}^{n} b_i \varkappa_{\phi^{-1}(E_i)\cap B}\Big)(1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big) dV(\zeta) \\
&= \int_{\mathbb{B}} u(\varphi(\zeta))(1-|\zeta|^2)^q\, K\big((1-|\Psi_b(\zeta)|^2)^n\big) dV(\zeta).
\end{aligned}$$

If u is a nonnegative Lebesgue measurable function, for $\zeta \in \mathbb{B}$ then there is a monotone increasing simple measurable function sequence $\{u_j\}$ such that

$$\lim_{j\to\infty} u_j(\zeta) = u(\zeta).$$

Thus,

$$\lim_{j\to\infty} \int_{\mathbb{B}} u_j(\zeta)\, d\lambda_{K,q,\varphi}(\zeta) = \int_{\mathbb{B}} u(\zeta)\, d\lambda_{K,q,\varphi}(\zeta).$$

Now let the function sequence $\{U_j(K,q,\phi)\} = \{u_j(\varphi(\zeta))(1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big)\}$, then $\{U_j(K,q,\phi)\}$ is a monotone increasing measurable function. Moreover,

$$\lim_{j\to\infty} U_j(K,q,\phi) = u(\varphi(\zeta))(1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big),$$

which implies that

$$\begin{aligned}\int_{\mathbb{B}} u(\zeta)\, d\lambda_{K,q,\varphi}(\zeta) &= \lim_{j\to\infty}\int_{\mathbb{B}} u_j(\zeta) d\lambda_{K,q,\varphi}(\zeta)\\ &= \lim_{j\to\infty}\int_{\mathbb{B}} u_j(\varphi(\zeta))(1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta)\\ &= \lim_{j\to\infty}\int_{\mathbb{B}} u(\varphi(\zeta))(1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta).\end{aligned}$$

This completes the proof. □

Lemma 2.5. *For $K \in \mathcal{RC}^+$ and $p > 0, q+n+1 > 0$. If (2.4) holds, then $u_w(\zeta) \in \mathcal{N}_K(p,q)$, where*

$$u_w(\zeta) = \frac{(1-|w|^2)}{(1-\langle \zeta, w\rangle)^{\frac{q+n+1}{p}+1}}.$$

Proof. Firstly, suppose that (2.4) holds, to show that $u_w(\zeta) \in \mathcal{N}_K(p,q)$, it suffices to show that there is $\varepsilon > 0$, such that

$$\sup_{b\in\mathbb{B}}\int_{\mathbb{B}} \frac{(1-|z|^2)^p(1-|\zeta|^2)^q}{|1-\langle\zeta,z\rangle|^{n+1+q+p}} K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta) \le \varepsilon, \ \ \forall\ z\in\mathbb{B}.$$

Now we let $\frac{1}{\sqrt{2}} < |\Psi_b(\zeta)| < 1$, by the fact that $(1-|\zeta|) \le |1-\langle\zeta,b\rangle|$ and Theorem 1.4.10 in [5], therefore

$$\begin{aligned}&\int_{\frac{1}{\sqrt{2}}<|\Psi_b(\zeta)|<1} \frac{(1-|z|^2)^p(1-|\zeta|^2)^q}{|1-\langle\zeta,z\rangle|^{n+1+q+p}} K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta)\\ &\le \varepsilon \int_{\mathbb{B}} (1-|\zeta|^2)^{-n-1} K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta)\\ &\le \varepsilon \int_0^1 \frac{t^{2n-1}}{(1-t^2)^{n+1}} K\big((1-t^2)^n\big)\, dt < \varepsilon. \qquad (2.5)\end{aligned}$$

At the same time,

$$\begin{aligned}&\int_{|\Psi_b(\zeta)|\le\frac{1}{\sqrt{2}}} \frac{(1-|z|^2)^p(1-|\zeta|^2)^q}{|1-\langle\zeta,z\rangle|^{n+1+q+p}} K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta)\\ &\le \int_{|w|\le\frac{1}{2}} \frac{(1-|z|^2)^p(1-|\Psi_b(w)|^2)^q(1-|b|^2)^{n+1}}{|1-\langle\Psi_b(w),z\rangle|^{n+1+q+p}|1-\langle w,b\rangle|^{2n+2}} K\big((1-|w|^2)^n\big)\, dV(w)\end{aligned}$$

$$\leq \varepsilon \int_{|w|\leq\frac{1}{2}} \frac{(1-|b|^2)^{n+1}}{(1-|\Psi_b(w)|^2)^{n+1}|1-\langle w,b\rangle|^{2n+2}} K\big((1-|w|^2)^n\big)\,dV(w)$$
$$\leq \varepsilon \int_{|w|\leq\frac{1}{2}} K\big((1-|w|^2)^n\big)\frac{dV(w)}{|1-\langle w,b\rangle|^{n+1}}$$
$$\leq \varepsilon \int_{|w|\leq\frac{1}{2}} K\big((1-|w|^2)^n\big)\frac{dV(w)}{(1-|w|^2)^{n+1}}$$
$$\leq \varepsilon \int_{\mathbb{B}} K\big((1-|w|^2)^n\big)\,dV(w) < \varepsilon. \tag{2.6}$$

Combining (2.5) and (2.6), it follows that

$$\sup_{b\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|z|^2)^p(1-|\zeta|^2)^q}{|1-\langle\zeta,z\rangle|^{n+1+q+p}} K\big((1-|\Psi_b(\zeta)|^2)^n\big)\,dV(\zeta) \leq \varepsilon,\ \ \forall\ z\in\mathbb{B}.$$

□

3. Main Results

3.1. **Boundedness.**

Theorem 3.1. *Let $K \in \mathcal{RC}^+$ and $0 < p, q < \infty$. Then the operator C_φ is bounded on $\mathcal{N}_K(p,q)$ if and only if*

$$\sup_{w,b\in\mathbb{B}} (1-|w|^2)^p\left(\int_{\mathbb{B}} \frac{(1-|\zeta|^2)^q}{|1-\langle\varphi(\zeta),w\rangle|^{q+n+1}} K(1-|\Psi_b(\zeta)|^2)dV(\zeta)\right) < \infty. \tag{3.1}$$

Proof. Let C_φ be the bounded operator in $\mathcal{N}_K(p,q)$. Consider the function

$$u_w(\zeta) = \frac{(1-|w|^2)}{(1-\langle\zeta,w\rangle)^{\frac{q+n+1}{p}+1}}.$$

Then by Lemma 2.5, we obtain

$$\int_{\mathbb{B}} |u_w(\zeta)|^p(1-|\zeta|^2)^q K(1-|\Psi_b(\zeta)|^2)\,dV(\zeta)$$
$$\leq \int_{\mathbb{B}} \frac{(1-|w|^2)^p(1-|\zeta|^2)^q}{|1-\langle\zeta,w\rangle|^{p+q+n+1}} K(1-|\Psi_b(\zeta)|^2)dV(\zeta) \leq \varepsilon,$$

which exactly

$$\|C_\varphi(u_w)\|_{\mathcal{N}_K(p,q)} \leq \|C_\varphi\|\|u_w\|_{\mathcal{N}_K(p,q)} \leq \varepsilon^{\frac{1}{p}}\|C_\varphi\|.$$

That is

$$\sup_{w,b\in\mathbb{B}} (1-|w|^2)^p \int_{\mathbb{B}} \frac{(1-|\zeta|^2)^q}{|1-\langle\varphi(\zeta),w\rangle|^{q+n+1}} K(1-|\Psi_b(\zeta)|^2)dV(\zeta) \leq \varepsilon\|C_\varphi\|^p.$$

Conversely, suppose that (3.1) holds, then by Lemma (2.3), there exists a constant ε such that

$$\frac{(1-|w|^2)^p}{K(1-|\Psi_b(w)|^2)} \int_{\mathbb{B}} \frac{d\lambda_{K,q,\varphi}(\zeta)}{|1-\langle\zeta,w\rangle|^{q+n+1}} \leq \varepsilon,\ \ \forall\ w,b\in\mathbb{B},$$

where

$$\lambda_{K,q,\varphi}=\int_{\varphi^{-1}(E)}(1-|\zeta|^2)^q\,K\big((1-|\Psi_b(\zeta)|^2)^n\big)dV(\zeta),\ \forall\ E\in\mathbb{B}.$$

Fixed $\delta>0$, so that

$$\frac{(1-|w|^2)^p}{K(1-|\Psi_b(w)|^2)}\int_{D(w,\delta)}\frac{d\lambda_{K,q,\varphi}(\zeta)}{|1-\langle\zeta,w\rangle|^{q+n+1}}\leq\varepsilon,\ \ \forall\ w,b\in\mathbb{B}.$$

Then, we have

$$\lambda_{K,q,\varphi}(D(w,\delta))\lesssim(1-|w|^2)^{q+n+1}K(1-|\Psi_b(w)|^2).$$

If $u\in\mathcal{N}_K(p,q)$, then

$$\begin{aligned}&\int_{\mathbb{B}}|u(\varphi(\zeta))|^p(1-|\zeta|^2)^qK(1-|\Psi_b(\zeta)|^2)dV(\zeta)\\&=\int_{\mathbb{B}}|u(\zeta)|^pd\lambda_{K,q,\varphi}(\zeta)\leq\sum_{j=1}^{\infty}\int_{D(w_j,\delta)}|u(\zeta)|^pd\lambda_{K,q,\varphi}(\zeta)\\&\leq\sum_{j=1}^{\infty}\sup_{\zeta\in D(w_j,\delta)}|u(\zeta)|^p\int_{D(w_j,\delta)}d\lambda_{K,q,\varphi}(\zeta)\\&\lesssim\sum_{j=1}^{\infty}\sup_{\zeta\in D(w_j,\delta)}|u(\zeta)|^p\{(1-|w_j|^2)^{q+n+1}K(1-|\Psi_b(w_j)|^2)\}\\&\lesssim\sum_{j=1}^{\infty}\int_{D(w_j,4\delta)}|u(\zeta)|^p(1-|\zeta|^2)^qK(1-|\Psi_b(\zeta)|^2)dV(\zeta)\\&\lesssim\|u\|^q_{\mathcal{N}_K(p,q)}.\end{aligned}$$

□

3.2. **Compactness.**

Theorem 3.2. *Let $K\in\mathcal{RC}^+$ and $0<p,q<\infty$. Then the operator C_φ is compact on $\mathcal{N}_K(p,q)$ if and only if*

$$\lim_{|w|\to1^-}\sup_{b\in\mathbb{B}}(1-|w|^2)^p\bigg(\int_{\mathbb{B}}\frac{(1-|\zeta|^2)^q}{|1-\langle\varphi(\zeta),w\rangle|^{q+n+1}}K(1-|\Psi_b(\zeta)|^2)dV(\zeta)\bigg)=0. \tag{3.2}$$

Proof. Let C_φ be compact on $\mathcal{N}_K(p,q)$. Then, for any sequence $\{\xi_j\}\subset\mathbb{B}$ with $\lim_{j\to\infty}|\xi_j|=1$. Take

$$h_j(\zeta)=\frac{(1-|\xi_j|)}{(1-\langle\zeta,\xi_j\rangle)^{\frac{q+n+1}{p}}}.$$

Since $\{h_j\}$ is bounded on $\mathcal{N}_K(p,q)$ and converges uniformly to 0 on any compact subset of $\mathbb{B}$. So, by the compactness of C_φ, we obtain

$$\begin{aligned}&(1-|w|^2)^p\int_{\mathbb{B}}\frac{(1-|\zeta|^2)^qK(1-|\Psi_b(\zeta)|^2)\,dV(\zeta)}{|1-\langle\varphi(\zeta),w\rangle|^{q+p+1}}\\&=\|C_\varphi(h_j)\|^p_{\mathcal{N}_K(p,q)}\to0,\ \text{as } j\to\infty.\end{aligned}$$

Conversely, assume that (3.2) holds. Then, we can choose the sequence $\{w_i\} \in \mathbb{B}$ from Lemma (2.2), such that

$$\sup_{b\in\mathbb{B}} \frac{(1-|w_i|^2)^p}{K(1-|\Psi_b(w_i)|^2)} \int_{\mathbb{B}} \frac{d\lambda_{K,q,\varphi}(\zeta)}{|1-\langle\zeta, w_i\rangle|^{q+n+p+1}} \to 0, \text{ as } i \to 0.$$

Thus, for any $\epsilon > 0$, there exists a positive integer N_0 such that

$$\sup_{b\in\mathbb{B}} \frac{(1-|w_i|^2)^p}{K(1-|\Psi_b(w_i)|^2)} \int_{\mathbb{B}} \frac{d\lambda_{K,q,\varphi}(\zeta)}{|1-\langle\zeta, w_i\rangle|^{q+n+p+1}} < \epsilon, \text{ when } i > N_0. \tag{3.3}$$

In this case, by (3.3) for all $a \in \mathbb{B}$ when $j > N_0$, we have

$$\lambda_{K,q,\varphi}(D(w_i,\delta) \lesssim \epsilon^p (1-|w|^2)^{q+n+p+1} K(1-|\Psi_b(\zeta)|^n). \tag{3.4}$$

Now we let $\{u_j\}$ be any sequence that converges to 0 uniformly on any compact subset of $\mathbb{B}$ with $\|u_j\|_{\mathcal{N}_K(p,q)} \le C$. Then, the sequence $\{u_j\}$ converges to 0 uniformly on $M = \bigcup_{k=1}^{N_0} \overline{D(w_k,\delta)}$. Thus, there exists a positive integer $\overline{N}_0$ such that

$$\sup_{\zeta\in M} |u_j(\zeta)| < \epsilon \text{ when } \ j > \overline{N}_0. \tag{3.5}$$

Otherwise,

$$\lambda_{K,q,\varphi}(\mathbb{B}) \le \int_{\mathbb{B}} (1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big)\, dV(\zeta) \le C. \tag{3.6}$$

Therefore, when $j > \overline{N}_0$, by Lemma 2.2-2.4, (3.4)-(3.6), for all $a \in \mathbb{B}$ we have

$$\begin{aligned}
&\int_{\mathbb{B}} |u_j(\varphi(\zeta))|^p (1-|w|^2)^q K(1-|\Psi_b(\zeta)|^n) dV(\zeta) \\
&= \int \mathbb{B} |u_j(\zeta)|^p d\lambda_{K,q,\varphi} \le \sum_{k=1}^{\infty} \int_{D(w_k,\delta)} |u_j(\zeta)|^p d\lambda_{K,q,\varphi} \\
&\le \sum_{k=1}^{N_0} \int_{D(w_k,\delta)} |u_j(\zeta)|^p d\lambda_{K,q,\varphi} + \sum_{k=N_0+1}^{\infty} \sup_{\zeta\in D(w_k,\delta)} |u_j(\zeta)|^p \lambda_{K,q,\varphi} \\
&\lesssim N_0\, \epsilon^p\, \lambda_{K,q,\varphi}(\mathbb{B}) + \epsilon^p \sum_{k=N_0+1}^{\infty} \sup_{\zeta\in D(w_k,\delta)} |u_j(\zeta)|^p (1-|\zeta|^2)^{q+n+1} K\big((1-|\Psi_b(\zeta)|^2)^n\big) \\
&\lesssim N_0\, \epsilon^p\, \lambda_{K,q,\varphi}(\mathbb{B}) + \epsilon^p \int_{D(w_k,4\delta)} |u_j|^p (1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big) dV(\zeta) \\
&\lesssim N_0\, \epsilon^p\, \lambda_{K,q,\varphi}(\mathbb{B}) + \epsilon^p \int_{\mathbb{B}} |u_j|^p (1-|\zeta|^2)^q K\big((1-|\Psi_b(\zeta)|^2)^n\big) dV(\zeta) \\
&\lesssim N_0\, \epsilon^p\, \lambda_{K,q,\varphi}(\mathbb{B}) + \epsilon^p \|u_j\|_{\mathcal{N}_K(p,q)} \lesssim \epsilon^p,
\end{aligned}$$

which exactly

$$\lim_{k\to\infty} \|C_\varphi(u_j)\|_{\mathcal{N}_K(p,q)} = 0.$$

In this case, the operator C_φ is compact on $\mathcal{N}_K(p,q)$, which completed the proof. □

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