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POSITIVE SOLUTIONS FOR A SINGULAR SUM FRACTIONAL DIFFERENTIAL SYSTEM

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Abstract. We investigate the existence of positive solutions for a singular sum fractional differential system under some boundary conditions by providing different conditions. Also, we give an example to illustrate one of our main results.

1. Preliminaries

It has been published many works on the existence of solutions for different singular fractional differential systems (see for example, [2], [3], [6], [7], [10] and [12]). In 2010, the existence of positive solutions for the singular Dirichlet problem

$$D^{\alpha}u(t) + f(t, u(t), D^{\mu}u(t)) = 0$$

with boundary conditions $u(0) = u(1) = 0$ is investigated, where $1 < \alpha < 2$, $0 < \mu \leq \alpha - 1$, f is a Caratheodory function on $[0, 1] \times (0, \infty) \times \mathbb{R}$ and D^{α} is Riemann-Liouville fractional derivative ([1]). In 2013, the singular problem

$$D^{\alpha}u + f(t, u, D^{\gamma}u, D^{\mu}u) + g(t, u, D^{\gamma}u, D^{\mu}u) = 0$$

with boundary conditions $u(0) = u'(0) = u''(0) = u'''(0) = 0$ is reviewed, where $3 < \alpha < 4$, $0 < \gamma < 1$, $1 < \mu < 2$, D^{α} is the Caputo fractional derivative and f is a Caratheodory function on $[0, 1] \times (0, \infty)^3$ ([4]).

By using main idea of [1] and [4], we investigate positive solutions for the singular differential system of equations

$$\begin{cases} D^{\alpha_1}u_1 + f_1(t, u_1, \dots, u_m, D^{\mu_1}u_1, \dots, D^{\mu_m}u_m) + g_1(t, u_1, \dots, u_m, D^{\mu_1}u_1, \dots, D^{\mu_m}u_m) = 0, \\ \dots\dots \\ D^{\alpha_m}u_m + f_m(t, u_1, \dots, u_m, D^{\mu_1}u_1, \dots, D^{\mu_m}u_m) + g_m(t, u_1, \dots, u_m, D^{\mu_1}u_1, \dots, D^{\mu_m}u_m) = 0, \end{cases} \tag{1.1}$$

with boundary conditions $u_i(0) = 0$, $u_i'(1) = 0$ and $\frac{d^k}{dt^k}[u_i(t)]_{t=0} = 0$ for $1 \leq i \leq m$ and $2 \leq k \leq n-1$, where $\alpha_i \geq 2$, $[\alpha_i] = n - 1$, $0 < \mu_i < 1$, D is the Caputo fractional derivative, f_i is a Caratheodory function, g_i satisfies Lipschitz condition and $f_i(t, x_1, \dots, x_{2m})$ is singular at $t = 0$ of for all $1 \leq i \leq m$.

We say that a map $f\colon [0, 1] \times D \subseteq [0, 1] \times D \to \mathbb{R}^n$ is Caratheodory whenever the function $t \mapsto f(t, x_1, \dots, x_n)$ is measurable for all $(x_1, \dots, x_n) \in D$ and $(x_1, \dots, x_n) \mapsto f(t, x_1, \dots, x_n)$ is continuous for almost all $t \in [0, 1]$ and for each compact $K \subseteq D$ there exists $\varphi_K \in L^1[0, 1]$ such that $|f(t, x_1, \dots, x_n)| \leq \varphi_K(t)$ for almost all $t \in [0, 1]$ and $(x_1, \dots, x_n) \in K$.

Put

$$\|x\|_1 = \int_0^1 |x(t|dt, \quad \|x\| = \sup\{|x(t)| : t \in [0, 1]\}, \quad \|(x_1, \dots, x_n)\|_* = \max\{\|x_1\|, \dots, \|x_n\|\},$$

$$\|(x_1, \dots, x_n)\|_{**} = \max\{\|x_1\|, \dots, \|x_n\|, \|x_1'\|, \dots, \|x_n'\|\}, \quad Y = C_{\mathbb{R}}([0, 1]), \quad X = C^1_{\mathbb{R}}([0, 1]).$$

By considering the problem (1.1), we assume the following hypotheses:

H1: $f_1, \dots, f_m$ are Caratheodory functions on $[0, 1] \times (0, \infty)^{2m}$ and there exists positive constants $m_1, \dots, m_m$ such that

$$f_i(t, x_1, \dots, x_{2m}) \geq m_i$$

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for almost all $t \in [0,1]$ and all $(x_1, \dots, x_{2m}) \in D := (0,\infty)^{2m}$.

H2: $g_1, \dots, g_m$ are nonnegative and

$$|g_i(t, x_1, \dots, x_{2m}) - g_i(t, y_1, \dots, y_{2m})| \le \sum_{k=1}^{2m} L_k^i |x_k - y_k|,$$

for almost all $t \in [0,1]$ and for all $(x_1, \dots, x_{2m}), (y_1, \dots, y_{2m}) \in D$, where $L_1^1, \dots, L_1^m, \dots, L_{2m}^1, \dots, L_{2m}^m$ in $[0,\infty)$ are constants such that

$$\frac{1}{\Gamma(\alpha_i - 1)}\Big(\sum_{k=1}^{m} L_k^i + \sum_{k=1}^{m} \frac{L_{m+k}^i}{\Gamma(2-\mu_i)}\Big) < 1.$$

H3: there exist some maps $\gamma_1, \dots, \gamma_m \in L^1([0,1])$, some non-increasing maps $p_1, \dots, p_m \in C_{\mathbb{R}}(D)$ with

$$\int_0^1 p_i\Big(M_1 t^{\alpha_1}, \dots, M_m t^{\alpha_m}, \frac{M_1(1-\mu_1)}{2} t^{1-\mu_1}, \dots, \frac{M_m(1-\mu_m)}{2} t^{1-\mu_m}\Big) dt < \infty$$

and some functions $h_1, \dots, h_m \in C_{\mathbb{R}}([0,\infty)^{2m})$ such that $\lim_{x\to\infty} \frac{h_i(x,\dots,x)}{x} = 0$, h_i is nondecreasing in all components and

$$f_i(t, x_1, \dots, x_{2m}) + g_i(t, x_1, \dots, x_{2m}) \le p_i(x_1, \dots, x_{2m}) + \gamma_i(t) h_i(x_1, \dots, x_{2m}),$$

for almost all $t \in [0,1]$ and all $(x_1, \dots, x_{2m}) \in D$, where $M_i = m_i \frac{\alpha_i - 1}{\Gamma(\alpha_i+1)}$ for all $1 \le i \le m$.

Now for each $1 \le i \le m$ and $n \ge 1$, put $f_{i,n}(t, x_1, \dots, x_{2m}) = f_i(t, \chi_1(x_1), \dots, \chi_n(x_{2m}))$, where $\chi_n(u) = u$ whenever $u \ge \frac{1}{n}$ and $\chi_n(u) = u$ whenever $u < \frac{1}{n}$.

It is easy to check that

$$f_{i,n}(t, x_1, \dots, x_{2m}) + g_i(t, x_1, \dots, x_{2m}) \le p_i\Big(\frac{1}{n}, \dots, \frac{1}{n}\Big) + \gamma_i(t) h_i\Big(x_1 + \frac{1}{n}, \dots, x_{2m} + \frac{1}{n}\Big),$$

$$f_{i,n}(t, x_1, \dots, x_{2m}) \ge m_i$$

and

$$f_{i,n}(t, x_1, \dots, x_{2m}) + g_i(t, x_1, \dots, x_{2m}) \le p_i(x_1, \dots, x_{2m}) + \gamma_i(t) h_i\Big(x_1 + \frac{1}{n}, \dots, x_{2m} + \frac{1}{n}\Big),$$

for all $(x_1, \dots, x_n) \in D$, $1 \le i \le m$ and almost all $t \in [0,1]$.

First, we investigate the regular fractional differential system

$$\begin{cases} D^{\alpha_1} u_1 + f_{1,n}(t, u_1, \dots, u_m, D^{\mu_1} u_1, \dots, D^{\mu_m} u_m) = 0 \\ \dots\dots \\ D^{\alpha_m} u_m + f_{m,n}(t, u_1, \dots, u_m, D^{\mu_1} u_1, \dots, D^{\mu_m} u_m) = 0, \end{cases} \tag{1.2}$$

with same boundary conditions in (1.1).

Now, we present some necessary notions.

According to [5], the Riemann-Liouville integral of order p for a function $f\colon (0,\infty) \to \mathbb{R}$ is

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds$$

if the right-hand side map is defined pointwise on $(0,\infty)$.

The Caputo fractional derivative of order $\alpha > 0$ for a function $f\colon (a,\infty) \to \mathbb{R}$ is defined by

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $n = [\alpha] + 1$; please, see [5].

Lemma 1.1 ([8]). *If $x \in C_{\mathbb{R}}[0,1] \cap L^1[0,1]$, then $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0, c_1, \dots, c_{n-1}$, where $0 < n-1 \le \alpha < n$.*

It has been proved in [11] that

$$\int_0^t (t-s)^{\alpha-1} s^\beta ds = B(\beta+1,\alpha)t^{\alpha+\beta}, \quad B(\beta,\alpha) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \beta > 0, \quad \alpha > -1.$$

We need the next result.

Lemma 1.2 ([9]). *Let M be a closed, convex and nonempty subset of a Banach space X, A a compact and continuous operator and B a contraction. Then, there exist $z \in M$ such that $z = Az + Bz$.*

Lemma 1.3. *Let $y \in L^1[0,1]$, $\alpha \geq 2$ and $n = [\alpha]+1$. Then, the unique solution of the equation $D^\alpha u(t) + y(t) = 0$ with boundary conditions $u'(1) = u(0) = u''(0) = \cdots = u^{n-1}(0) = 0$ is $u(t) = \int_0^1 G_\alpha(t,s)y(s)ds$, where $t \in [0,1]$ and*

$$G_\alpha(t,s) = \begin{cases} \dfrac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq t \leq s \leq 1 \\[2ex] \dfrac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Proof. By Using Lemma 1.1 and the boundary conditions, we get

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)ds + c_1 t$$

and so $u'(1) = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)ds + c_1$. Since $u'(1) = 0$, $c_1 = \frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds$. Thus, $u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)ds$. Hence, we conclude that $u(t) = \int_0^1 G_\alpha(t,s)y(s)ds$, where $G_\alpha(t,s) = \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ whenever $0 \leq s \leq t \leq 1$ and $G_\alpha(t,s) = \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ whenever $0 \leq t \leq s \leq 1$. □

Consider the Green function $G_\alpha(t,s)$ as in Lemma 1.3.

If $0 < t \leq s < 1$, then it is clear that $G_\alpha(t,s) > 0$.

If $0 < s < t < 1$, then $\frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} > 0$ if and only if $\alpha - 1 > \frac{(t-s)^{\alpha-1}}{t(1-s)^{\alpha-2}}$ and so $G_\alpha(t,s) > 0$. One can check that $G_\alpha(t,s) > 0$, $G_\alpha(t,s) \leq \frac{1}{\Gamma(\alpha-1)}$ and $\int_0^1 G_\alpha(t,s)ds \leq \frac{1}{\Gamma(\alpha)}$, for all $t,s \in (0,1)$. Also, $\int_0^1 G_\alpha(t,s)ds \geq \frac{t}{\Gamma(\alpha)} - \frac{t^\alpha}{\Gamma(\alpha+1)} \geq \frac{t^\alpha(\alpha-1)}{\Gamma(\alpha+1)}$ and $\frac{\partial}{\partial t}G_\alpha(t,s) > 0$ for all $t,s \in (0,1)$. Moreover, $G_\alpha, \frac{\partial}{\partial t}G_\alpha \in C_{\mathbb{R}}([0,1]\times[0,1])$, $\frac{\partial}{\partial t}G_\alpha(t,s) \leq \frac{1}{\Gamma(\alpha-1)}$, for all $t,s \in [0,1]$ and $\int_0^1 \frac{\partial}{\partial t}G_\alpha(t,s) \geq \frac{1-t^{\alpha-1}}{\Gamma(\alpha)}$, for all $t \in [0,1]$.

Suppose that $x \in C^1_{\mathbb{R}}[0,1]$ and $0 \leq \mu \leq 1$. Since $D^\mu x(t) = \frac{1}{\Gamma(2-\mu)}\int_0^t (t-s)^{-\mu}x'(s)ds$ for all $0 \leq t \leq 1$, $|D^\mu x| \leq \frac{\|x'\|}{\Gamma(1-\mu)}\int_0^t (t-s)^{-\mu}ds = \frac{\|x'\|}{\Gamma(2-\mu)}t^{1-\mu}$ and so $|D^\mu x| \leq \frac{\|x'\|}{\Gamma(2-\mu)}$ and $D^\mu x \in C_{\mathbb{R}}[0,1]$.

Now, put

$$P = \{(x_1,\ldots,x_m) \in X^m : x_i(t) \geq 0 \text{ and } x_i'(t) \geq 0 \text{ for all } t \in [0,1] \text{ and } 0 \leq i \leq m\}.$$

For each $n \geq 1$ and $0 \leq i \leq m$, define the maps

$$\Phi_{i,n}(x_1,\ldots,x_m)(t) = \int_0^1 G_{\alpha_i}(t,s) f_{i,n}(s, x_1(s),\ldots,x_m(s), D^{\mu_1}x_1(s),\ldots,D^{\mu_m}x_m(s))ds,$$

and

$$\Psi_i(x_1,\ldots,x_m)(t) = \int_0^1 G_{\alpha_i}(t,s) g_i(s, x_1(s),\ldots,x_m(s), D^{\mu_1}x_1(s),\ldots,D^{\mu_m}x_m(s))ds,$$

$$T_n(x_1,\ldots,x_m)(t) = \begin{pmatrix} \Phi_{1,n}(x_1,\ldots,x_m)(t) \\ \vdots \\ \Phi_{m,n}(x_1,\ldots,x_m)(t) \end{pmatrix}$$

and $\Psi(x_1,\ldots,x_m)(t) = \begin{pmatrix} \Psi_1(x_1,\ldots,x_m)(t) \\ \vdots \\ \Psi_m(x_1,\ldots,x_m)(t) \end{pmatrix}$ for all $(x_1,\ldots,x_m) \in P$.

Lemma 1.4. *The map $\Psi\colon P \to P$ is a contraction.*

Proof. It is easy to check that $\Psi_i(x_1, \dots, x_m)(t) \geq 0$ and

$$\Psi_i'(x_1, \dots, x_m)(t) = \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t,s) g_i(s, x_1(s), \dots, x_m(s), D^{\mu_1} x_1(s), \dots, D^{\mu_m} x_m(s)) ds \geq 0$$

for all $t \in [0,1]$, $(x_1, \dots, x_m) \in P$ and $0 \leq i \leq m$.

Note that,

$$\begin{aligned}
&\|\Psi_i(x_1, \dots, x_m) - \Psi_i(y_1, \dots, y_m)\| \\
&= \sup_{t\in[0,1]} \Big| \int_0^1 G_{\alpha_i}(t,s)[g_i(s, x_1(s), \dots, x_m(s), D^{\mu_1} x_1(s), \dots, D^{\mu_m} x_m(s)) \\
&\quad - g_i(s, y_1(s), \dots, y_m(s), D^{\mu_1} y_1(s), \dots, D^{\mu_m} y_m(s))] ds \Big| \\
&\leq \Big| \int_0^1 G_{\alpha_i}(t,s) ds \Big| (L_1^i \|x_1 - y_1\| + \cdots + L_m^i \|x_m - y_m\| \\
&\quad + L_{m+1}^i \|D^{\mu_1} x_1 - D^{\mu_1} y_1\| + \cdots + L_{2m}^i \|D^{\mu_m} x_m - D^{\mu_m} y_m\|) \\
&\leq \frac{1}{\Gamma(\alpha_i)} (L_1^i \|x_1 - y_1\| + \cdots + L_m^i \|x_m - y_m\| \\
&\quad + \frac{L_{m+1}^i}{\Gamma(2-\mu_1)} \|x_1' - y_1'\| + \cdots + \frac{L_{2m}^i}{\Gamma(2-\mu_m)} \|x_m' - y_m'\|) \\
&\leq \frac{1}{\Gamma(\alpha_i)} \Big(\sum_{k=1}^m L_k^i + \sum_{k=1}^m \frac{L_{m+k}^i}{\Gamma(2-\mu_i)} \Big) \|(x_1, \dots, x_m) - (y_1, \dots, y_m)\|_{**} \\
&\leq \frac{1}{\Gamma(\alpha_i - 1)} \left(\sum_{k=1}^m L_k^i + \sum_{k=1}^m \frac{L_{m+k}^i}{\Gamma(2-\mu_i)} \right) \|(x_1, \dots, x_m) - (y_1, \dots, y_m)\|_{**}
\end{aligned}$$

for all $0 \leq i \leq m$ and so

$$\begin{aligned}
&\|\Psi(x_1, \dots, x_m) - \Psi(y_1, \dots, y_m)\|_* = \max_{1\leq i\leq m} \|\Psi_i(x_1, \dots, x_m) - \Psi_i(y_1, \dots, y_m)\| \\
&\leq \max_{1\leq i\leq m} \left\{ \frac{1}{\Gamma(\alpha_i - 1)} \left(\sum_{k=1}^m L_k^i + \sum_{k=1}^m \frac{L_{m+k}^i}{\Gamma(2-\mu_i)} \right) \right\} \|(x_1, \dots, x_m) - (y_1, \dots, y_m)\|_{**}.
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}
&\|\Psi'(x_1, \dots, x_m) - \Psi'(y_1, \dots, y_m)\|_* \\
&\leq \max_{1\leq i\leq m} \left\{ \frac{1}{\Gamma(\alpha_i - 1)} \left(\sum_{k=1}^m L_k^i + \sum_{k=1}^m \frac{L_{m+k}^i}{\Gamma(2-\mu_i)} \right) \right\} \|(x_1, \dots, x_m) - (y_1, \dots, y_m)\|_{**}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\|\Psi(x_1, \dots, x_m) - \Psi(y_1, \dots, y_m)\|_{**} \\
&\leq \max_{1\leq i\leq m} \left\{ \frac{1}{\Gamma(\alpha_i - 1)} \left(\sum_{k=1}^m L_k^i + \sum_{k=1}^m \frac{L_{m+k}^i}{\Gamma(2-\mu_i)} \right) \right\} \|(x_1, \dots, x_m) - (y_1, \dots, y_m)\|_{**}.
\end{aligned}$$

Since $\max_{1\leq i\leq m} \left\{ \frac{1}{\Gamma(\alpha_i-1)} \left(\sum_{k=1}^m L_k^i + \sum_{k=1}^m \frac{L_{m+k}^i}{\Gamma(2-\mu_i)} \right) \right\} < 1$, Ψ is a contraction mapping. □

Lemma 1.5. *For each $n \geq 1$, T_n is a completely continuous operator on P.*

Proof. Let $n \geq 1$, $(x_1, \ldots, x_m) \in P$ and $1 \leq i \leq m$. Choose a positive constant m_i such that

$$f_{i,n}(t, x_1(t), \ldots, x_m(t), D^{\mu_1} x_1(t), \ldots, D^{\mu_m} x_m(t)) \geq m_i$$

for almost all $t \in [0,1]$.

Since G_{α_i} and $\frac{\partial}{\partial t} G_{\alpha_i}$ are nonnegative and continuous on $[0,1] \times [0,1]$ for all $1 \leq i \leq m$, we conclude that $\Phi_{i,n}(x_1, \ldots, x_m)(t) \geq 0$ and $(\Phi_{i,n}(x_1, \ldots, x_m))'(t) \geq 0$ for all $t \in [0,1]$ and $1 \leq i \leq m$. Hence, T_n maps P into P.

Now, we prove that T_n is continuous.

Let $\{(x_{1,k}, \ldots, x_{m,k})\}$ be a convergent sequence in P with $\lim_{k\to\infty}(x_{1,k}, \ldots, x_{m,k}) = (x_1, \ldots, x_m)$. In this case, we get $\lim_{k\to\infty} x_{i,k} = x_i$ and $\lim_{k\to\infty} x'_{i,k} = x'_i$ uniformly on $[0,1]$ $(i = 1, 2, \ldots, m)$. But, $|D^{\mu_i} x_{i,k}(t) - D^{\mu_i} x_i(t)| \leq \frac{\|x'_{i,k} - x'_i\|}{\Gamma(2-\mu_i)}$ for all $t \in [0,1]$ and $1 \leq i \leq m$. Thus, we conclude that $\lim_{k\to\infty} D^{\mu_i} x_{i,k}(t) = D^{\mu_i} x_i(t)$ uniformly on $[0,1]$. Hence,

$$\lim_{k\to\infty} f_{i,n}(t, x_{1,k}(t), \ldots, x_{m,k}(t), D^{\mu_1} x_{1,k}(t), \ldots, D^{\mu_m} x_{m,k}(t))$$
$$= f_{i,n}(t, x_1(t), \ldots, x_m(t), D^{\mu_1} x_1(t), \ldots, D^{\mu_m} x_m(t)).$$

Since $f_{i,n} \in Car([0,1] \times \mathbb{R}^{2m})$, $\{(x_{1,k}, \ldots, x_{m,k})\}$ is bounded in X_m there exist a map $\varphi_i \in L^1[0,1]$ such that

$$m_i \leq f_{i,n}(t, x_{1,k}(t), \ldots, x_{m,k}(t), D^{\mu_1} x_{1,k}(t), \ldots, D^{\mu_m} x_{m,k}(t)) \leq \varphi_i(t) \tag{1.3}$$

for almost all $t \in [0,1]$, $1 \leq i \leq m$ and $k \geq 1$.

By using the Lebesgue dominated convergence theorem, we conclude that

$$|\Phi_{i,n}(x_{1,k}, \ldots, x_{m,k})(t) - \Phi_{i,n}(x_1, \ldots, x_m)(t)|$$
$$\leq \frac{1}{\Gamma(\alpha_i)} \int_0^1 |f_{i,n}(s, x_{1,k}(s), \ldots, x_{m,k}(s), D^{\mu_1} x_{1,k}(s), \ldots, D^{\mu_m} x_{m,k}(s))$$
$$- f_{i,n}(s, x_1(s), \ldots, x_m(s), D^{\mu_1} x_1(s), \ldots, D^{\mu_m} x_m(s))| ds,$$

and

$$|(\Phi_{i,n}(x_{1,k}, \ldots, x_{m,k}))'(t) - (\Phi_{i,n}(x_1, \ldots, x_m))'(t)|$$
$$\leq \frac{1}{\Gamma(\alpha_i - 1)} \int_0^1 |f_{i,n}(s, x_{1,k}(s), \ldots, x_{m,k}(s), D^{\mu_1} x_{1,k}(s), \ldots, D^{\mu_m} x_{m,k}(s))$$
$$- f_{i,n}(s, x_1(s), \ldots, x_m(s), D^{\mu_1} x_1(s), \ldots, D^{\mu_m} x_m(s))| ds.$$

Hence, $\lim_{k\to\infty} \left| (\Phi_{i,n}(x_{1,k}, \ldots, x_{m,k}))^j (t) - (\Phi_{i,n}(x_1, \ldots, x_m))^j (t) \right| = 0$ uniformly on $[0,1]$ for $j = 0, 1$. Thus, $\|T_n(x_{1,k}, \ldots, x_{m,k})(t) - T_n(x_1, \ldots, x_m)(t)\|_{**} \to 0$ and so T_n is continuous.

Now, we prove that T_n maps bounded sets to relatively compact subsets.

Let $\{(x_{1,k}, \ldots, x_{m,k})\}$ be a bounded sequence in P. Choose a positive number S such that $\|x_{i,k}\| \leq S$ and $\left\| x'_{i,k} \right\| \leq S$ for all $1 \leq i \leq m$ and for $k \geq 1$. Since $\|D^{\mu_i} x_{i,k}\| \leq \frac{1}{\Gamma(2-\mu_i)}$ for all $1 \leq i \leq m$, there exist a map $\varphi_i \in L^1[0,1]$ such that (1.3) holds for almost all $t \in [0,1]$, $1 \leq i \leq m$ and $k \geq 1$.

Note that

$$0 \leq \Phi_{i,n}(x_{1,k}, \ldots, x_{m,k})(t)$$
$$= \int_0^1 G_{\alpha_i}(t,s) f_{i,n}(s, x_{1,k}(s), \ldots, x_{m,k}(s), D^{\mu_1} x_{1,k}(s), \ldots, D^{\mu_m} x_{m,k}(s)) ds$$
$$\leq \frac{1}{\Gamma(\alpha_i)} \int_0^1 \varphi_i(s) ds = \frac{\|\varphi_i\|_1}{\Gamma(\alpha_i)}$$

and

$$0 \leq (\Phi_{i,n}(x_{1,k}, \ldots, x_{m,k}))'(t)$$
$$= \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t,s) f_{i,n}(s, x_{1,k}(s), \ldots, x_{m,k}(s), D^{\mu_1} x_{1,k}(s), \ldots, D^{\mu_m} x_{m,k}(s)) ds$$
$$\leq \frac{1}{\Gamma(\alpha_i - 1)} \int_0^1 \varphi_i(s) ds = \frac{\|\varphi_i\|_1}{\Gamma(\alpha_i - 1)}$$

for all $1 \leq i \leq m$. Thus, $\|T_n(x_{1,k},\dots,x_{m,k})(t)\|_{**} \leq B$, where $B = \max_{1\leq i\leq m} \frac{\|\varphi_i\|_1}{\Gamma(\alpha_i-1)}$. This implies that $\{T_n(x_{1,k},\dots,x_{m,k})\}$ is bounded in X^m. Let $0 \leq t_1 \leq t_2 \leq 1$ and $1 \leq i \leq m$. Then, we have

$$\begin{aligned}
&\left|(\Phi_{i,n}(x_{1,k},\dots,x_{m,k}))'(t_2) - (\Phi_{i,n}(x_{1,k},\dots,x_{m,k}))'(t_1)\right| \\
&\leq \frac{t_2-t_1}{\Gamma(\alpha_i-1)}\int_0^1 (1-s)^{\alpha_i-2} f_{i,n}(s,x_{1,k}(s),\dots,x_{m,k}(s),D^{\mu_1}x_{1,k}(s),\dots,D^{\mu_m}x_{m,k}(s))ds \\
&+\frac{1}{\Gamma(\alpha_i)}\Big|\int_0^{t_2}(t_2-s)^{\alpha_i-1} f_{i,n}(s,x_{1,k}(s),\dots,x_{m,k}(s),D^{\mu_1}x_{1,k}(s),\dots,D^{\mu_m}x_{m,k}(s))ds \\
&-\int_0^{t_1}(t_1-s)^{\alpha_i-1} f_{i,n}(s,x_{1,k}(s),\dots,x_{m,k}(s),D^{\mu_1}x_{1,k}(s),\dots,D^{\mu_m}x_{m,k}(s))ds\Big| \\
&\frac{\|f_{i,n}\|_1}{\Gamma(\alpha_i-1)}(t_2-t_1) + \frac{1}{\Gamma(\alpha_i)}[\int_0^{t_1}((t_2-s)^{\alpha_i-1}-(t_1-s)^{\alpha_i-1})\times \\
&f_{i,n}(s,x_{1,k}(s),\dots,x_{m,k}(s),D^{\mu_1}x_{1,k}(s),\dots,D^{\mu_m}x_{m,k}(s))ds \\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha_i-2} f_{i,n}(s,x_{1,k}(s),\dots,x_{m,k}(s),D^{\mu_1}x_{1,k}(s),\dots,D^{\mu_m}x_{m,k}(s))ds] \\
&\leq \frac{\|\varphi_i\|_1}{\Gamma(\alpha_i-1)}(t_2-t_1) + \frac{1}{\Gamma(\alpha_i)}\left[\int_0^{t_1}\left((t_2-s)^{\alpha_i-1}-(t_1-s)^{\alpha_i-1}\right)\varphi_i(s)ds + (t_2-t_1)^{\alpha_i-1}\|\varphi_i\|_1\right].
\end{aligned}$$

Let $\varepsilon > 0$ be given. Since the function $|t-s|^{\alpha_i-1}$ is uniformly continuous on $[0,1]\times[0,1]$, there exist $\delta > 0$ such that $(t_2-s)^{\alpha_i-1}-(t_1-s)^{\alpha_i-1} < \varepsilon$ for all $0 \leq t_1 \leq t_2 \leq 1$ with $t_2 - t_1 < \delta$ and $0 \leq s \leq t_1$. If $0 \leq t_1 \leq t_2 \leq 1$ with $t_2 - t_1 < \min\{\delta, \varepsilon\}$, then we have

$$\left|(\Phi_{i,n}(x_{1,k},\dots,x_{m,k}))'(t_2) - (\Phi_{i,n}(x_{1,k},\dots,x_{m,k}))'(t_1)\right| < \frac{3\varepsilon\|\varphi_i\|_1}{\Gamma(\alpha_i)}.$$

Thus,

$$\|T_n'(x_{1,k},\dots,x_{m,k})(t_2) - T_n'(x_{1,k},\dots,x_{m,k})(t_1)\|_* < \max_{1\leq i\leq m}\frac{3\varepsilon\|\varphi_i\|_1}{\Gamma(\alpha_i)}.$$

This implies that $\{T_n'(x_{1,k},\dots,x_{m,k})\}$ is equi-continuous on $[0,1]$. Now by using the Arzela-Ascoli theorem, $\{T_n(x_{1,k},\dots,x_{m,k})\}$ is relatively compact and so T_n is completely continuous. □

2. Main Results

Now, we are ready to provide our main results about the problem (1.1).

Theorem 2.1. *Assume that hypotheses* H1 *and* H2 *hold. Then, the problem* (1.2) *with the boundary conditions in* (1.1) *has a solution* $(x_{1,n},\dots,x_{m,n})$ *in* P *such that* $x_{i,n}(t) \geq \frac{m_i t^{\alpha_i}(\alpha_i-1)}{\Gamma(\alpha_i+1)}$, *for all* $t \in [0,1]$ *and* $1 \leq i \leq m$.

Proof. By using Lemma 1.4, the mapping $\Psi\colon P \to P$ is a contraction. Also by using Lemma 1.5, the operator $T_n\colon P \to P$ is a completely continuous one. Now by using Lemma 1.2, there exists $(x_{1,n},\dots,x_{m,n}) \in P$ such that $(x_{1,n},\dots,x_{m,n}) = T_n(x_{1,n},\dots,x_{m,n}) + \Psi(x_{1,n},\dots,x_{m,n})$. Thus, $x_{i,n} = \Phi_{i,n}(x_{1,n},\dots,x_{m,n}) + \Psi_i(x_{1,n},\dots,x_{m,n})$ for all $1 \leq i \leq m$. Hence,

$$\begin{aligned}
x_{i,n}(t) &= \int_0^1 G_{\alpha_i}(t,s) f_{i,n}(s,x_1(s),\dots,x_m(s),D^{\mu_1}x_1(s),\dots,D^{\mu_m}x_m(s))ds \\
&+\int_0^1 G_{\alpha_i}(t,s) g_i(s,x_1(s),\dots,x_m(s),D^{\mu_1}x_1(s),\dots,D^{\mu_m}x_m(s))ds
\end{aligned}$$

for all $1 \leq i \leq m$. By using the assumptions, we get $x_{i,n}(t) \geq \frac{m_i t^{\alpha_i}(\alpha_i-1)}{\Gamma(\alpha_i+1)}$ for all $t \in [0,1]$ and $1 \leq i \leq m$. One can check that the element $(x_{1,n},\dots,x_{m,n}) \in P$ is a solution for the problem (1.2) with the boundary conditions in (1.1). □

Lemma 2.1. *Assume that hypotheses* H1, H2 *and* H3 *hold. If* $(x_{1,n},\dots,x_{m,n})$ *is a solution for the problem* (1.2) *with the boundary conditions in* (1.1), *then* $\{(x_{1,n},\dots,x_{m,n})\}_{n\geq 1}$ *is relatively compact in* P.

Proof. As we found in the last result,

$$x_{i,n}(t) = \int_0^1 G_{\alpha_i}(t,s) f_{i,n}(s, x_{1,n}(s), \dots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \dots, D^{\mu_m} x_{m,n}(s)) ds$$
$$+ \int_0^1 G_{\alpha_i}(t,s) g_i(s, x_{1,n}(s), \dots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \dots, D^{\mu_m} x_{m,n}(s)) ds$$

for all $n \geq 1$, $t \in [0,1]$ and $1 \leq i \leq m$. Thus,

$$x'_{i,n}(t) \geq m_i \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t,s) ds \geq \frac{m_i(1 - t^{\alpha i - 1})}{\Gamma(\alpha_i)},$$

for all $t \in [0,1]$. Hence,

$$D^{\mu_i} x_{i,n}(t) = \frac{1}{\Gamma(1-\mu_i)} \int_0^t (t-s)^{-\mu_i} x'_{i,n}(s) ds$$
$$\geq \frac{m_i}{\Gamma(\alpha_i)\Gamma(1-\mu_i)} \int_0^t (t-s)^{-\mu_i}(1 - s^{\alpha_i - 1}) ds > \frac{m_i}{\Gamma(\alpha_i)\Gamma(1-\mu_i)} \int_0^t (t-s)^{-\mu_i}(1-s) ds$$

for all $t \in [0,1]$. Thus,

$$D^{\mu_i} x_{i,n}(t) > \frac{m_i t^{1-\mu_i}}{\Gamma(\alpha_i)\Gamma(2-\mu_i)} - \frac{m_i t^{2-\mu_i}}{\Gamma(\alpha_i)\Gamma(3-\mu_i)}$$
$$= \frac{m_i t^{1-\mu_i}}{\Gamma(\alpha_i)} \left(\frac{\Gamma(3-\mu_i) - t\Gamma(2-\mu_i)}{\Gamma(2-\mu_i)\Gamma(3-\mu_i)} \right) = \frac{m_i t^{1-\mu_i}}{\Gamma(\alpha_i)} \left(\frac{2-\mu_i-t}{\Gamma(3-\mu_i)} \right) \geq \frac{m_i t^{1-\mu_i}(1-\mu_i)}{\Gamma(\alpha_i)\Gamma(3-\mu_i)}$$

for all $t \in [0,1]$. Since $\Gamma(3-\mu_i) \leq 2$, we get $D^{\mu_i} x_{i,n}(t) \geq \frac{m_i t^{1-\mu_i}(1-\mu_i)}{2\Gamma(\alpha_i)}$.

Now, put

$$M_i = m_i \min\Big\{ \frac{1}{\Gamma(\alpha_i)}, \frac{\alpha_i - 1}{\Gamma(\alpha_i + 1)} \Big\}.$$

Then, $x_{i,n}(t) \geq M_i t^{\alpha_i}$ and $D^{\mu_i} x_{i,n}(t) \geq \frac{M_i(1-\mu_i)}{2} t^{1-\mu_i}$ for all $n \geq 1$, $t \in [0,1]$ and $1 \leq i \leq m$. Hence,

$$p_i(x_{1,n}(t), \dots, x_{m,n}(t), D^{\mu_1} x_{1,n}(t), \dots, D^{\mu_m} x_{m,n}(t))$$
$$\leq p_i\Big(M_1 t^{\alpha_1}, \dots, M_m t^{\alpha_m}, \frac{M_1(1-\mu_1)}{2} t^{1-\mu_1} \dots, \frac{M_m(1-\mu_m)}{2} t^{1-\mu_m} \Big)$$

for all $n \geq 1$, $t \in [0,1]$ and $1 \leq i \leq m$. This implies that

$$0 \leq x'_{i,n}(t) = \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t,s) f_{i,n}(s, x_{1,n}(s), \dots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \dots, D^{\mu_m} x_{m,n}(s)) ds$$
$$+ \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t,s) g_i(s, x_{1,n}(s), \dots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \dots, D^{\mu_m} x_{m,n}(s)) ds$$
$$\leq \frac{1}{\Gamma(\alpha_i - 1)} \int_0^1 p_i\Big(M_1 s^{\alpha_1}, \dots, M_m s^{\alpha_m}, \frac{M_1(1-\mu_1)}{2} s^{1-\mu_1} \dots, \frac{M_m(1-\mu_m)}{2} s^{1-\mu_m} \Big) ds$$
$$+ \frac{1}{\Gamma(\alpha_i - 1)} \int_0^1 \gamma_i(s) h_i(x_{1,n}(s), \dots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \dots, D^{\mu_m} x_{m,n}(s)) ds$$

for all $n \geq 1$, $t \in [0,1]$ and $1 \leq i \leq m$.

Also, we have

$$\int_0^1 p_i\Big(M_1 s^{\alpha_1}, \dots, M_m s^{\alpha_m}, \frac{M_1(1-\mu_1)}{2} s^{1-\mu_1}, \dots, \frac{M_m(1-\mu_m)}{2} s^{1-\mu_m} \Big) ds := \Lambda_i < \infty$$

for all $1 \leq i \leq m$. If $\eta_n = \|(x_{1,n}, \dots, x_{m,n})\|_{**}$, then $\|x_{i,n}\| \leq \eta_n$ and $\left\|x'_{i,n}\right\| \leq \eta_n$ for all i and n. Thus, $|D^{\mu_i} x_{i,n}(t)| \leq \frac{\eta_n}{\Gamma(2-\mu_i)}$ for all $n \geq 1$, $t \in [0,1]$ and $1 \leq i \leq m$ and so

$$0 \leq x'_{i,n}(t)$$
$$\leq \frac{1}{\Gamma(\alpha_i - 1)} \left(\Lambda_i + h_i(1+\eta_n, \dots, 1+\eta_n, 1 + \frac{\eta_n}{\Gamma(2-\mu_1)}, \dots, 1 + \frac{\eta_n}{\Gamma(2-\mu_m)}) \right) \|\gamma_i\|_1$$

and so $0 \leq x_{i,n}(t) = \int_0^t x'_{i,n}(s) ds$ for all $n \geq 1$, $t \in [0,1]$ and $1 \leq i \leq m$.

Similarly, we obtain

$$0 \le x_{i,n}(t)$$
$$\le \frac{1}{\Gamma(\alpha_i - 1)} \left(\Lambda_i + h_i(1+\eta_n, \dots, 1+\eta_n, 1+\frac{\eta_n}{\Gamma(2-\mu_1)}, \dots, 1+\frac{\eta_n}{\Gamma(2-\mu_m)})\right) \|\gamma_i\|_1$$

and $\eta_n \le \frac{1}{\Gamma(\alpha_i-1)} \left(\Lambda_i + h_i(1+\eta_n, \dots, 1+\eta_n, 1+\frac{\eta_n}{\Gamma(2-\mu_1)}, \dots, 1+\frac{\eta_n}{\Gamma(2-\mu_m)})\right) \|\gamma_i\|_1$ for all i.

Since $\lim_{x\to\infty} \frac{h_i(x,\dots,x)}{x} = 0$ for all $1 \le i \le m$, there exists $L_i > 0$ such that

$$\frac{1}{\Gamma(\alpha_i - 1)} \left(\Lambda_i + h_i(1+\nu_i, \dots, 1+\nu_i, 1+\frac{\nu_i}{\Gamma(2-\mu_1)}, \dots, 1+\frac{\nu_i}{\Gamma(2-\mu_m)})\right) \|\gamma_i\|_1 < \nu_i$$

for all $\nu_i > L_i$. If $L = \max\{L_1, \dots, L_m\}$, then

$$\frac{1}{\Gamma(\alpha_i - 1)} \left(\Lambda_i + h_i(1+\nu, \dots, 1+\nu, 1+\frac{\nu}{\Gamma(2-\mu_1)}, \dots, 1+\frac{\nu}{\Gamma(2-\mu_m)})\right) \|\gamma_i\|_1 < \nu$$

for all $\nu > L$. Thus, $\eta_n = \|(x_{1,n}, \dots, x_{m,n})\|_{**} = \max_{1\le i\le m}\{\|x_{i,n}\|, \|x'_{i,n}\|\} < L$ which implies $\{\|(x_{1,n}, \dots, x_{m,n})\|_{**}\}$ is bounded in X_m.

Now, put

$$B_i := h_i\Big(1+L, \dots, 1+L, 1+\frac{L}{\Gamma(2-\mu_1)}, \dots, 1+\frac{L}{\Gamma(2-\mu_m)}\Big)$$

and

$$F_i(t) := p_i\Big(M_1 t^{\alpha_1}, \dots, M_m t^{\alpha_m}, \frac{M_1(1-\mu_1)}{2} t^{1-\mu_1} \dots, \frac{M_m(1-\mu_m)}{2} t^{1-\mu_m}\Big),$$

for all i and almost all $t \in [0,1]$.

Then, we have $\Lambda_i = \int_0^1 F_i(t)dt$ and

$$\begin{aligned} &f_{i,n}(t, x_{1,n}(t), \dots, x_{m,n}(t), D^{\mu_1} x_{1,n}(t), \dots, D^{\mu_m} x_{m,n}(t)) \\ &+g_i(t, x_{1,n}(t), \dots, x_{m,n}(t), D^{\mu_{1,n}} x_1(t), \dots, D^{\mu_m} x_{m,n}(t)) \\ &\le F_i(t) + B_i \gamma_i(t). \end{aligned}$$

If $0 \le t_1 \le t_2 \le 1$, then

$$\begin{aligned} |x'_{i,n}(t_2) - x'_{i,n}(t_1)| = \Big| \int_0^1 &(\frac{\partial}{\partial t} G_{\alpha_i}(t_2, s) - \frac{\partial}{\partial t} G_{\alpha_i}(t_2, s)) \times \\ &[f_{i,n}(s, x_{1,n}(s), \dots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \dots, D^{\mu_m} x_{m,n}(s)) \\ &+g_i(s, x_{1,n}(s), \dots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \dots, D^{\mu_m} x_{m,n}(s))]ds\Big| \end{aligned}$$

$$\le \frac{1}{\Gamma(\alpha_i - 1)}[(t_2 - t_1) \int_0^1 F_i(s) + B_i \gamma_i(s) ds + \int_0^{t_1} ((t_2 - s)^{\alpha_i - 2} - (t_1 - s)^{\alpha_i - 2})(F_i(s) + B_i \gamma_i(s)\ )ds$$
$$+ \int_{t_1}^{t_2} (t_2 - s)^{\alpha_i - 2} \left(F_i(s) + B_i \gamma_i(s)\right) ds]$$
$$\le \frac{1}{\Gamma(\alpha_i - 1)}[(t_2 - t_1)(\Lambda_i + B_i\|\gamma_i\|_1) + \int_0^{t_1} ((t_2 - s)^{\alpha_i - 2} - (t_1 - s)^{\alpha_i - 2})(F_i(s) + B_i \gamma_i(s))ds$$
$$+(t_2 - t_1)^{\alpha_i - 2}(\Lambda_i + B_i\|\gamma_i\|_1)].$$

Let $\epsilon_i > 0$ be given. Choose $\delta(\epsilon_i) > 0$ such that $(t_2 - s)^{\alpha_i - 2} - (t_1 - s)^{\alpha_i - 2} < \epsilon_i$ for all $0 \le t_1 < t_2 \le 1$ with $t_2 - t_1 < \delta(\epsilon_i)$ and $0 \le s \le t$. If we put

$$0 < \delta < \min\{\delta(\epsilon_1), \dots, \delta(\epsilon_m),\ \sqrt[\alpha_1 - 2]{\epsilon_1}, \dots,\ \sqrt[\alpha_m - 2]{\epsilon_m}\},$$

then $|x'_{i,n}(t_2) - x'_{i,n}(t_1)| \le \frac{3\ \epsilon_i}{\Gamma(\alpha_i - 1)}(\Lambda_i + B_i \|\gamma_i\|_1)$ for all $1 \le i \le m$. Hence, $\{(x_{1,n}, \dots, x_{m,n})'\}$ is equi-continuous and so $\{(x_{1,n}, \dots, x_{m,n})\}_{n\ge1}$ is relatively compact in X^m. □

Theorem 2.2. *Assume that hypotheses* H1, H2 *and* H3 *hold. Then the system* (1.1) *has a solution* $(x_1, \dots, x_m)$ *in* P *such that* $D^{\mu_i} x_i(t) \ge \frac{M_i(1-\mu_i)}{2} t^{1-\mu_i}$ *and* $x_i(t) \ge M_i t^{\alpha_i}$ *for all* $t \in [0,1]$ *and* $1 \le i \le m$.

Proof. By Theorem 2.1, for each $n \geq 1$ the system (1.2) with the boundary conditions in (1.1) has a solution $(x_{1,n}, \ldots, x_{m,n}) \in P$. By Lemma 2.1, $\{(x_{1,n}, \ldots, x_{m,n})\}_{n\geq 1}$ is relatively compact in X^m. By using the Arzela-Ascoli theorem, there exists $(x_1, \ldots, x_m)$ such that $\lim_{n\to\infty}(x_{1,n}, \ldots, x_{m,n}) = (x_1, \ldots, x_m)$. It is obvious that $(x_1, \ldots, x_m)$ satisfies the boundary conditions of the problem (1.1), $D^{\mu_i} x_{i,n} \to D^{\mu_i} x_i$ and

$$\begin{aligned}
&\lim_{n\to\infty} f_{i,n}(t, x_{1,n}(t), \ldots, x_{m,n}(t), D^{\mu_1} x_{1,n}(t), \ldots, D^{\mu_m} x_{m,n}(t)) \\
&+g_i(t, x_{1,n}(t), \ldots, x_{m,n}(t), D^{\mu_1} x_{1,n}(t), \ldots, D^{\mu_m} x_{m,n}(t)) \\
&= f_i(t, x_1(t), \ldots, x_m(t), D^{\mu_1} x_1(t), \ldots, D^{\mu_m} x_m(t)) \\
&+g_i(t, x_1(t), \ldots, x_m(t), D^{\mu_1} x_1(t), \ldots, D^{\mu_m} x_m(t))
\end{aligned}$$

for almost all $t \in [0, 1]$ and $1 \leq i \leq m$ and so $(x_1, \ldots, x_m) \in P$. Now, suppose that $K := \sup_{n\geq 1} \|(x_{1,n}, \ldots, x_{m,n})\|_{**}$. Then, we have $\|D^{\mu_i} x_{i,n}\| \leq \frac{K}{\Gamma(2-\mu_i)}$ for all n and $1 \leq i \leq m$. Hence,

$$\begin{aligned}
0 &\leq G_{\alpha_i}(t, s)[f_{i,n}(s, x_{1,n}(s), \ldots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \ldots, D^{\mu_m} x_{m,n}(s)) \\
&\quad +g_i(s, x_{1,n}(s), \ldots, x_{m,n}(s), D^{\mu_1} x_{1,n}(s), \ldots, D^{\mu_m} x_{m,n}(s))] \\
&\leq \frac{1}{\Gamma(\alpha_i - 1)} \left(F_i(s) + h_i(1 + K, \ldots, 1 + K, 1 + \frac{K}{\Gamma(2-\mu_i)}, \ldots, 1 + \frac{K}{\Gamma(2-\mu_i)})\gamma_i(s) \right)
\end{aligned}$$

for almost all $(t, s) \in [0, 1] \times [0, 1]$, $n \geq 1$ and $1 \leq i \leq m$. Now by using the Lebesgue dominated theorem, we conclude that

$$\begin{aligned}
x_i(t) &= \int_0^1 G_{\alpha_i}(t, s) f_i(s, x_1(s), \ldots, x_m(s), D^{\mu_1} x_1(s), \ldots, D^{\mu_m} x_m(s)) ds \\
&\quad + \int_0^1 G_{\alpha_i}(t, s) g_i(s, x_1(s), \ldots, x_m(s), D^{\mu_1} x_1(s), \ldots, D^{\mu_m} x_m(s)) ds
\end{aligned}$$

for all $1 \leq i \leq m$ and $t \in [0, 1]$, and this completes the proof. □

Next example illustrates our last result.

Example 2.1. Let us study the system

$$\begin{cases}
D^{\frac{5}{2}} x_1 + \frac{1}{t^{\frac{2}{3}}}(2 + a_1 x_1 + a_2 x_2 + a_3 D^{\frac{1}{3}} x_1 + a_4 D^{\frac{1}{2}} x_2) \\
\ +(0.1e^{\frac{1}{1+x_1}} + 0.2e^{\frac{1}{1+x_2}} + 0.1e^{\frac{1}{1+D^{\frac{1}{3}}x_1}} + 0.2e^{\frac{1}{1+D^{\frac{1}{2}}x_2}}) = 0 \\
D^{\frac{7}{3}} x_2 + \frac{1}{t^{\frac{1}{2}}}(1 + b_1 x_1 + b_2 x_2 + b_3 D^{\frac{1}{3}} x_1 + b_4 D^{\frac{1}{2}} x_2) \\
\ +(0.2e^{\frac{1}{1+x_1}} + 0.2e^{\frac{1}{1+x_2}} + 0.3e^{\frac{1}{1+D^{\frac{1}{3}}x_1}} + 0.1e^{\frac{1}{1+D^{\frac{1}{2}}x_2}}) = 0
\end{cases}$$

with boundary condition $x_1(0) = x_2(0) = 0$, $x_1'(1) = x_2'(1) = 0$ and $x_1''(0) = x_2''(0) = 0$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are positive constants.

Consider the functions

$$f_1(t, x_1, x_2, x_3, x_4) = \frac{1}{t^{\frac{2}{3}}}(2 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4),$$

$$f_2(t, x_1, x_2, x_3, x_4) = \frac{1}{t^{\frac{1}{2}}}(1 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4),$$

$$g_1(t, x_1, x_2, x_3, x_4) = p_1(x_1, x_2, x_3, x_4) = 0.1e^{\frac{1}{1+x_1}} + 0.2e^{\frac{1}{1+x_2}} + 0.1e^{\frac{1}{1+x_3}} + 0.2e^{\frac{1}{1+x_4}},$$

$$g_2(t, x_1, x_2, x_3, x_4) = p_2(x_1, x_2, x_3, x_4) = 0.2e^{\frac{1}{1+x_1}} + 0.2e^{\frac{1}{1+x_2}} + 0.3e^{\frac{1}{1+x_3}} + 0.1e^{\frac{1}{1+x_4}},$$

$$h_1(x_1, x_2, x_3, x_4) = 2 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4,$$

$$h_2(x_1, x_2, x_3, x_4) = 1 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4,$$

$\lambda_1(t) = \frac{1}{t^{\frac{2}{3}}}$ and $\lambda_2(t) = \frac{1}{t^{\frac{1}{2}}}$. Put $m = 2$, $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{7}{3}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{3}$, $L_1^1 = 0.1$, $L_2^1 = 0.2$, $L_3^1 = 0.1$, $L_4^1 = 0.2$, $L_1^2 = 0.2$, $L_2^2 = 0.2$, $L_3^2 = 0.3$, $L_4^2 = 0.1$, $m_1 = 2$ and $m_2 = 1$.

One can check that f_1 and f_2 are Caratheodory functions,

$$f_1(t, x_1, x_2, x_3, x_4) \geq 2, \quad f_2(t, x_1, x_2, x_3, x_4) \geq 1$$

for all $(x_1, x_2, x_3, x_4) \in (0, \infty)^4$ and almost all $t \in [0, 1]$, g_1 and g_2 are nonnegative,

$$|g_1(t, x_1, x_2, x_3, x_4) - g_1(t, y_1, y_2, y_3, y_4)| \leq \sum_{i=1}^{4} L_i^1 |x_i - y_i|$$

and

$$|g_2(t, x_1, x_2, x_3, x_4) - g_2(t, y_1, y_2, y_3, y_4)| \leq \sum_{i=1}^{4} L_i^2 |x_i - y_i|$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in (0, \infty)^4$ and $t \in [0, 1]$.

Also, we have

$$\frac{1}{\Gamma(\alpha_1 - 1)} \Big(\sum_{k=1}^{2} L_k^1 + \sum_{k=1}^{2} \frac{L_{2+k}^1}{\Gamma(2 - \mu_1)} \Big) = \frac{1}{\Gamma(\frac{3}{2} - 1)} (0.1 + 0.2 + \frac{0.1}{\Gamma(\frac{5}{3})}) + \frac{0.2}{\Gamma(\frac{5}{3})}) < 1$$

and

$$\frac{1}{\Gamma(\alpha_2 - 1)} \Big(\sum_{k=1}^{2} L_k^2 + \sum_{k=1}^{2} \frac{L_{2+k}^2}{\Gamma(2 - \mu_2)} \Big) = \frac{1}{\Gamma(\frac{4}{3} - 1)} (0.2 + 0.2 + \frac{0.3}{\Gamma(\frac{3}{2})}) + \frac{0.1}{\Gamma(\frac{3}{2})}) < 1.$$

Note that the maps p_1 and p_2 are non-increasing respect to all components.

If

$$M_1 := m_1 \frac{\alpha_1 - 1}{\Gamma(\alpha_1 + 1)} = 2 \times \frac{\frac{3}{2}}{\Gamma(\frac{7}{2})} = \frac{3}{\Gamma(\frac{7}{2})}, \quad M_2 := m_2 \frac{\alpha_2 - 1}{\Gamma(\alpha_2 + 1)} = 1 \times \frac{\frac{4}{3}}{\Gamma(\frac{10}{3})} = \frac{4}{3\Gamma(\frac{10}{3})},$$

then

$$\int_0^1 p_1 \Big(M_1 t^{\alpha_1}, M_2 t^{\alpha_2}, \frac{M_1(1 - \mu_1)}{2} t^{1-\mu_1}, \frac{M_2(1 - \mu_2)}{2} t^{1-\mu_2} \Big) dt < \infty$$

and

$$\int_0^1 p_2 \Big(M_1 t^{\alpha_1}, M_2 t^{\alpha_2}, \frac{M_1(1 - \mu_1)}{2} t^{1-\mu_1}, \frac{M_2(1 - \mu_2)}{2} t^{1-\mu_2} \Big) dt < \infty.$$

Also, the functions h_1 and h_2 are non-decreasing respect to all components,

$$\lim_{x \to \infty} \frac{h_1(x, \ldots, x)}{x} = \lim_{x \to \infty} \frac{2 + a_1 x + a_2 x + a_3 x + a_4 x}{x} = 0$$

and

$$\lim_{x \to \infty} \frac{h_2(x, \ldots, x)}{x} = \lim_{x \to \infty} \frac{1 + b_1 x + b_2 x + b_3 x + b_4 x}{x} = 0.$$

Now by using Theorem 2.2, the problem (2.1) has a positive solution.

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