A Method to Construct Approximate Fuzzy Voronoi Diagram for Fuzzy Numbers of Dimension Two

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Abstract: In this paper, we propose an approximate "fuzzy Voronoi" diagram (FVD)for fuzzy numbers of dimension two (FNDT) by designing an extension of crisp Voronoi diagram for fuzzy numbers. The fuzzy Voronoi sites are defined as fuzzy numbers of dimension two. In this approach, the fuzzy numbers have a convex continuous differentiable shape. The proposed algorithm has two stages: in the first stage we use the Fortune's algorithm in order to construct a "fuzzy Voronoi" diagram for membership values of FNDTs that are equal to 1. In the second stage, we propose a new algorithm based on the Euclidean distance between two fuzzy numbers in order to construct the approximate "fuzzy Voronoi" diagram for values of the membership of FNDTs that are smaller than 1. The experimental results are presented for a particular shape, the fuzzy ellipse numbers.

Keywords: approximated fuzzy Voronoi diagram, fuzzy numbers of dimension two, computational geometry, fuzzy arithmetic, bisector median, path planning.

1 Introduction

Many studies have been made relating to Voronoi diagram and its application [1]. Usually, the main algorithms deal with exact Voronoi diagram. Only few of them deal with approximated Voronoi diagram. In some of paper, 2D objects has been taken into account in order to construct Voronoi diagram. An algorithm to construct Voronoi diagram for convex sites is proposed in [2]. The authors start with the post-office problem where the main question is how to process a set of sites in the plane in order to find the closest site to a query point. The study in [2] treats points as disjoint convex polygons.

In [3] the authors proposed Voronoi diagrams of ellipses using an incremental algorithm that constructs also the exact Delauny graph. A distance from a point to an ellipse is defined to be the minimum Euclidean distance to the boundary of the ellipse with negative value in case the point is inside the ellipse.

Most of the papers treat the problem of disjoint objects. In some papers, the objects can intersect [4] or one object can be inside another object [4]. In the latter case, the circles can intersect each other and circles can contain other circles [4]. In [5] the authors used an incremental algorithm in order to construct Voronoi diagrams for convex objects. A polynomial approximation of the curves is suggested. An algorithm that computes an approximate version of Voronoi diagram is introduced in [6]. The cells in these approximated diagrams are cubes or difference of two cubes. Other papers deal with weighted Voronoi diagrams [7] using the Euclidean distance functions. Each point p from a given set P of points in the plane has an associated weight that expresses the level of influence over this neighborhood.

Fuzzy objects are an important part in the fuzzy set theory. 1-D fuzzy sets have been extended to fuzzy plane and fuzzy plane geometry (circles and polygons) in [9]. Only very few papers deal with fuzzy Voronoi diagram. In ([10]- [11]) the authors proposed an algorithm that deals with fuzzy objects in fuzzy Voronoi diagrams. The "Fuzzy Voronoi" diagram is an extension of the Voronoi diagram based on Fortune's algorithm for crisp cases. The "Fuzzy Voronoi" diagram

has been extended for a set of fuzzy circles, rather than fuzzy points, of the same radius. The authors proved that the boundaries of this diagram are constructed by intersecting a number of hyperbolae and they proposed an algorithm that computes the exact formula [11]. The geometric approach is presented in [12]. The authors define the basic fuzzy geometric shapes (point, line, circle, ellipse and polygon) using the 2-D support space on the level of fuzzy set. A fuzzy measure for trapezoidal fuzzy number is proposed in [13].

Optimal Path Planning for Mobile Robot Navigation is an important area of research in computer science. Usually, obstacles are approximated by rectangular or circular cylinders and Voronoi diagram can be an effective solution [14]. An interesting approach is optimal path planning in an uncertain (fuzzy) environment and 3D space that conduct to idea of fuzzy Voronoi diagram. Fuzzy solution can be very effective for various problems [15].

We present a different algorithm that computes an approximated fuzzy Voronoi diagram for fuzzy numbers of dimension two. It is possible to extend the proposed algorithm to n-dimensional fuzzy numbers using α-cuts.

2 Fuzzy Numbers of Dimension Two

Before we proceed to the main results described in this paper we make a short review of the crisp Voronoi diagram [1] and FNDT [16]. Some proposal for fuzzy distance are investigated by some authors ([17]- [19]). We will use Euclidean distance in fuzzy workspace and some of notations adapted from [11] for Voronoi cell and Voronoi diagram.

Let P be a discrete subset in the metric space X. The Voronoi cell $V(p)$ is defined as the set of all points x in X that have the distance to p lower or equal to the distance from x to other points q in P [11].

$$V(p) = \{x \in X | \forall q \in P, d(x,p) \leq d(x,q)\} \tag{1}$$

The Voronoi diagram of P is the set of all Voronoi cells denoted by $V(p)$. The points p are named Voronoi sites of the set P ([10]- [11]). A fuzzy numbers of dimension n can be defined according to ([9], [16]). Let's consider a fuzzy graph $G \subset R \times R... \times R$, a functional relation in R^n having the membership function $\mu_G(x_1, x_2, ..., x_n) \in R^n$. We define an n-dimensional fuzzy number $F[A]^n$ as any non-empty convex fuzzy subset of $G \in R^n$ that meets the following conditions [16]:

- A is a normal fuzzy set, i.e. there is an $x_0 \in R^n$, $\mu_G(x_0) = 1$;

- A is a convex set, that is membership function is convex hypersurface, i.e. $\mu_A(\alpha x + (1 - \alpha)y) \geq min(\mu_A(x), \mu_A(y))$, $\alpha \in (0,1]$;

- The membership function is a convex hypersurface that for $\forall a \in R$, there is $\mu_A(x) = a$.

In particular, for $n=2$ we have fuzzy numbers of dimension two (FNDT), as they are defined in ([9], [16]). A general shape is shown in Fig. 1a. In Fig. b we showed the cut of this surface (top view) for a level $\alpha \in (0,1]$.

The Fuzzy Voronoi diagram involves computing distances in fuzzy spaces. The results of calculus depend of the metrics used for the fuzzy space.An important point is the selection of the shape of FNDT, derivable in all the points or not: convex polygon, rectangular, square, circle, ellipse or closed curves. In our approach, a continuous convex closed curve of FNDT is considered. The circle or ellipsis is a particular case.

Various metrics in $\tilde{F}(\Re)$ have been proposed in the literature, most of them based on the Hausdorff distance ([18]- [19]). Other authors proposed definitions for measures in the Euclidean
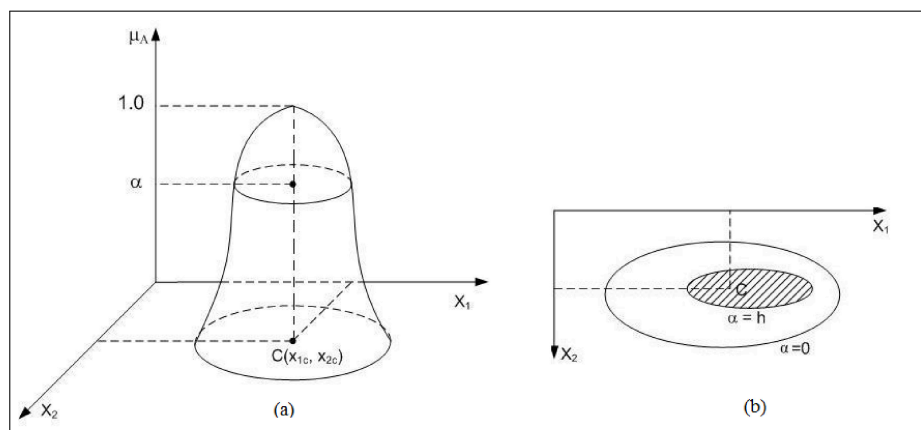
Figure 1: (a) A fuzzy number of dimension 2; (b) Top view of α-cut of FNDT (general case)

space, function that capture more information about vagueness, α-cut that transform fuzzy number into an interval number and distance index based on centroid of fuzzy number. The α-level set A_α of a fuzzy number is a non fuzzy set defined by:

$$A_\alpha = \{x \in \Re | \mu(x) \geq \alpha\}, \quad \mu_A(x) = \sup_{a \in (0,1]} \{\alpha \cdot G_{A_\alpha}(x)\} \tag{2}$$

where $G_{A_\alpha}(x)$ is the characteristic of the function A_α.

3   Approximated Fuzzy Voronoi Diagram (AFVD) for Fuzzy Numbers of Dimension Two

In the next paragraphs we define the FVD for a set of sites that are fuzzy numbers of dimension two. Let $\tilde{p}$ be a subset of $\Re^2$, which is constructed from fuzzy numbers of dimension two and let $\tilde{p} \in \tilde{P}$ be a FNDT. A fuzzy Voronoi cell of $\tilde{p}$ is defined, in a similar manner as in [11], by $\tilde{V}(\tilde{p})$:

$$\tilde{V}(\tilde{p}) = \{(x, \mu_{\tilde{p}}(\tilde{p})) | \forall \tilde{q} \in \tilde{P}, d(x,p) \leq d(x,q)\} \tag{3}$$

We note $p = (p_x, p_y)$ and $q = (q_x, q_y)$ two crisp points that belong to a FNDT at α-cut level. We omitted α from the following formulae in order to simplify the notations. The Euclidean distance between two points p and q that belong to a FNDT at α-cut level is given by:

$$d_E(p,q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2} \tag{4}$$

This definition is consistent with the results obtained in the particular degenerated case when FNDT become a single point or when $\alpha = 1$. Let us consider two FNDT ellipse shapes not necessary equal. Each level of the α-cuts level will produce two ellipses in the same plane. The fuzzy Voronoi diagram for n FNDT is the union of Voronoi diagrams of fuzzy α-cuts for all n FNDT. Each α-cut of FNDT become a set of ellipses for which a Voronoi diagram is required.

It is easy to demonstrate that given two disjoint continuous convex closed curves A and B, for any two points inside of continuous convex closed curves, the bisector median of these two points are bordered by other two bisector median of two points situated on the contour of the curves. A conclusion of this assertion is that to construct the bisector median of all the points on surface of two continuous convex closed curves it is enough to construct the bisector median for all the points on the contour of the curve that is the border of the bisector median for all the points.
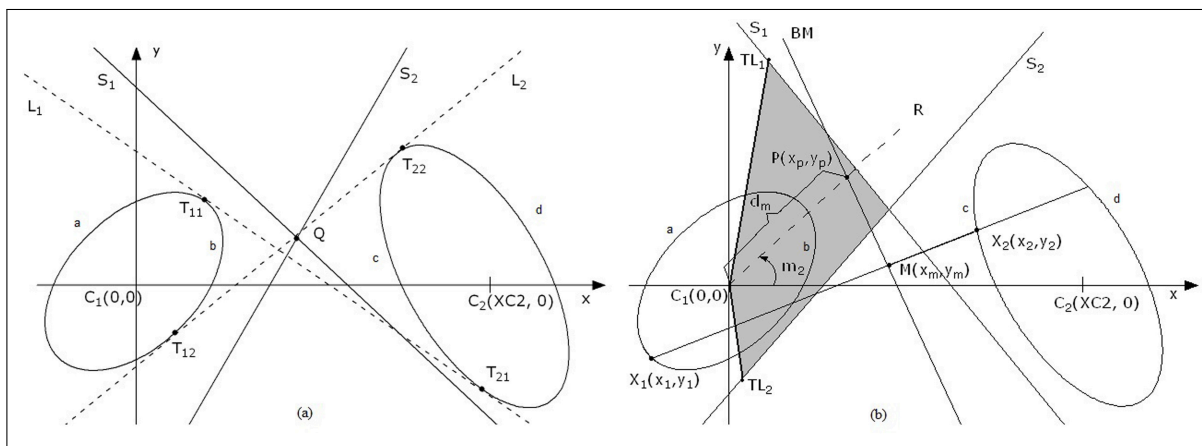
Figure 2: (a) The construction of the bisector median for two points located on the FNDT and the asymptotes of the FVD; (b) The construction of APVD for first curve

For each α-cut, we can compute the fuzzy Voronoi diagram using only the contour of the curve from FNDT obtained by this α-cut. In what follows, we will denote by $h_k \in [0,1)$ the α-cut level of index k.

The starting point for our approach is the calculus of bisector median for two points, one of them situated on the first curve and the second one situated on the second curve. Let's denote by $X_1(x_1, y_1)$ and $X_2(x_2, y_2)$ these two points. The equation of the bisector median is given by a line, perpendicular on the middle of the segment $\overline{X_1 X_2}$. If the slope of the line $X_1 X_2$ is m, the slope of the bisector median of $\overline{X_1 X_2}$ will be $m' = -1/m = (x_2 - x_1)/(y_2 - y_1)$. The equation bisector median (BM) is a line that pass by the point $M(x_m, y_m)$, the middle of the segment $\overline{X_1 X_2}$ and it has the slope m'. After few manipulations, the BMs equation si given by:

$$y = -\frac{x_2 - x_1}{y_2 - y_1} \cdot x + \frac{x_2^2 + y_2^2 - x_1^2 - y_1^2}{2 \cdot (y_2 - y_1)} \tag{5}$$

The fuzzy Voronoi diagram for two FNDT consists from all the bisector medians for all the points situated on the two closed curves. The asymptotes for the envelopes of all bisector medians is given by the bisector median of internal tangents of the two closed convex curves. If we apply counterclockwise rotation and translation for any point with φ angle in Euclidean space, for any two ellipses we have simplified parametric equations of two ellipses (as in Fig. 2(a) and Fig. 2(b)).

Let's consider two convex closed curve that after eventually rotation and translation looks like in Fig. 2(a). The origin of axis is located in the center point of the first unimodal FNDT and his center point of the second unimodal FNDT is located on the axis $0X$ (Fig 2(a)).

We consider two points situated on this curves $X_1(x_1, y_1)$ and $X_2(x_2, y_2)$ as in Fig. 2b. The bisector median of these points will be given by formula (5). Let's consider a line that pass in the point $C_1(0, 0)$, with m_2 incrementally from point TL_2 and TL_1 arbitrary chosen on the asymptotic limits for fuzzy Voronoi Diagram S_1 and S_2. The two lines intersect in the point $P(x_p, y_p)$ given by the solution of the system:

$$\begin{cases} y_p = m_1 x_p + n_1 \\ y_p = m_2 x_p + n_2 \end{cases} \tag{6}$$

$$\begin{cases} m_1 = -(x_2 - x_1)/(y_2 - y_1), n_1 = (x_2^2 + y_2^2 - x_1^2 - y_1^2)/(2 \cdot (y_2 - y_1)) \\ m_2 = const., n_2 = 0 \end{cases} \tag{7}$$

$$\begin{cases} x_p = \frac{x_2^2 + y_2^2 - x_1^2 - y_1^2}{2 \cdot (x_2 - x_1 + m_2 \cdot (y_2 - y_1))} \\ y_p = m_2 x_p \end{cases} \tag{8}$$

If $m_1 \neq m_2$, the two line intersect in the point $P(x_p, y_p)$, where $x_p = (n_1 - n_2)/(m_1 - m_2)$ and $y_p = m_1((n_1 - n_2)/(m_1 - m_2)) + n_1$. In the case $m_1 = m_2$, the two lines are parallel, so no intersection point exists. In our algorithm we simply skip this situation and we go to the next increment.

Let's consider the bisector median for points C_1 and C_2. This crisp Voronoi diagram for two points C_1 and C_2 split conventionally the plane in two halves, left (where first curve is located) and right (where the curve is located). The envelope of the fuzzy Voronoi Diagram for left half-plane is given by the minimum of distance between points P and C_1. For a given m_1, this distance is the intersection of the line R (Fig. 2(b)) and one of the bisector median for all the lines given by formula (5). The minimum of distance is given by:

$$D_m = d^2(C_1, P) = d_m^2 = x_p^2 + y_p^2 = m_1^2 \cdot x_p^2 \tag{9}$$

The function D_m reach the minimum when x_p^2 reach the minimum, that is we must find (x_1, y_1) and (x_2, y_2) for which this minimum is obtained. If the curves are given by parametric equations, this mean that we must find the pair (t_1, t_2) for which $D_m = F(x_1(t_1), y_1(t_1), x_2(t_2), y_2(t_2)) = G(t_1, t_2), t_1 \in \angle(T_{11}, a, T_{12}), t_2 \in \angle(T_{21}, c, T_{22})$ reach the minimum for a given m_2. For each $m_{2j} \in [m_{min}, m_{max}], j = 1, ...n$ we will find an P_j. The union of all the P_j will give the left envelope of the AFVD for these two FNDT at h_k level of α-cut. The envelope is asymptotic limited by the bisector median of the internal tangents.

$$FV_{left} = \bigcup_{j=1}^{n} P_j \tag{10}$$

For the right side, the formulas (6)-(9) are a bit changed, taking into account the equation of the line R that must pass in the point $C_2(XC2, 0)$. After few calculations, the formulas become:

$$\begin{cases} x_p = \frac{x_2^2 + y_2^2 - x_1^2 - y_1^2 - q \cdot (y_2 - y_1)}{2 \cdot (x_2 - x_1 + m_2 \cdot (y_2 - y_1))}, \ q = 2 \cdot m_2 \cdot XC2 = const. \\ y_p = m_2 x_p + n2, \ n2 = -m_2 \cdot XC2 \end{cases} \tag{11}$$

$$D_m = d^2(CA_2, P) = d_m^2 \tag{12}$$

$$d_m^2 = (x_p - XC2)^2 + y_p^2 = (1 + m_2^2) \cdot x_p^2 + 2 \cdot (m_2 n_2 - SC2) \cdot x_p + XC2^2 + n_2^2) \tag{13}$$

AFVD use numerical methods to find the minimum of function given by formulas (7)-(13). Let's suppose that we have the equation of curves given by parametric formula. We can convert the equation of curves into parametric formulas. The quadratic form (circle, ellipse) can be easily converted to parametric form. We taken into account two options. The first one use a minimization with constrains of the function D_m as a function of two parameters in order to find (t_1, t_2) and so one (x_p, y_p).

$$min_{t_1, t_2} F(t_1, t_2) \quad such \ that \quad t_{1,min} \leq t_{1,max}, t_{2,min} \leq t_2 \leq t_{2,max} \tag{14}$$

It is a problem o constrained nonlinear optimization that can be solved by many methods: interior-point, SQP, active-set, trust-region-reflective. We chosen to use function $fmincon$ implemented in $MATLAB$ toolbox.

The second option is possible in the case of twice differentiable curves. We can find the extremum of the function $F(t_1, t_2)$ by calculating the stationary points using first partial derivative and thereafter by select the points that are local minimum using the second derivative. The saddle and the maximum points are eliminated.

Both numerical solutions have a problem: how to choose the initial point in order find the global minimum. The surface of equations (13)-(11) can have many local minima, so we must start with initial solution close enough to global minimum. In order to overcome this situation, we propose to use a brute-force algorithm (BFI) to find the initial points. We split the domain of t_i, i=1,2 in n_i parts. For each combination (t_{ij}, t_{ik}) we find a bisector median and a intersection points. In total we will have maximum $n_1 * n_2$ lines that intersect the line R. Using formula (14), the pairs (t_{10}, t_{20}) for which the minimum of $F(t_1, t_2)$ is obtained is the starting point for minimization in the next refinement stage.

The same reasoning is used for the second curve that have the center $C_2(XC2, 0)$. After both envelopes have been found, we rotate and translate the curves in order to come in initial position. In our experiments we used the first method and the 'trust-region-reflective' algorithm.

4   Algorithm for Approximated Fuzzy Voronoi Diagram and Experimental Results

We can summarize the algorithm for APVD as follow.

Step 1: Construct the crisp Voronoi diagram for crisp points that are the centers of convex closed curves (Fortune's algorithm)P.

Step 2: For each level h_k of α-cuts do the two following steps..

Step 3: For each Voronoi cell that includes a convex closed curve A_i, construct the fuzzy border generated by all adjacent crisp Voronoi cells. The fuzzy border is generated by AFVD described previously.

Step 4: Calculate the intersection of the fuzzy borders and apply the norm max for all fuzzy points.

Step 5: If $h_k \neq 0.0$, go to Step 2, else Stop the algorithm.

In our experimental results we used ellipses given in a parametric form. It is easy to transform the parametric form into quadratic form, and reverse. The experimental results, AFVD for six ellipses, are presented in Fig. 3a and Fig. 3b. In Fig. 3b, the solid line inside the grey area represents the crisp Voronoi diagram for centers C_i of closed curves (ellipses).

Unbounded cells have at infinite the limit determined by internal tangent of adjacent cells corresponding to crisp Voronoi cells of the centers of closed curves (ellipses).

Computational time for Crisp Voronoi (Fortune's algorithm) is given by the known result $O(n \log n)$. Our approach considers a less efficient algorithm, based on construction of bisector lines, for which the computational time is $O(n^2 \log n)$.

5   Conclusions and Future Works

In this paper, we introduced Approximated fuzzy Voronoi diagram for fuzzy numbers of dimension two. We developed an original algorithm for computing fuzzy Voronoi diagram for fuzzy numbers of dimension two with closed convex derivable curve shape. We used α-cuts in order to develop the diagram and the boundaries of Voronoi edges of fuzzy cells.
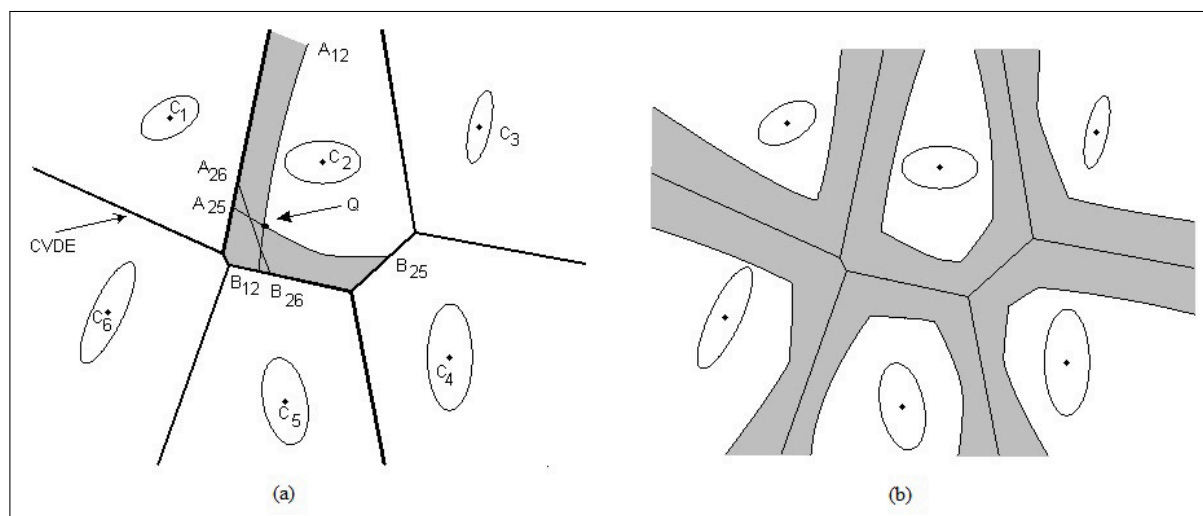
Figure 3: (a)The maximum norm for all fuzzy points inside a fuzzy Voronoi cell (CVDE: Crisp
Voronoi Diagram-Edges); (b)AFVD for six ellipses

We considered not only the contour of the fuzzy numbers of dimension two at h level of α-cuts
but also points inside this area. The same dimension for all the fuzzy object is a particular case
of the presented algorithm. The fuzzy numbers have the same size but different sizes are taken
into account.

The experimental results presented the fuzzy Voronoi diagram for FNDT not only for two
sites but also for six sites with applicability and algorithm extension to n sites.

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