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THE ROLE OF COMPLETE PARTS IN TOPOLOGICAL POLYGROUPS

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Abstract. A topological polygroup is a polygroup P together with a topology on P such that the polygroup's binary hyperoperation and the polygroup's inverse function are continuous with respect to the topology. In this paper, we present some facts about complete parts in polygroups and we use these facts to obtain some new results in topological polygroups. We define the concept of cp-resolvable topological polygroups. A non-empty subset X of a topological polygroup is called cp-resolvable if there exist disjoint dense subsets A and B such that at least one of them is a complete part. Then, we investigate a few properties of cp-resolvable topological polygroups.

1. Introduction

Hypergroup that is based on the notion of hyperoperation was introduced by Marty in [13] and studied extensively by many mathematicians. Applications of hypergroups have mainly appeared in special subclasses. For example, polygroups that form an important subclass of hypergroups are studied by Comer [3, 4]. Polygroups have been applied in many area, such as geometry, lattices, combinatorics and color scheme. There exists a rich bibliography on polygroups [6]. This book contains the principal definitions endowed with examples and the basic results of the theory. Aghabozorgi et al. [1] defined perfect and solvable polygroups. Also see [11]. Till now, only a few papers treated the notion of topological hyperstructures, for example see [2, 8, 10]. Heidari et al. [9] defined the notion of topological polygroups. By considering the relative topology on subpolygroups they proved some properties of them. In particular, they proved the topological isomorphism theorems of topological polygroups. The purpose of this paper is as stated in the abstract.

2. Basic definitions and results

Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. Then, the mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* and $(H, \circ)$ is called a *hypergroupoid*. If A and B are two non-empty subsets of H and $x \in H$, then we define

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ A = \{x\} \circ A.$$

If $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$ (*associativity axiom*), then $(H, \circ)$ is called a *semihypergroup* and it is called a *quasihypergroup* if for every $x \in H$, we have $x \circ H = H = H \circ x$ (*reproduction axiom*). The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. Thus, a hypergroup is a generalization of a group. Let $(H, \circ)$ be a semihypergroup and A be a non-empty subset of H. We say that A is a *complete part* of H if for any natural number n and for all $a_1, a_2, \ldots, a_n \in H$, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

The complete parts are introduced for the first time by Koskas[12] and are studied by Corsini [5] and many others. A special subclass of hypergroups is the class of polygroups. We recall the following

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definition from [3]. A *polygroup* is a system $P =< P, \circ, e, ^{-1} >$, where $\circ : P \times P \to \mathcal{P}^*(P)$, $e \in P$, $^{-1}$ is a unitary operation on P and the following axioms hold for all $x, y, z \in P$:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$;
(2) $e \circ x = x \circ e = x$;
(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$. A non-empty subset K of a polygroup P is a *subpolygroup* of P if and only if $a, b \in K$ implies $a \circ b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. The subpolygroup N of P is *normal* in P if and only if $a^{-1} \circ N \circ a \subseteq N$ for every $a \in P$.

Proposition 2.1. *Let A and B be non-empty subsets of a polygroup $P =< P, \circ, e, ^{-1} >$ such that A is a complete part and $x \in P$. Then,*

(1) *$x^{-1} \circ x \circ A = x \circ x^{-1} \circ A = A$;*
(2) *A^{-1} is a complete part;*
(3) *$x \circ A$ and $A \circ x$ are complete parts;*
(4) *$B \subseteq x^{-1} \circ A$ if and only if $x \circ B \subseteq A$.*

Remark 1. Suppose that A is a complete part of a polygroup P, $n, m \in \mathbb{N}$ and $(x_1, x_2, \ldots, x_n) \in P^n, (y_1, y_2, \ldots, y_m) \in P^m$. By Proposition 2.1, $\prod\limits_{i=1}^{n} x_i \circ A \circ \prod\limits_{j=1}^{m} y_j$ and $\prod\limits_{i=1}^{n} x_i \circ A^{-1} \circ \prod\limits_{j=1}^{m} y_i$ are complete parts. Let $\{A_s\}_{s \in S}$ be a family of complete parts of P. Then, we observe that $\bigcup\limits_{s \in S} A_s$ is a complete part and if $I = \bigcap\limits_{s \in S} A_s \neq \emptyset$, then I is a complete part.

Proposition 2.2. *If A and B are two non-empty subsets of a polygroup P such that A is a complete part, then $A \circ B$ and $B \circ A$ are complete parts.*

Proof. Suppose that $n \in \mathbb{N}$ and $\prod\limits_{i=1}^{n} a_i \cap (A \circ B) \neq \emptyset$ where $(a_1, a_2, \ldots, a_n) \in P^n$. Then, there exists $b \in B$ such that $\prod\limits_{i=1}^{n} a_i \cap (A \circ b) \neq \emptyset$. By Proposition 2.1, $A \circ b$ is a complete part and so $\prod\limits_{i=1}^{n} a_i \subseteq A \circ b \subseteq A \circ B$, i.e., $A \circ B$ is a complete part. Similarly, $B \circ A$ is a complete part. □

We observe that if A is a complete part and B_i, C_j are non-empty subsets of P for $i = 1, 2, ..., n,\ j = 1, ..., m$, where $n, m \in \mathbb{N}$, then $C_1 \circ ... \circ C_m \circ A \circ B_1 \circ B_2 ... \circ B_n$ is a complete part.

Proposition 2.3. *Let P be a polygroup and B_α be a complete part for every $\alpha \in I$. Then, $a \circ \bigcap\limits_{\alpha \in I} B_\alpha = \bigcap\limits_{\alpha \in I} (a \circ B_\alpha)$ and $(\bigcap\limits_{\alpha \in I} B_\alpha) \circ a = \bigcap\limits_{\alpha \in I} (B_\alpha \circ a)$.*

Proof. Clearly, $a \circ \bigcap\limits_{\alpha \in I} B_\alpha \subseteq \bigcap\limits_{\alpha \in I} a \circ B_\alpha$. Conversely, suppose that $t \in \bigcap\limits_{\alpha \in I} a \circ B_\alpha$. So, for every $\alpha \in I$, $t \in a \circ B_\alpha$. Now, if $s \in I$, then $t \in a \circ B_s$. Hence, there exists $b_s \in B_s$ such that $t \in a \circ b_s$. Thus, for every $\alpha \in I$, $t \in a \circ b_s \cap a \circ B_\alpha$. So, for every $\alpha \in I$, $a \circ b_s \subseteq a \circ B_\alpha$. Therefore, by Proposition 2.1, for every $\alpha \in I$, $b_s \in a^{-1} \circ a \circ b_s \subseteq a^{-1} \circ a \circ B_\alpha = B_\alpha$, i.e., $b_s \in \bigcap\limits_{\alpha \in I} B_\alpha$. Thus, $t \in a \circ b_s \subseteq a \circ (\bigcap\limits_{\alpha \in I} B_\alpha)$, and so $a \circ \bigcap\limits_{\alpha \in I} B_\alpha = \bigcap\limits_{\alpha \in I} (a \circ B_\alpha)$. Similarly the equality $(\bigcap\limits_{\alpha \in I} B_\alpha) \circ a = \bigcap\limits_{\alpha \in I} (B_\alpha \circ a)$ holds. □

Remark 2. Being complete part is necessary in Proposition 2.3 as it is illustrated in the following example.

Example 1. Consider the set of integer numbers $\mathbb{Z}$ and define the hyperoperation $\circ$ on it as follows: for every $m \in \mathbb{Z}$, $m \circ 0 = m$ and if $m, n \in \mathbb{Z} \setminus \{0\}$, then

$$m \circ n = \begin{cases} \mathbb{E}, & m + n \in \mathbb{E} \\ \mathbb{E}^c, & m + n \in \mathbb{E}^c \end{cases}$$

where $\mathbb{E} = 2\mathbb{Z}$. If $A = \{1, 2\}$ and $B = \{1, 4\}$, then A and B are not complete parts and $1 \circ A = 1 \circ B = \mathbb{Z}$. But the equality $1 \circ (A \cap B) = 2\mathbb{Z}$ holds.

The following result is a direct consequence of Proposition 2.3.

Corollary 2.4. Let A and C be non-empty subsets of a polygroup P. If B_α is a complete part for every $\alpha \in I$, then $A \circ (\bigcap\limits_{\alpha \in I} B_\alpha) \circ C = \bigcap\limits_{\alpha \in I} (A \circ B_\alpha \circ C)$.

3. On topological polygroups and cp-resolvable topological polygroups

By using complete parts in topological polygroups, some interesting results were obtained by a few authors, in the classical case, see [2, 8, 9, 10, 14]. In this section, we study subpolygroups and dense subsets of a topological polygroup. Let (H, τ) be a topological space. Then, the family $\mathcal{B} = \{S_V \mid V \in \tau\}$, where $S_V = \{U \in \mathcal{P}^*(H) \mid U \subseteq V\}$ is a base for a topology on $\mathcal{P}^*(H)$. This topology is denoted by τ^* [10]. Let (H, τ) be a topological space. We consider the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$.

Let H be a topological space and $A \subset Y \subset H$, $int_Y A$ denotes the interior of A in the subspace Y, and the closure of A in the subspace Y is denoted by $cl_Y A$; the interior of A in H is denoted by A°, and the closure of A in H is denoted by $\overline{A}$.

Definition 3.1. [9] Let $P =< P, \circ, e, ^{-1} >$ be a polygroup and (P, τ) be a topological space. Then, the system $P = (P, \circ, e, ^{-1}, \tau)$ is called a topological polygroup if the mappings $\mu : P \times P \to \mathcal{P}^*(P)$ and $\iota : P \to P$ defined by $\mu(x, y) = x \circ y$ and $\iota(x) = x^{-1}$ are continuous.

Let U be an open subset of a topological polygroup P such that U is a complete part. Then, $a \circ U$ and $U \circ a$ are open subsets of P for every $a \in P$ [9]. Let P be a topological polygroup such that every open subset of P is a complete part. Let $\mathcal{U}$ be an open base at e. Then, the families $\{x \circ U \mid x \in P\ , U \in \mathcal{U}\}$ and $\{U \circ x \mid x \in P\ , U \in \mathcal{U}\}$ are open bases for P [9].

Theorem 3.2. [9] *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and $\mathcal{U}$ be a base at e. Then, the following assertions hold.*

1. *For every $U \in \mathcal{U}$ and $x \in U$ there exists $V \in \mathcal{U}$ such that $x \circ V \subseteq U$.*
2. *For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.*
3. *For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.*

Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup. We denote by $\nu_P(e)$ the set of all neighborhoods of e.

Theorem 3.3. [9] *Let P be a topological polygroup such that every open subset of P is a complete part. Then, for every $U \in \nu_P(e)$ there exists $V \in \nu_P(e)$ such that $\overline{V} \subseteq U$.*

Let every open subset of a topological polygroup P be a complete part. If F is a compact subset of P, then for every $a \in P$, $a \circ F$ and $F \circ a$ are compact [14].

Theorem 3.4. [14] *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part, F be a compact subset of P and G be a closed subset of P. Then, the sets $F \circ G$ and $G \circ F$ are closed in P.*

Lemma 3.5. [9] *If H is a subpolygroup of a topological polygroup $P = (P, \circ, e, ^{-1}, \tau)$ and every open subset of P is a complete part, then $\overline{H}$ is a subpolygroup of P.*

Theorem 3.6. [9] *Let P be topological polygroup such that every open subset of P is a complete part. Then, a subpolygroup K of P is open if and only if its interior is non-empty.*

Theorem 3.7. [9] *Let P be a topological polygroup such that every open subset of P is a complete part. Then, every open subpolygroup is closed.*

Theorem 3.8. [14] *Let H be a non-empty subset of a topological polygroup $P = (P, \circ, e, ^{-1}, \tau)$ and every open subset of P is a complete part. Then, we have*

1. $\overline{H} = \bigcap\limits_{U \in\ \nu_P(e)} U \circ H = \bigcap\limits_{U \in \nu_P(e)} H \circ U = \bigcap\limits_{U \in \nu_P(e), V \in \nu_P(e)} U \circ H \circ V;$
2. *If H is a normal subpolygroup, then $\overline{H}$ is a normal subpolygroup.*

Lemma 3.9. *If P is a topological polygroup such that every open subset of P is a complete part, then* $\overline{a \circ A} = a \circ \overline{A}$.

Proof. By Theorem 3.8 and Proposition 2.3, $\overline{a \circ A} = \bigcap\limits_{U \in \nu_P(e)} (a \circ A) \circ U = \bigcap\limits_{U \in \nu_P(e)} a \circ (A \circ U) = a \circ \bigcap\limits_{U \in \nu_P(e)} (A \circ U) = a \circ \overline{A}$. □

Remark 3. Begin complete part is necessary in Lemma 3.9 as it is illustrated in the following example.

Example 2. Suppose that the multiplication table for a polygroup $P =< P, \circ, 1, ^{-1} >$, where $P = \{1, 2\}$, $1^{-1} = 1$ and $2^{-1} = 2$ is

$\circ$	1	2
1	$\{1\}$	$\{2\}$
2	$\{2\}$	$\{1,2\}$

If $\tau = \{\emptyset, \{1\}, \{1, 2\}\}$, then $P = (P, \circ, ^{-1}, \tau)$ is a topological polygroup. If $A = \{1\}$, then $\overline{A} = \{1, 2\}$, $2 \circ A = \{2\}$ and $\overline{2 \circ A} = \{2\}$. But the equality $2 \circ \overline{A} = \{1, 2\}$ holds.

Let X be a topological space and $x \in X$. Recall that a family β of open subsets of X is called a base for X at the point x if all elements of β contain x and, for every neighborhood O of x, there exists $U \in \beta$ such that $x \in U \subseteq O$, and that the *character* of X at the point x is denoted by $\chi(x, X)$ where $\chi(x, X) = min\{\ |\beta| \mid \beta\ is\ a\ base\ for\ X\ at\ the\ point\ x\}$ and that if X has a countable base at each point $x \in X$, then X is called first-countable [7].

Proposition 3.10. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and Y is a dense subspace of P. Then, $\chi(y, Y) = \chi(y, P)$ for each $y \in Y$.*

Proof. Suppose that β is a base for P at a point $y \in Y$ such that $\chi(y, P) = |\beta|$. Then, the family $\beta_Y = \{U \cap Y |\ U \in \beta\ \}$ is a base for Y at y. Thus,
$\chi(y, Y) = \min\{\ |\beta_1| \mid \beta_1 \text{ is a base for } Y \text{ at } y\} \leq |\beta_Y| \leq |\beta| = \chi(y, P)$.

Conversely, let β_Y be a base for Y at a point y such that $\chi(y, Y) = |\beta_Y|$. For every $U \in \beta_Y$, choose an open set V_U in P such that $U = V_U \cap Y$. We claim that the family $\beta = \{\ V_U |\ U \in \beta_Y\}$ is a base for P at the point y. Indeed, we have

(1) For every $V_U \in \beta$, $y \in V_U$.
(2) Let O be an open subset of P containing y. Then, by Theorem 3.2, there exists $W_1 \in \nu_P(e)$ such that $y \circ W_1 \subseteq O$. So, by Theorem 3.3, there exists $W \in \nu_P(e)$ such that $\overline{W} \subseteq W_1$. Hence, $y \circ \overline{W} \subseteq y \circ W_1 \subseteq O$. Since $W_2 = y \circ W \cap Y$ is an open subset of Y and $y \in W_2$, there exists $U \in \beta_Y$ such that $U \subseteq W_2$. Since $U = V_U \cap Y$ and $\overline{Y} = P$, it follows that $\overline{V_U} = \overline{U} \subseteq \overline{y \circ W} = y \circ \overline{W} \subseteq O$ by Lemma 3.9. Thus, $y \in V_U \subseteq O$ and so $\chi(y, P) = \min\{\ |\beta_1| \mid \beta_1$ is a base for P at $y\} \leq |\beta| \leq |\beta_Y| = \chi(y, Y)$.

□

Lemma 3.11. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and let H be a subpolygroup of P. Then, every open subset of H is a complete part.*

Proof. Suppose that $U \in \mathcal{P}^*(H)$ is an open subest of H and for $n \in \mathbb{N}$, $\prod\limits_{i=1}^{n} a_i \cap U \neq \emptyset$, where $a_i \in H$. Thus, there exists an open subset V of P such that $U = V \cap H$. Hence, $\prod\limits_{i=1}^{n} a_i \cap V \neq \emptyset$. So, $\prod\limits_{i=1}^{n} a_i \subseteq V$. On the other hand $\prod\limits_{i=1}^{n} a_i \subseteq H$ and hence, $\prod\limits_{i=1}^{n} a_i \subseteq V \cap H = U$. □

Lemma 3.12. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part. If Y is a first-countable subpolygroup of P, then $\overline{Y}$ is also first-countable.*

Proof. Suppose that $K = \overline{Y}$. Then, by Lemma 3.5, K is a subpolygroup of P and so by Proposition 3.10, $\chi(e, K) = \chi(e, Y) \leq \aleph_0$. □

Definition 3.13. Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and $U \in \nu_P(e)$. A subset A of P is called *U-disjoint*, if $b \notin a \circ U$, for any distinct $a, b \in A$.

EXAMPLE 3. Suppose that the multiplication table for a topological polygroup $P = (P, \tau, \circ, 1, ^{-1})$ where $P = \{e, a, b\}$, $e^{-1} = e$, $a^{-1} = a$ and $b^{-1} = b$ is

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{e, a\}$

and $\tau = \{\emptyset, \{e, a\}, \{e, a, b\}\}$. Let $A = \{a, b\}$, $U = \{e, a\}$. Then, $a \circ U = \{a, e\}$, $b \circ U = \{b\}$ and $b \notin a \circ U$, $a \notin b \circ U$. Thus, A is U-disjoint but $A \cap U \neq \emptyset$.

We recall that a family $\{A_s\}_{s \in S}$ of subsets of a topological space X is called discrete if every point $x \in X$ has a neighborhood that intersects at most one set of the given family [7].

Lemma 3.14. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and $U, V \in \nu_P(e)$ such that $V^4 \subseteq U$ and $V^{-1} = V$. If a subset A of P is U-disjoint, then the family of open sets $\{a \circ V \mid a \in A\}$ is discrete in P.*

Proof. Suppose that $x \in P$, we claim that $x \circ V$ intersects at most one element of the family $\{a \circ V \mid a \in A\}$. Suppose to the contrary that, there exist distinct elements $a, b \in A$ such that $x \circ V \cap a \circ V \neq \emptyset$ and $x \circ V \cap b \circ V \neq \emptyset$. Let $t \in x \circ V \cap a \circ V$, hence there exists $v_1, v_2 \in V$ such that $t \in x \circ v_1$, $t \in a \circ v_2$. Thus, $x \in t \circ v_1^{-1}$, $a \in t \circ v_2^{-1}$ that is $x^{-1} \circ a \subseteq v_1 \circ t^{-1} \circ t \circ v_2^{-1} \subseteq V \circ t^{-1} \circ t \circ V^{-1} = V \circ V$ since V is a complete part. Similarly, we can prove that $b^{-1} \circ x \subseteq V \circ V$. So, $b^{-1} \circ a \subseteq b^{-1} \circ x \circ x^{-1} \circ a \subseteq V^4 \subseteq U$. Then, $a \in b \circ b^{-1} \circ a \subseteq b \circ U$ which is a contradiction. □

Let $(P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and N be a normal subpolygroup of P. Let $\pi : P \to P/N$ be the natural mapping defined by $\pi(x) = N \circ x$. Recall that $(P/N, \overline{\tau})$ is a topological space, where $\overline{\tau}$ is the quotient topology induced by π, that is $\overline{\tau} = \{U \subseteq P/N \mid \pi^{-1}(U) \subseteq P \text{ is open}\}$.

Lemma 3.15. [9] *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and N be a normal subpolygroup of P. Let $\pi : P \to P/N$ be the natural mapping. Then,*

(1) *$\pi^{-1}(\{N \circ x \mid x \in X\}) = N \circ X$ for every nonempty subset X of P and $\pi^{-1}(\pi(X)) = N \circ X$;*
(2) *$\{N \circ x \mid x \in X\} = \{N \circ y \mid y \in N \circ X\}$ for every nonempty subset X of P;*
(3) *If every open subset of P is a complete part, then the natural mapping π is open.*

Theorem 3.16. [9] *Let N be a normal subpolygroup of a topological polygroup $P = (P, \circ, e, ^{-1}, \tau)$ and every open subset of P is a complete part. Then, $< P/N, \odot, N, ^{-I}, \overline{\tau} >$ is a topological polygroup, where $N \circ x \odot N \circ y = \{N \circ z | z \in x \circ y\}$ and $(N \circ x)^{-I} = N \circ x^{-1}$.*

Theorem 3.17. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and N is a normal subpolygroup of P. Let $\pi : P \to P/N$ be the natural mapping. Then, the family $\{\pi(U \circ x) \mid U \in \nu_P(e)\}$ is a local base of the space P/N at the point $N \circ x \in P/N$.*

Proof. Since $U \in \nu_P(e)$ is a complete part, it follows that $U \circ x$ is an open subset of P. Then, for every $y \in N$, $y \circ (U \circ x)$ is open. So, $\pi^{-1}(\pi(U \circ x)) = N \circ (U \circ x) = \bigcup\limits_{y \in N} y \circ (U \circ x)$ is an open subset of P. Thus, $\pi(U \circ x)$ is an open subset of P/N by Lemma 3.15. Now, suppose that W is an open neighborhood of $N \circ x$ in P/N and $O = \pi^{-1}(W)$. Clearly $O \subseteq P$ is open. Since $N \circ x \subseteq W$, it follows that $x \in \pi^{-1}(N \circ x) \subseteq \pi^{-1}(W) = O$. So, there exists $U \in \nu_P(e)$ such that $U \circ x \subseteq O$. Therefore, $N \circ x \in \pi(U \circ x) \subseteq \pi(O) = W$ and this completes the proof. □

Lemma 3.18. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part. Let N be a normal subpolygroup of P and $\pi : P \to P/N$ be the natural mapping. If U , $V \in \nu_P(e)$ and $V \circ V^{-1} \subseteq U$, then $\overline{\pi(V)} \subseteq \pi(U)$.*

Proof. Suppose that $\pi(x) \in \overline{\pi(V)}$, where $x \in P$. We prove that $\pi(x) \in \pi(U)$. Since V is a complete part, it follows that $x \circ V \subseteq P$ is open. By Lemma 3.15, since π is open, it follows that $\pi(x \circ V)$ is open in P/N. Hence, $\pi(x \circ V) \cap \pi(V) \neq \emptyset$. Thus, there exists a $y \in P$ such that $\pi(y) \in \pi(x \circ V) \cap \pi(V)$. Therefore, there exist $t \in x \circ V$ and $z \in V$ such that $\pi(y) = \pi(t) = N \circ t$ and $\pi(y) = \pi(z) = N \circ z$. Since $t \in x \circ V$, there exists $b \in V$ such that $t \in x \circ b$. So, $x \in t \circ b^{-1}$ and $x \in N \circ x \subseteq N \circ t \circ b^{-1} = N \circ z \circ b^{-1} \subseteq N \circ V \circ V^{-1} \subseteq N \circ U$. Then, $\pi(x) \in \pi(N \circ U) = \pi(U)$ and this completes the proof. □

Theorem 3.19. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part , N is a normal subpolygroup of P and $W \in \nu_{P/N}(N)$. Then, there exists $V \in \nu_{P/N}(N)$ such that $\overline{V} \subseteq W$.*

Proof. Suppose that $\pi : P \to P/N$ is the natural mapping. Since π is continuous, it follows that $U = \pi^{-1}(W) \subseteq P$ is open and $e \in U$. Then, $U \in \nu_P(e)$. Thus, by Theorem 3.2, there exists $V_1 \in \nu_P(e)$ such that $V_1 \circ V_1^{-1} \subseteq U$. Hence, by Lemma 3.18, $\overline{\pi(V_1)} \subseteq \pi(U) = W$. Since π is open and $V_1 \in \nu_P(e)$, $V = \pi(V_1) \in \nu_{P/N}(N)$ and this completes the proof. □

Recall that a continuous mapping $f : X \to Y$ is *perfect* if f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X.

Theorem 3.20. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and let N be a compact normal subpolygroup of P. Then, the natural mapping $\pi : P \to P/N$ is perfect.*

Proof. Suppose that $H \subseteq P$ is a closed subset. Then, by Theorem 3.4, $N \circ H$ is closed in P. Since $\pi^{-1}(\pi(H)) = N \circ H$, so $\pi^{-1}(\pi(H)) \subseteq P$ is closed. Then, $\pi(H)$ is closed in P/N. Hence, π is closed. Now, suppose that $y \in P/N$. Then, there exists $x \in P$ such that $\pi(x) = y$. Since N is compact, it follows that $\pi^{-1}(y) = \pi^{-1}(\pi(x)) = N \circ x$ is compact. Therefore, $\pi^{-1}(y)$ is compact. □

Lemma 3.21. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and N is a normal subpolygroup of P.*

(1) *If P is compact, then P/N is compact.*
(2) *If N and P/N are compact, then P is compact.*

Proof. (1) Since π is continuous and P is compact, it follows that $\pi(P) = P/N$ is compact.

(2) Since N is compact, by Theorem 3.20, the natural mapping $\pi : P \to P/N$ is perfect. Then, $\pi^{-1}(P/N) = P$ is compact [[7], Theorem 3.7.2]. □

Definition 3.22. Let X be a non-empty subset of a topological polygroup $P = (P, \circ, e, ^{-1}, \tau)$. Then, X is called *cp-resolvable* if there exist two disjoint dense subsets of X such that at least one of them is a complete part of P.

Lemma 3.23. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and let H be a cp-resolvable subpolygroup of P. If $x \in P$, then $x \circ H$ is cp-resolvable.*

Proof. Suppose that A and B are disjoint dense subspaces of H such that one of them is a complete part of P. Then, by Lemma 3.9, $cl_{x\circ H}(x \circ A) = \overline{x \circ A} \cap x \circ H = x \circ \overline{A} \cap x \circ H = x \circ H$ since $H = cl_H(A) = \overline{A} \cap H$. Similarly, $cl_{x\circ H}(x \circ B) = x \circ H$. Without lose of generality we can assume that A is a complete part of P. Thus, by Proposition 2.1, $x \circ A$ is a complete part and we observe that $x \circ A \cap x \circ B = \emptyset$. Thus, $x \circ H$ is cp-resolvable. □

We recall that if N is a normal subpolygroup of P, then the relation

$$x, y \in P,\ xN_Py \text{ if and only if } x^{-1} \circ y \cap N \neq \emptyset$$

is an equivalence relation. The equivalence class of the element $x \in P$ is denoted by $N_P(x)$ [6].

Lemma 3.24. [6] *If N is a normal subpolygroup of P, then $x \circ N = N_P(x)$.*

Theorem 3.25. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part and let $\{P_\alpha | \alpha \in I\}$ be a pairwise disjoint family of non-empty subsets of P such that every P_α is cp-resolvable. If $P = \bigcup_{\alpha\in I} P_\alpha$, then P is cp-resolvable.*

Proof. Since each P_α is cp-resolvable, there exist disjoint dense subsets A_α and B_α of P_α such that A_α is a complete part of P and so $\overline{A_\alpha \cap P_\alpha} \cap P_\alpha = cl_{P_\alpha} A_\alpha = P_\alpha$. If $A = \bigcup_{\alpha\in I} A_\alpha$ and $B = \bigcup_{\alpha\in I} B_\alpha$, then A is a complete part by Remark 1 and A and B are disjoint dense subsets of P. Thus, P is cp-resolvable. □

Theorem 3.26. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part. Then, the following conditions hold.*

(1) *If H is a proper dense subpolygroup of P and H is a complete part, then P is cp-resolvable.*
(2) *If a normal subpolygruop N of P is cp-resolvable, then P is cp-resolvable.*
(3) *If P contains a non-closed normal subpolygroup N which is a complete part, then P is cp-resolvable.*

Proof. (1) Suppose that $A = H$ and $B = P \setminus H$. Since the subpolygroup A is not closed, then by Theorem 3.7, A is not open. Thus, $int(A) = \emptyset$. Therefore, $\overline{A} = \overline{B} = P$.

(2) By Lemma 3.24, P is equal to the disjoint unions $(\bigcup_{x \notin N} x \circ N) \cup N$. Thus, P is cp-resolvable by Lemma 3.23 and Theorem 3.25.

(3) By Theorem 3.8, $B = \overline{N}$ is a normal subpolygroup of P, and so by (1); B is cp-resolvable. Therefore, by (2), P is cp-resolvable. □

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