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PETER-WEYL THEOREM FOR HOMOGENEOUS SPACES OF COMPACT GROUPS

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Abstract. This paper presents a structured formalism for a constructive generalization of the Peter-Weyl Theorem over homogeneous spaces of compact groups. Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on the compact left coset space G/H. We then present an abstract T_H-version of the Peter-Weyl Theorem for the Hilbert function space $L^2(G/H,\mu)$.

1. Introduction

The abstract aspects of harmonic analysis over homogeneous spaces of compact non-Abelian groups or precisely left coset (resp. right coset) spaces of non-normal subgroups of compact non-Abelian groups is placed as building blocks for classical harmonic analysis [5,7], coherent states analysis [8,11], theoretical and particle physics [1]. Over the last decades, abstract and computational aspects of Plancherel formulas over symmetric spaces have achieved significant popularity in geometric analysis, mathematical physics and scientific computing (computational engineering), see [2–4, 6, 12, 13] and references therein.

The Peter-Weyl theorem is a fundamental result in the theory of classical harmonic analysis, applying to compact topological groups that are not necessarily abelian. It was initially proved by Hermann Weyl and Fritz Peter, in the setting of a compact topological groups [15]. The theorem is a collection of results generalizing the significant facts about the decomposition of the regular representations of finite groups, as presented by F. G. Frobenius and Issai Schur, see [1, 9, 10] and classical references therein. The theorem has three parts. The first part states that the matrix coefficients of irreducible representations of a compact groups G are dense in the space $\mathcal{C}(G)$ of continuous complex-valued functions on G, and thus also in the space $L^2(G)$ of square-integrable functions. The second part asserts the complete reducibility of unitary representations of G. The final part then asserts that the regular representation of G on $L^2(G)$ decomposes as the direct sum of all irreducible unitary representations. Moreover, the matrix coefficients of the irreducible unitary representations form an orthonormal basis of $L^2(G)$.

Let G be a compact group and H be a closed subgroup of G. Also, let G/H be the left coset space of H in G and $\widehat{G/H}$ be the abstract dual space of G/H. Let μ be the normalized G-invariant measure over the homogeneous space G/H with respect to the probability measures of H and G, associated to the Weil's formula. Then we present a structured formalism for a constructive generalization of the Peter-Weyl Theorem for the Hilbert function space $L^2(G/H,\mu)$.

The paper is organized as follows. Section 2 is devoted to fixing notations and a brief summary on the non-Abelian Fourier analysis of compact groups, general formalism of the Peter-Weyl theorem, and preliminaries and classical results on harmonic analysis of compact homogeneous spaces. Then we present a systematic study of abstract harmonic analysis over the Hilbert function space $L^2(G/H,\mu)$. In section 4, using the abstract notion of the dual space $\widehat{G/H}$ of the homogeneous space G/H, we prove that the Hilbert function space $L^2(G/H,\mu)$ satisfies a canonical decomposition into a direct sum of some closed and mutually orthogonal subspaces. This decomposition coincides with the Peter-Weyl

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decomposition, when H is a normal subgroup of G. This result can be considered as a generalization of the Peter-Weyl Theorem for homogeneous spaces of compact groups.

2. Preliminaries and Notations

Let $\mathcal{H}$ be a separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Hilbert-Schmidt operator if for one, hence for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$ we have $\sum_k \|Te_k\|^2 < \infty$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathrm{HS}(\mathcal{H})$ and for $T \in \mathrm{HS}(\mathcal{H})$ the Hilbert-Schmidt norm of T is $\|T\|_{\mathrm{HS}}^2 = \sum_k \|Te_k\|^2$. The set $\mathrm{HS}(\mathcal{H})$ is a self adjoint two sided ideal in $\mathcal{B}(\mathcal{H})$ and if $\mathcal{H}$ is finite-dimensional we have $\mathrm{HS}(\mathcal{H}_\pi) = \mathcal{B}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ called trace-class, whenever $\|T\|_{\mathrm{tr}} = \mathrm{tr}[|T|] < \infty$, where $\mathrm{tr}[T] = \sum_k \langle Te_k, e_k\rangle$ and $|T| = (TT^*)^{1/2}$, see [14].

Let G be a compact group with the Haar measure dx, H be a closed subgroup of G with the left Haar measure dh. The left coset space G/H is considered as a locally compact homogeneous space that G acts on it from the left and $q : G \to G/H$ given by $x \mapsto q(x) := xH$ is the surjective canonical mapping. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where (see Proposition 2.48 of [1]) $f \in \mathcal{C}(G)$ and

$$T_H(f)(xH) = \int_H f(xh)dh. \tag{2.1}$$

Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$, for Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x = \mu$, for $x \in G$. If G is compact, the homogeneous space G/H has a G-invariant measure μ, which satisfies the following Weil's formula, for $f \in L^1(G)$ (see [1])

$$\int_{G/H} T_H(f)(xH)d\mu(xH) = \int_G f(x)dx. \tag{2.2}$$

If μ is a G-invariant measure on the homogeneous space G/H and $p \geq 1$, the notation $L^p(G/H, \mu)$ stands for the Banach space of all equivalence classes of μ-measurable complex valued functions $\phi : G/H \to \mathbb{C}$ such that $\|\phi\|_{L^p(G/H,\mu)} < \infty$.

Each irreducible representation of G is finite dimensional and every unitary representation of G is a direct sum of irreducible representations, see [1, 9]. The set of of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. This definition of $\widehat{G}$ is in essential agreement with the classical definition when G is Abelian, since each character of an Abelian group is a one dimensional representation of G. If π is any unitary representation of G, for $u, v \in \mathcal{H}_\pi$ the functions $\pi_{u,v}(x) = \langle \pi(x)u, v\rangle$ are called matrix elements of π. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then π_{ij} means π_{e_i,e_j}. The notation $\mathcal{E}_\pi$ is used for the linear span of the matrix elements of π and the notation $\mathcal{E}$ is used for the linear span of $\bigcup_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$. The Peter-Weyl Theorem (see [1, 9]) guarantees that if G is a compact group, $\mathcal{E}$ is uniformly dense in $\mathcal{C}(G)$, $L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$, and $\{d_\pi^{-1/2}\pi_{ij} : i, j = 1...d_\pi, [\pi] \in \widehat{G}\}$ is an orthonormal basis for $L^2(G)$. Using the Peter-Weyl Theorem, for $f \in L^2(G)$ we have

$$f = \sum_{[\pi]\in\widehat{G}} \sum_{i,j=1}^{d_\pi} c_{ij}^\pi(f)\pi_{ij}, \tag{2.3}$$

where $c_{i,j}^\pi(f) = d_\pi\langle f, \pi_{ij}\rangle_{L^2(G)}$.

3. Abstract Harmonic Analysis over Homogeneous Spaces of Compact Groups

Throughout this article we assume that H is a closed subgroup of a compact group G with normalized Haar measures dh and dx respectively.

We start this section with an extension of the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ for other function spaces related to the homogeneous space G/H. If $p = 1$, it is easy to check that $\|T_H(f)\|_{L^1(G/H,\mu)} \leq \|f\|_{L^1(G)}$.

Proposition 3.1. *Let H be a closed subgroup of a compact group G. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ is a uniformly continuous.*

Proof. Let $f\in\mathcal{C}(G)$ and $x\in G$. Then we have

$$|T_H(f)(xH)|=\left|\int_H f(xh)dh\right|\le\int_H|f(xh)|dh\le\|f\|_{\sup}\left(\int_H dh\right)=\|f\|_{\sup},$$

which implies $\|T_H(f)\|_{\sup}\le\|f\|_{\sup}$. □

Next we prove that the linear map T_H is norm-decreasing in L^2-spaces.

Theorem 3.1. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H associated to the Weil's formula. The linear map $T_H:\mathcal{C}(G)\to\mathcal{C}(G/H)$ has a unique extension to a bounded linear map from $L^2(G)$ onto $L^2(G/H,\mu)$.*

Proof. We shall show that, each $f\in\mathcal{C}(G)$ satisfies $\|T_H(f)\|_{L^2(G/H,\mu)}\le\|f\|_{L^2(G)}$. Let $f\in\mathcal{C}(G)$. Using compactness of H and the Weil's formula we have

$$\begin{aligned}
\|T_H(f)\|^2_{L^2(G/H,\mu)}&=\int_{G/H}|T_H(f)(xH)|^2d\mu(xH)\\
&=\int_{G/H}\left|\int_H f(xh)dh\right|^2d\mu(xH)\\
&\le\int_{G/H}\left(\int_H|f(xh)|dh\right)^2d\mu(xH)\\
&\le\int_{G/H}\int_H|f(xh)|^2dhd\mu(xH)\\
&=\int_{G/H}\int_H|f|^2(xh)dhd\mu(xH)\\
&=\int_{G/H}T_H(|f|^2)(xH)d\mu(xH)=\int_G|f(x)|^2dx=\|f\|^2_{L^2(G)}.
\end{aligned}$$

Thus, we can extend T_H to a bounded linear operator from $L^2(G)$ onto $L^2(G/H,\mu)$, which we still denote it by T_H. □

Let $\mathcal{J}^p(G,H):=\{f\in L^p(G):T_H(f)=0\}$. Then, $\mathcal{J}^2(G,H)^\perp$ is the orthogonal completion of the closed subspace $\mathcal{J}^2(G,H)$ in $L^2(G)$.
As an immediate consequence of Proposition 3.1 we deduce the following corollary.

Corollary 3.1. *Let H be a closed subgroup of a compact group G and μ be a G-invariant measure on G/H. The linear map $T_H:L^2(G)\to L^2(G/H,\mu)$ is partial isometric.*

Proof. Let $\varphi\in L^2(G/H,\mu)$ and $\varphi_q:=\varphi\circ q$. Then, we have $\varphi_q\in L^2(G)$. Indeed,

$$\begin{aligned}
\|\varphi_q\|^2_{L^2(G)}&=\int_G|\varphi_q(x)|^2dx\\
&=\int_{G/H}T_H\left(|\varphi_q|^2\right)(xH)d\mu(xH)\\
&=\int_{G/H}\left(\int_H|\varphi_q(xh)|^2dh\right)d\mu(xH)\\
&=\int_{G/H}\left(\int_H|\varphi(xhH)|^2dh\right)d\mu(xH)\\
&=\int_{G/H}\left(\int_H|\varphi(xH)|^2dh\right)d\mu(xH)\\
&=\int_{G/H}|\varphi(xH)|^2\left(\int_H dh\right)d\mu(xH)\\
&=\int_{G/H}|\varphi(xH)|^2d\mu(xH)=\|\varphi\|^2_{L^2(G/H,\mu)}.
\end{aligned}$$

Also $T_H^*(\varphi) = \varphi_q$ and $T_H T_H^*(\varphi) = \varphi$. Because using the Weil's formula, for all $f \in L^2(G)$ we achieve

$$\begin{aligned}
\langle T_H^*(\varphi), f\rangle_{L^2(G)} &= \langle \varphi, T_H(f)\rangle_{L^2(G/H,\mu)} \\
&= \int_{G/H} \varphi(xH)\overline{T_H(f)(xH)}d\mu(xH) \\
&= \int_{G/H} \varphi(xH)T_H(\overline{f})(xH)d\mu(xH) \\
&= \int_{G/H} T_H(\varphi_q.\overline{f})(xH)d\mu(xH) = \int_G \varphi_q(x)\overline{f}(x)dx = \langle \varphi_q, f\rangle_{L^2(G)}.
\end{aligned}$$

Now a straightforward calculation implies $T_H = T_H T_H^* T_H$. Then by Theorem 2.3.3 of [14], T_H is a partial isometric operator. □

We can conclude the following corollaries as well.

Corollary 3.2. *Let H be a closed subgroup of a compact group G. Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^\perp$ respectively. Then, for each $f \in L^2(G)$ and a.e. $x \in G$ we have*

$$P_{\mathcal{J}^2(G,H)^\perp}(f)(x) = T_H(f)(xH), \qquad P_{\mathcal{J}^2(G,H)}(f)(x) = f(x) - T_H(f)(xH). \tag{3.1}$$

Corollary 3.3. *Let H be a compact subgroup of a locally compact group G and μ be a G-invariant measure on G/H. The following statements hold.*

(1) $\mathcal{J}^2(G,H)^\perp = \{\psi_q : \psi \in L^2(G/H,\mu)\}$.
(2) *For all $f \in \mathcal{J}^2(G,H)^\perp$ and each $h \in H$ we have $R_h f = f$.*
(3) *For all $\psi \in L^2(G/H,\mu)$ we have $\|\psi_q\|_{L^2(G)} = \|\psi\|_{L^2(G/H,\mu)}$.*
(4) *For all $f, g \in \mathcal{J}^2(G,H)^\perp$ we have $\langle T_H(f), T_H(g)\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}$.*

Remark 3.1. *Invoking Corollary 3.3 one can regard $L^2(G/H,\mu)$ as a closed subspace of $L^2(G)$, that is the subspace consists of all $f \in L^2(G)$ which satisfies $R_h f = f$ for all $h \in H$. Then Theorem 3.1 and Proposition 3.1 guarantees that the linear map*

$$T_H : L^2(G) \to L^2(G/H,\mu) \subset L^2(G)$$

is an orthogonal projection.

4. Peter-Weyl Theorem for Homogeneous Spaces of Compact Groups

For a closed subgroup H of G, define

$$H^\perp := \left\{[\pi] \in \widehat{G} : \pi(h) = I \quad \text{for all } h \in H\right\}, \tag{4.1}$$

If G is Abelian, each closed subgroup H of G is normal and the locally compact group G/H is Abelian and so $\widehat{G/H}$ is precisely the set of all characters (one dimensional irreducible representations) of G which are constant on H, that is precisely $H^\perp$. If G is a non-Abelian group and H is a closed normal subgroup of G, then the dual space $\widehat{G/H}$ which is the set of all unitary equivalence classes of unitary representations of G/H, has meaning and it is well-defined. Indeed, G/H is a non-Abelian group. In this case, the map $\Phi : \widehat{G/H} \to H^\perp$ defined by $\sigma \mapsto \Phi(\sigma) := \sigma \circ q$ is a Borel isomorphism and $\widehat{G/H} = H^\perp$, see [1]. Thus if H is normal, $H^\perp$ coincides with the classic definitions of the dual space either when G is Abelian or non-Abelian.

For a closed subgroup H of G and a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, define

$$T_H^\pi := \int_H \pi(h)dh, \tag{4.2}$$

where the operator valued integral (4.2) is considered in the weak sense. In other words,

$$\langle T_H^\pi \zeta, \xi\rangle = \int_H \langle \pi(h)\zeta, \xi\rangle dh, \quad \text{for } \zeta, \xi \in \mathcal{H}_\pi. \tag{4.3}$$

The function $h \mapsto \langle \pi(h)\zeta, \xi\rangle$ is bounded and continuous on H. Since H is compact, the right integral is the ordinary integral of a function in $L^1(H)$. Hence, T_H^π defines a bounded linear operator on $\mathcal{H}_\pi$ with $\|T_H^\pi\| \leq 1$.

Remark 4.1. *Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Let $(\sigma, \mathcal{H}_\sigma)$ be a continuous unitary representation of G such that $[\pi] = [\sigma]$. Let $S : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ be the unitary operator which satisfies $\sigma(x)S = S\pi(x)$ for all $x \in G$. Then we have*

$$\begin{aligned} ST_H^\pi &= S\left(\int_H \pi(h)dh\right) \\ &= \int_H S\pi(h)dh \\ &= \int_H \sigma(h)Sdh = \left(\int_H \sigma(h)dh\right) S = T_H^\sigma S, \end{aligned}$$

which implies that $T_H^\sigma \neq 0$ as well. Thus we deduce that the non-zero property of T_H^π depends only on $[\pi]$, that is the unitary equivalence class of $(\pi, \mathcal{H}_\pi)$.

Let

$$\mathcal{K}_\pi^H := \{\zeta \in \mathcal{H}_\pi : \pi(h)\zeta = \zeta \ \forall h \in H\}. \tag{4.4}$$

Then, $\mathcal{K}_\pi^H$ is a closed subspace of $\mathcal{H}_\pi$ and $\mathcal{R}(T_H^\pi) = \mathcal{K}_\pi^H$, where

$$\mathcal{R}(T_H^\pi) = \{T_H^\pi \zeta : \zeta \in \mathcal{H}_\pi\}.$$

It is easy to see that $[\pi] \in H^\perp$ if and only if $\mathcal{K}_\pi^H = \mathcal{H}_\pi$.

Proposition 4.1. *Let H be a closed subgroup of a compact group G and $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G. Then,*

1. *The operator T_H^π is an orthogonal projection onto $\mathcal{K}_\pi^H$.*
2. *The operator T_H^π is unitary if and only if $[\pi] \in H^\perp$.*

Proof. (1) Using compactness of H, it can be easily checked that $(T_H^\pi)^* = T_H^\pi$. As well as we achieve that

$$\begin{aligned} T_H^\pi T_H^\pi &= \left(\int_H \pi(h)dh\right)\left(\int_H \pi(t)dt\right) \\ &= \int_H \pi(h)\left(\int_H \pi(t)dt\right)dh \\ &= \int_H \left(\int_H \pi(h)\pi(t)dt\right)dh \\ &= \int_H \left(\int_H \pi(ht)dt\right)dh = \int_H T_H^\pi dt = T_H^\pi. \end{aligned}$$

(2) Since T_H^π is a projection, the operator T_H^π is unitary if and only if $T_H^\pi = I$. The operator T_H is identity if and only if $\pi(h) = I$ for all $h \in H$. Thus, T_H^π is unitary if and only if $[\pi] \in H^\perp$.

□

Definition 4.1. *Let H be a closed subgroup of a compact group G. Then we define the dual space of G/H, as the subset of $\widehat{G}$ which is given by*

$$\widehat{G/H} := \left\{[\pi] \in \widehat{G} : T_H^\pi \neq 0\right\} = \left\{[\pi] \in \widehat{G} : \int_H \pi(h)dh \neq 0\right\}. \tag{4.5}$$

Evidently, any closed subgroup H of G satisfies

$$H^\perp \subset \widehat{G/H}. \tag{4.6}$$

Next we shall show that the reverse inclusion of (4.6) holds, if and only if H is a normal subgroup of G.

Theorem 4.1. *Let H be a closed normal subgroup of a compact group G. Then*

$$\widehat{G/H} = H^\perp.$$

Proof. Let H be a closed normal subgroup of a compact group G. It is sufficient to show that $\widehat{G/H} \subset H^{\perp}$. Let $[\pi] \in \widehat{G/H}$ be given. Due to the normality of H in G, for all $x \in G$ the map $\tau_x : H \to H$ given by $h \mapsto \tau_x(h) := x^{-1}hx$ belongs to $Aut(H)$ and $x^{-1}Hx = H$. Invoking compactness of G we have $d(\tau_x(h)) = dh$, for $x \in G$. Now, for $x \in G$ we get

$$\begin{aligned}\int_H \pi(h)dh &= \int_{xHx^{-1}} \pi(\tau_x(h))d(\tau_x(h)) \\ &= \int_H \pi(\tau_x(h))dh \\ &= \int_H \pi(x)^*\pi(h)\pi(x)dh \\ &= \pi(x)^*\left(\int_H \pi(h)dh\right)\pi(x) = \pi(x)^*T_H^\pi\pi(x).\end{aligned}$$

Therefore $\pi(x)T_H^\pi = T_H^\pi\pi(x)$ for $x \in G$, which implies $T_H^\pi \in \mathcal{C}(\pi)$. Irreducibility of π guarantees that $T_H^\pi = \alpha I$ for some non-zero $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Thus, for $t \in H$ we can write

$$\begin{aligned}\pi(t) &= \alpha^{-1}\pi(t)\alpha I \\ &= \alpha^{-1}\pi(t)T_H^\pi \\ &= \alpha^{-1}\int_H \pi(th)dh \\ &= \alpha^{-1}\int_H \pi(h)dh = \alpha^{-1}T_H^\pi = I,\end{aligned}$$

which implies $[\pi] \in H^{\perp}$. □

Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G such that $T_H^\pi \neq 0$. Then the functions $\pi^H_{\zeta,\xi} : G/H \to \mathbb{C}$ defined by

$$\pi^H_{\zeta,\xi}(xH) := \langle \pi(x)T_H^\pi\zeta, \xi\rangle \quad \text{for } xH \in G/H, \tag{4.7}$$

for $\xi, \zeta \in \mathcal{H}_\pi$ are called H-matrix elements of $(\pi, \mathcal{H}_\pi)$.

For $xH \in G/H$ and $\zeta, \xi \in \mathcal{H}_\pi$, we have

$$\begin{aligned}|\pi^H_{\zeta,\xi}(xH)| &= |\langle \pi(x)T_H^\pi\zeta, \xi\rangle| \\ &\leq \|\pi(x)T_H^\pi\zeta\|\|\xi\| \leq \|T_H^\pi\zeta\|\|\xi\| \leq \|\zeta\|\|\xi\|.\end{aligned}$$

Also we can write

$$\pi^H_{\zeta,\xi}(xH) = \langle \pi(x)T_H^\pi\zeta, \xi\rangle = \pi_{T_H^\pi\zeta,\xi}(x). \tag{4.8}$$

Invoking definition of the linear map T_H and also T_H^π we have

$$\begin{aligned}T_H(\pi_{\zeta,\xi})(xH) &= \int_H \pi_{\zeta,\xi}(xh)dh \\ &= \int_H \langle \pi(xh)\zeta, \xi\rangle dh \\ &= \int_H \langle \pi(x)\pi(h)\zeta, \xi\rangle dh = \langle \pi(x)T_H^\pi\zeta, \xi\rangle,\end{aligned}$$

which implies that

$$T_H(\pi_{\zeta,\xi}) = \pi^H_{\zeta,\xi}. \tag{4.9}$$

Theorem 4.2. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure and $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G such that $T_H^\pi \neq 0$. Then*

1. *The subspace $\mathcal{E}_\pi(G/H)$ depends on the unitary equivalence class of π.*
2. *The subspace $\mathcal{E}_\pi(G/H)$ is a closed left invariant subspace of $L^1(G/H, \mu)$.*

Proof. (1) Let $(\sigma, \mathcal{H}_\sigma)$ be a continuous unitary representation of G such that $[\pi] = [\sigma]$. Let $S : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ be the unitary operator which satisfies $\sigma(x)S = S\pi(x)$ for all $x \in G$. Remark 4.1 guarantees that $ST_H^\pi = T_H^\sigma S$ and also $T_H^\sigma \neq 0$. Thus for $x \in G$ and $\zeta, \xi \in \mathcal{H}_\pi$ we can write

$$\begin{aligned}\pi^H_{\zeta,\xi}(xH) &= \langle \pi(x)T_H^\pi \zeta, \xi\rangle_{\mathcal{H}_\pi}\\ &= \langle S^{-1}\sigma(x)ST_H^\pi\zeta, \xi\rangle_{\mathcal{H}_\pi}\\ &= \langle \sigma(x)ST_H^\pi\zeta, S\xi\rangle_{\mathcal{H}_\sigma}\\ &= \langle \sigma(x)T_H^\sigma S\zeta, S\xi\rangle_{\mathcal{H}_\sigma} = \sigma^H_{S\zeta,S\xi}(xH),\end{aligned}$$

which implies that $\mathcal{E}_\pi(G/H) = \mathcal{E}_\sigma(G/H)$.
(2) It is straightforward. □

If ζ, ξ belongs to an orthonormal basis $\{e_i\}$ for $\mathcal{H}_\pi$, H-matrix elements of $[\pi]$ with respect to an orthonormal basis $\{e_j\}$ changes in the form

$$\pi^H_{ij}(xH) = \pi^H_{e_j,e_i}(xH) = \langle \pi(x)T_H^\pi e_j, e_i\rangle, \quad \text{for } \ xH \in G/H. \tag{4.10}$$

The linear span of the H-matrix elements of a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ satisfying $T_H^\pi \neq 0$, is denoted by $\mathcal{E}_\pi(G/H)$ which is a subspace of $\mathcal{C}(G/H)$.

Definition 4.2. *Let H be a closed subgroup of a compact group G and $[\pi] \in \widehat{G/H}$. An ordered orthonormal basis $\mathfrak{B} = \{e_\ell : 1 \le \ell \le d_\pi\}$ of the Hilbert space $\mathcal{H}_\pi$ is called H-admissible, if it is an extension of an orthonormal basis $\{e_\ell : 1 \le \ell \le d_{\pi,H}\}$ of the closed subspace $\mathcal{K}_\pi^H$, which equivalently means that $d_{\pi,H}$-first elements of $\mathfrak{B}$ be an orthogonal basis of $\mathcal{K}_\pi^H$.*

Let $[\pi] \in \widehat{G/H}$ and $\mathfrak{B}_\pi = \{e_\ell : 1 \le \ell \le d_\pi\}$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$. Then, each $\pi_{i\ell}$ with $1 \le i \le d_\pi$ and $1 \le \ell \le d_{\pi,H}$, is a well-defined continuous function over G/H. Let $\mathcal{E}_\pi^\ell(G/H)$ be the subspace of $\mathcal{C}(G/H)$ spanned by the set $\mathfrak{B}_\pi^\ell := \{\sqrt{d_\pi}\pi_{i\ell} : 1 \le i \le d_\pi\}$.

Proposition 4.2. *Let $[\pi] \in \widehat{G/H}$, $\mathfrak{B}_\pi$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$, and $1 \le \ell \neq \ell' \le d_{\pi,H}$. Then*
1. *$\dim \mathcal{E}_\pi^\ell(G/H) = d_\pi$ and $\mathfrak{B}_\pi^\ell$ is an orthonormal basis for $\mathcal{E}_\pi^\ell(G/H)$.*
2. *$\mathcal{E}_\pi^\ell(G/H)$ is a closed left translation invariant subspace of $\mathcal{C}(G/H)$.*
3. *$\mathcal{E}_\pi^{\ell'}(G/H) \perp \mathcal{E}_\pi^\ell(G/H)$.*

Proof. (1) Let $1 \le i, i' \le d_\pi$. Then by Theorem 27.19 of [10] we get

$$\langle \pi_{i\ell}, \pi_{i'\ell}\rangle_{L^2(G/H,\mu)} = \langle \pi_{i\ell}, \pi_{i'\ell}\rangle_{L^2(G)} = d_\pi^{-1}\delta_{ii'}.$$

Since $\dim \mathcal{E}_\pi^\ell(G/H) \le d_\pi$ we achieve that $\mathfrak{B}_\pi^\ell$ is an orthonormal basis for $\mathcal{E}_\pi^\ell(G/H)$ and hence $\dim \mathcal{E}_\pi^\ell(G/H) = d_\pi$.
(2) It is straightforward.
(3) Let $1 \le i, i' \le d_\pi$. Applying Theorem 27.19 of [10] we get

$$\langle \pi_{i\ell}, \pi_{i'\ell'}\rangle_{L^2(G/H,\mu)} = \langle \pi_{i\ell}, \pi_{i'\ell'}\rangle_{L^2(G)} = d_\pi^{-1}\delta_{ii'}\delta_{\ell\ell'},$$

which completes the proof. □

The following theorem shows that H-admissible bases lead to orthogonal decompositions of the subspace $\mathcal{E}_\pi(G/H)$.

Theorem 4.3. *Let H be a closed subgroup of a compact group G. Let $[\pi] \in \widehat{G/H}$ and $\mathfrak{B}_\pi = \{e_{\ell,\pi} : 1 \le \ell \le d_\pi\}$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$. Then $\mathfrak{B}_\pi(G/H) := \{\sqrt{d_\pi}\pi_{i\ell} : 1 \le i \le d_\pi, 1 \le \ell \le d_{\pi,H}\}$ is an orthonormal basis for the Hilbert space $\mathcal{E}_\pi(G/H)$ and hence it satisfies the following direct sum decomposition*

$$\mathcal{E}_\pi(G/H) = \bigoplus_{\ell=1}^{d_{\pi,H}} \mathcal{E}_\pi^\ell(G/H). \tag{4.11}$$

Proof. It is straightforward to check that $\mathfrak{B}_\pi(G/H)$ spans the subspace $\mathcal{E}_\pi(G/H)$. Then Proposition 4.2 guarantees that $\mathfrak{B}_\pi(G/H)$ is an orthonormal set in $\mathcal{E}_\pi(G/H)$. Since $\dim \mathcal{E}_\pi(G/H) \leq d_{\pi,H} d_\pi$ we deduce that it is an orthonormal basis for $\mathcal{E}_\pi(G/H)$, which automatically implies the decomposition (4.11). □

Next proposition lists basic properties of H-matrix elements.

Proposition 4.3. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G. Then,*

(1) *$T_H^\pi = 0$ if and only if $\mathcal{E}_\pi(G) \subseteq \mathcal{J}^2(G,H)$.*
(2) *If $T_H^\pi \neq 0$ then $T_H(\mathcal{E}_\pi(G)) = \mathcal{E}_\pi(G/H)$ and $T_H^*(\mathcal{E}_\pi(G/H)) \subseteq \mathcal{E}_\pi(G)$.*
(3) *$\mathcal{E}_\pi(G) \subseteq \mathcal{J}^2(G,H)^\perp$ if and only if $\pi(h) = I$ for all $h \in H$.*

Then we can prove the following orthogonality relation concerning the functions in $\mathcal{E}(G/H)$.

Theorem 4.4. *Let H be a closed subgroup of a compact group G, μ be a normalized G-invariant measure on G/H and $[\pi] \neq [\sigma] \in \widehat{G/H}$. The closed subspaces $\mathcal{E}_\pi(G/H)$ and $\mathcal{E}_\sigma(G/H)$ are orthogonal to each other as subspaces of the Hilbert space $L^2(G/H,\mu)$.*

Proof. Let $\psi \in \mathcal{E}_\pi(G/H)$ and $\varphi \in \mathcal{E}_\sigma(G/H)$. Then we have $\psi_q \in \mathcal{E}_\pi(G)$ and also $\varphi_q \in \mathcal{E}_\sigma(G)$. Using Proposition 4.3, Corollary 3.3, and Theorem 27.15 of [10], we get

$$\langle \varphi, \psi \rangle_{L^2(G/H,\mu)} = \langle \varphi_q, \psi_q \rangle_{L^2(G)} = 0.$$

which completes the proof. □

We can define

$$\mathcal{E}(G/H) := \text{the linear span of} \bigcup_{[\pi] \in \widehat{G/H}} \mathcal{E}_\pi(G/H). \tag{4.12}$$

Next theorem presents some analytic aspects of the function space $\mathcal{E}(G/H)$.

Theorem 4.5. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then,*

(1) *The linear operator T_H maps $\mathcal{E}(G)$ onto $\mathcal{E}(G/H)$.*
(2) *$\mathcal{E}(G/H)$ is $\|.\|_{L^2(G/H,\mu)}$-dense in $L^2(G/H,\mu)$.*
(3) *$\mathcal{E}(G/H)$ is $\|.\|_{\sup}$-dense in $\mathcal{C}(G/H)$.*

Proof. (1) It is straightforward.
(2) Let $\phi \in L^2(G/H,\mu)$ and also $f \in L^2(G)$ with $T_H(f) = \phi$. Then by $\|.\|_{L^2(G)}$-density of $\mathcal{E}(G)$ in $L^2(G)$ we can pick a sequence $\{f_n\}$ in $\mathcal{E}(G)$ such that $f = \|.\|_{L^2(G)} - \lim_n f_n$. By Proposition 4.3 we have $\{T_H(f_n)\} \subseteq \mathcal{E}(G/H)$. Then continuity of the linear map $T_H : L^2(G) \to L^2(G/H,\mu)$ implies

$$\phi = T_H(f) = \|.\|_{L^2(G/H,\mu)} - \lim_n T_H(f_n),$$

which completes the proof.
(3) Invoking uniformly boundedness of T_H, uniformly density of $\mathcal{E}(G)$ in $\mathcal{C}(G)$, and the same argument as used in (1), we get $\|.\|_{\sup}$-density of $\mathcal{E}(G/H)$ in $\mathcal{C}(G/H)$. □

The following theorem can be considered as an abstract extension of the Peter-Weyl Theorem for homogeneous spaces of compact groups.

Theorem 4.6. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The Hilbert space $L^2(G/H,\mu)$ satisfies the following orthogonality decomposition*

$$L^2(G/H,\mu) = \bigoplus_{[\pi] \in \widehat{G/H}} \mathcal{E}_\pi(G/H). \tag{4.13}$$

Proof. Using Peter-Weyl Theorem, Proposition 4.3, and since the bounded linear map $T_H : L^2(G) \to L^2(G/H,\mu)$ is surjective we achieve that each $\varphi \in L^2(G/H,\mu)$ has a decomposition to elements of $\mathcal{E}_\pi(G/H)$ with $[\pi] \in \widehat{G/H}$, namely

$$\varphi = \sum_{[\pi] \in \widehat{G/H}} c_\pi \varphi_\pi, \tag{4.14}$$

with $\varphi_\pi \in \mathcal{E}_\pi(G/H)$ for all $[\pi] \in \widehat{G/H}$. Since the subspaces $\mathcal{E}_\pi(G/H)$ with $[\pi] \in \widehat{G/H}$ are mutually orthogonal we conclude that decomposition (4.14) is unique for each φ, which guarantees (4.13). $\square$

We immediately deduce the following corollaries.

Corollary 4.1. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. For each $[\pi] \in \widehat{G/H}$, let $\mathfrak{B}_\pi = \{e_{\ell,\pi} : 1 \leq \ell \leq d_\pi\}$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$. Then we have the following statements.*

(1) *The Hilbert space $L^2(G/H, \mu)$ satisfies the following direct sum decomposition*

$$L^2(G/H,\mu) = \bigoplus_{[\pi]\in\widehat{G/H}} \bigoplus_{\ell=1}^{d_{\pi,H}} \mathcal{E}_\pi^\ell(G/H), \tag{4.15}$$

(2) *The set $\mathfrak{B}(G/H) := \{\pi_{i\ell} : 1 \leq i \leq d_\pi, 1 \leq \ell \leq d_{\pi,H}\}$ constitutes an orthonormal basis for the Hilbert space $L^2(G/H,\mu)$.*

(3) *Each $\varphi \in L^2(G/H,\mu)$ decomposes as the following*

$$\varphi = \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{\ell=1}^{d_{\pi,H}} \sum_{i=1}^{d_\pi} \langle \varphi, \pi_{i\ell}\rangle_{L^2(G/H,\mu)} \pi_{i\ell}, \tag{4.16}$$

where the series is converges in $L^2(G/H,\mu)$.

Remark 4.2. *Let H be a closed normal subgroup of a compact group G. Also, let μ be the normalized G-invariant measure over G/H associated to the Weil's formula. Then G/H is a compact group and the normalized G-invariant measure μ is a Haar measure of the quotient compact group G/H. By Theorem 4.1, we deduce that $\widehat{G/H} = H^\perp$, and for each $[\pi] \in \widehat{G/H}$ we get $T_H^\pi = I$ and $d_{\pi,H} = d_\pi$. Thus we obtain*

$$L^2(G/H) = \bigoplus_{[\pi]\in H^\perp} \mathcal{E}_\pi(G/H),$$

which precisely coincides with the decomposition associated to applying the Peter-Weyl Theorem to the compact quotient group G/H.

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