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Investment Opportunities, Uncertain Implicit Transaction Costs and Maximum Downside Risk in Dynamic Stochastic Financial Optimization

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ABSTRACT

A dynamic stochastic methodology in optimal portfolio selection that maximizes investment opportunities and minimizes maximum downside risk while taking into account implicit transaction costs incurred in initial trading and in subsequent rebalancing of the portfolio is proposed. The famous mean-variance (MV) model (Markowitz, 1952) and the mean absolute deviation (MAD) model (Konno and Yamazaki, 1991) both penalize gains (upside deviations) and losses (downside deviations) in the same way. However, investors are concerned about downside deviations and are happy of upside deviations. Hence the proposed model penalizes only downside deviations and, instead, maximizes upside deviations. The methodology maintains transaction cost at the investor's prescribed level. Dynamic stochastic programming is employed with stochastic data given in the form of a scenario tree. Consideration a set of discrete scenarios of asset returns and implicit transaction costs is given, taking deviation around each return scenario. Model validation is done by comparing its performance with those of the MV, MAD and minimax models. The results show that the proposed model generates optimal portfolios with least risk, highest portfolio wealth and minimum implicit transaction costs.

Keywords: Investment Opportunities, Downside Risk, Uncertain Implicit Transaction Costs
JEL Classifications: C01, C58, D81, G11

1. INTRODUCTION

Individual investors, investment managers and fund managers are all concerned with achieving optimal portfolios of a set of investment assets. Models for portfolio selection have evolved over the years starting with Markowitz's (1952) MV formulation to more recent stochastic optimization forms (Hiller and Eckstein, 1993; Vladimirou and Zenios, 1997). Regardless of whether portfolios are selected for a bank's derivative mix, an investor's equity holdings or a firm's asset and liability management, the common objective in all models is the minimization of some measure of risk while maximizing some reward measure. The MV model has enjoyed popularity over the years despite its criticisms. The MV portfolio analysis has the following simplifying assumptions:

i. The assets' returns are multivariate normally distributed,
ii. The investor's utility function is quadratic, and
iii. There are no transaction costs.

None of these is exactly true in actual markets. Many studies show that returns from hedge funds are not normally distributed (Brooks and Kat, 2002). Pratt (1964) concludes that a quadratic utility function is very unlikely because it implies increasing absolute risk aversion. Volatility treats risks and opportunities equally yet investors are concerned about downside deviations (losses) and are happy of upside deviations (gains). Hakansson (1971) explains that in the absence of transaction costs, myopic policies are sufficient to achieve optimality. The incorporation of transaction costs in any model provides essential "friction" which without it the optimization has complete freedom to reallocate the portfolio every time-period, which (if implemented) can result in significantly poorer realized performance than forecast, due to excessive transaction costs (Hakansson, 1971). These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D' Hondt and Giraud, 2008). In this study, a stochastic multi-stage upside-downside deviation (SMUDTC) model is proposed that takes into account a risk-averse investor's view of minimizing maximum downside risk while maximizing upside deviations (gains) in an uncertain environment. The model captures uncertainty in both portfolio risk and gain by way of scenarios, which is a representative and comprehensive set of possible realizations of the future. This is achieved by taking deviations around each return scenario. The SMUDTC model also takes into account uncertainty of implicit trading costs incurred by the investor during initial trading and in subsequent rebalancing of the portfolio.

2. LITERATURE REVIEW

As a way of overcoming the limitations of the MV model, alternative risk measures were developed. Konno and Yamazaki (1991) propose the MAD model in order to overcome the problem of computational difficulty inherent in the MV model. The MAD model does not require calculation of the variance-covariance matrix of asset returns and results in optimal portfolios with fewer assets (Simaan, 1997). Similar to the MV formulation, the MAD model penalizes both upside deviations and downside deviations. However, upside deviations are desirable to any investor while downside deviations are not. Thus the proposed model maximizes upside deviations and minimizes downside deviations in the presence of implicit transaction costs. The models, MV and MAD, are both deterministic.

A number of researchers in the literature have studied optimal portfolio selection in the presence of transaction costs. Gulpinar et al. (2004), incorporate proportional transaction costs in a multi-period MV formulation. Glen (2011) considers a MV portfolio rebalancing strategy with transaction costs comprising fixed charges and variable costs that include market impact. The variable transaction costs are assumed to be non-linear functions of the traded value. However, implicit transaction costs follow a random-walk process and hence, the use of a non-linear function to approximate such costs seems inappropriate. Xia and Tian (2012) estimate implicit transaction costs in Shenzhen A-stock market using the daily closing prices, and examine the variation of the cost of Shenzhen A-stock market from 1992 to 2010. The Bayesian Gibbs sampling method proposed by Hasbrouck (2009) is used to analyze implicit costs in the bull and bear markets. Kozmik (2012) discusses asset allocation with transaction costs formulated as multi-stage stochastic programming model. Transaction costs are regarded as proportional to the value of assets bought or sold, but no implicit trading costs are considered in the model. Conditional value-at-risk is employed as a risk measure. Brown and Smith (2011) study the problem of dynamic portfolio optimization in discrete-time finite-horizon setting and also consider proportional transaction costs. Lynch and Tan (2010) study portfolio selection problem with multiple risky assets. Analytic frameworks are developed for the case with many assets taking into account proportional transaction costs. While the study of optimal asset allocation has received fair consideration in the literature, it is important such studies to have accounted for implicit transaction costs for they are invisible and dependent on the chosen strategy. These costs are difficult to measure and can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt and Giraud, 2008). The model being proposed addresses the impact of implicit transaction costs by employing dynamic stochastic programming which takes into account scenarios and stages. Uncertainty of asset returns, implicit trading costs and risk is accounted for by using a set of scenarios. The model employs stochastic programming with recourse by rebalancing portfolio compositions at discrete time-intervals as new information on asset returns become available. This study's contributions include:

a. The development of a multi-stage stochastic model that maximizes portfolio gains and minimizes maximum downside risk in the presence of uncertain implicit transaction costs incurred during initial trading and in subsequent rebalancing of portfolios,
b. The development of a strategy that captures uncertainty of stock prices and corresponding implicit trading costs by way of scenarios in uncertain environments.

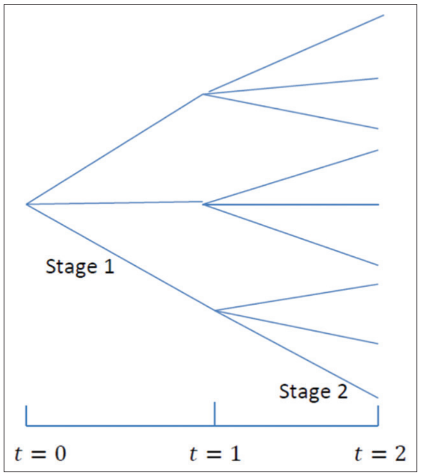
3. PROBLEM STATEMENT

A multi-period discrete-time optimal portfolio strategy is determined over an investment horizon $[0,T]$. The planning phase $[0.t]$ where $t<T$, consists of non-overlapping time-intervals indexed by $t = 1,2,\ldots,\tau$, and $(\tau,T]$ is the period to maturity of the investment. Initial investment takes place at $t = 0$, and during the period $(0,\tau]$, the investor makes adjustments to his portfolio at each of the τ periods as new information on assets' returns become available. This adjustment of the portfolio results in the investor incurring some transaction costs as he buys and sells shares of some securities. These transaction costs tend to erode the benefits of investment, hence the need to minimize them. Thus, the study considers an investor who is interested in maximizing portfolio gains (upside deviations) while keeping portfolio losses (downside deviations) to the minimum and implicit transaction costs at some prescribed level. This is achieved by employing dynamic stochastic programming in which uncertainty about future events is described by a discrete probability distribution of random parameters carried by a finite number of scenarios with prescribed probabilities. It is assumed that this discrete probability distribution is a reliable substitute of the true underlying probability distribution. Each complete realization of all uncertain parameters is a scenario along the multi-period horizon.

3.1. Scenario Generation

Let $I=\{i:i=1,2,\ldots,n\}$ be a set of risky assets for an investment. The information available about the single uncertain parameter, the risky active yield, is a set of scenarios R_{ist}, where $s\in\Omega=\{s: s=1,2,\ldots,S\}$ is a finite set of discrete scenarios. Each scenario has an associated probability p_s such that $\sum_{s=1}^{S} p_s = 1$. Scenarios are arranged in the form of a tree spanning along the succession periods and being of length equal to the planning horizon τ. Each path represents a scenario, s, and each node of the tree represents a time when a decision is made and implemented. The decision process is non-anticipative, that is, a decision at a particular stage does not depend on the future realization of the random events. The recourse decision at period t is dependent on the outcome of period $t-1$. Given the event history up to time t, R_t, the uncertainty in period $t+1$ is characterised by finitely many possible outcomes for the observations R_{t+1}. Below is an example of a scenario tree with a three-three branching structure in a 2-time period (Figure 1).

Figure 1: A scenario tree



3.2. Model Constraints

An investor who has capital W_0 to spend in his initial portfolio is considered. This capital is distributed among the n-securities of the initial portfolio. Let $x_t = [x_{1,1,t}, x_{1,2,t} \ldots x_{2,1,t}, x_{2,2,t} \ldots, x_{i,s,t}]^T$, $i = 1,\ldots,n; s = 1,\ldots,S; t = 1,\ldots,\tau$, be the investor's optimal strategy to be achieved at the end of the planning horizon. It is noted that x_{ist} is the proportion of wealth, W_t, of period t allocated to buy shares of asset i of scenario s in period t. Observe that $\sum_{i=1}^{n} x_{it} = \sum_{s=1}^{S} x_{ist} = 1,\ t = 1,2,\ldots,\tau.$ If a_{ist} and v_{ist} are the proportions of wealth, W_t, used to buy and sell, respectively, shares of security i of scenario s of period t, then

$x_{ist} = x_{i,s,t-1} + a_{ist} - v_{ist}, i = 1,2,\ldots,n; s = 1,2,\ldots,S; t = 1\ldots,\tau$. It is assumed that the investor cannot buy and sell the same asset at each time when portfolio rebalancing takes place. That is, $a_{ist} \cdot v_{ist} = 0$. The expected return of asset i of period t is given by $r_{it} = \sum_{s\in Q} p_s \cdot R_{ist} \cdot x_{ist}, i = 1,\ldots,n; t = 1,\ldots,\tau,$ where $Q \subset \Omega$ is a set of scenarios of asset i of period t. Thus, the gross expected portfolio return of period is given by $r_{pt} = \sum_{s=1}^{S} p_s \cdot R_{ist} \cdot x_{ist},\ i = 1,2,\ldots,n; t = 1,2,\ldots,\tau.$ Suppose that the portfolio is self-financing, this results in the constraint.

$$0 \le v_{it} \le x_{it}, \tag{1}$$

Where $v_{it} = \sum_{s\in Q} v_{ist} \cdot p_s$ and $x_{it} = \sum_{s\in Q} x_{ist} \cdot p_s$. The constraint ensures that the volume of asset i of period t sold for portfolio rebalancing must not exceed the volume of the asset in the portfolio. In a self-financing portfolio being rebalanced, the amount of money got from selling asset i of period t should be at most the amount of money used to buy asset j ($i \neq j$) of the same period. Hence the following results:

$$0 \le \sum_{i\in A} a_{ist} \le \sum_{j=1, j\neq i}^{n} v_{jst}, s = 1,\ldots,S, t = 1,\ldots,\tau, \tag{2}$$

Where set A contains all assets for which volumes have been bought. The investor ensures that no short-selling takes place by having the constraint

$$0 \le x_{ist} \le U_{ist},\ i = 1,\ldots, n; s = 1,\ldots, S; t = 1,\ldots,\tau, \tag{3}$$

Where U_{ist} is the maximum proportion allowed for each asset i of scenario s in period t. Let the transaction cost rates for buying and selling shares of asset i in scenario s of period t during portfolio rebalancing be k_{ist} and l_{ist} respectively. Thus either $k_{ist} \cdot a_{ist} = 0$ or $l_{ist} \cdot v_{ist} = 0$ or both are zero. This results in the transaction cost for buying or selling shares of asset i of scenario s of period t being $k_{ist} \cdot a_{ist} + l_{ist} \cdot v_{ist}$. The expected transaction cost of the portfolio of period becomes.

$$\sum_{s=1}^{S} p_s \cdot \{k_{ist} \cdot a_{ist} + l_{ist} \cdot v_{ist}\} \cdot R_{ist},\ \ i = 1,\ldots,n; t = 1,\ldots,\tau.$$

Thus getting the net expected portfolio return, N_{pt}, of period t as

$$N_{pt} = r_{pt} - \sum_{s=1}^{S} p_s \cdot \{k_{ist} \cdot a_{ist} + l_{ist} \cdot v_{ist}\} \cdot R_{ist}, i = 1,\ldots,n; t = 1,\ldots,\ ,$$

with the net portfolio wealth of period t given by

$$W_t = (1 + N_{pt}) \cdot W_{t-1}, t = 1,\ldots,\tau \tag{3}$$

3.3. Portfolio Risk

The study assumes that the investor intends to choose a portfolio with minimum of the maximum downside deviations of asset returns relative to expected portfolio return. Let the downside risk of asset i of scenario s in period t be defined by $M_{ist} = |min\ [0, R_{ist} - r_{pt}]|$.

This gives the expected portfolio risk in period t as $\sum_{s=1}^{S} p_s M_{ist} x_{ist}$. The expected portfolio risk for the period $[0,\tau]$ becomes

$$\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{S} p_s M_{ist} x_{ist}$$

Letting $Z_t = \sum_{s=1}^{S} p_s M_{ist} x_{ist}$ results in the expected portfolio risk for the entire planning phase as

$$H_p = \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t$$

3.4. Portfolio Gain

Portfolio gain is defined as an upside deviation of an asset return from expected portfolio return at any period t. The study assumes that the investor wants to maximize upside deviations of asset returns resulting in better investment opportunities. Let the upside deviation of asset i of scenario s of period t be defined by

$$N_{ist} = \max\ [0, R_{ist} - r_{pt}]$$

This gives the expected upside deviation of the portfolio of period t as $\sum_{s=1}^{S} p_s N_{ist} x_{ist}$. Hence the expected upside deviation of the portfolio for all time-periods becomes

$$\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{S} p_s N_{ist} x_{ist}$$

If $Y_t = \sum_{s=1}^{S} p_s N_{ist} x_{ist}$, then the expected portfolio gain for the whole rebalancing period is given by

$$\varphi_p = \frac{1}{\tau}\sum_{t=1}^{\tau} Y_t$$

3.5. The Multi-stage Stochastic Optimization (SMUDTC) Model

There is a bi-criteria objective to be satisfied in the study by maximizing portfolio gain while minimizing maximum downside risk. Since the intention of the investor is to minimize portfolio risk, it is achieved by maximizing the negative of the downside risk, that is, maximizing $-H_p = -\frac{1}{\tau}\sum_{t=1}^{\tau} Z_t$. This results in a single objective where the investor maximizes the sum of the negative loss and gain of the portfolio of any period t. Thus, the objective function becomes

$$\textit{Maximize} \qquad \left(\varphi_p - H_p\right) = \frac{1}{\tau}\left(\sum_{t=1}^{\tau} Y_t - \sum_{t=1}^{\tau} Z_t\right).$$

Letting λ to be the minimum acceptable transaction cost and θ to be the minimum portfolio expected return, the following multi-stage stochastic model with uncertain implicit transaction costs is obtained:

$$\begin{aligned}
\text{Maximize} \quad & \frac{1}{\tau}\left(\sum_{t=1}^{\tau} Y_t - \sum_{t=1}^{\tau} Z_t\right)\\
\text{Subject to} \quad & W_t = (1+N_{pt})\, W_{t-1},\ t=1,\dots,\tau,\\
& \theta \le N_{pt},\ t=1,\dots,\tau\\
& 0 = Z_t - \sum_{s=1}^{S} p_s M_{ist} x_{ist}, i = 1,\dots,n; t = 1,\dots,\tau,\\
& 0 = -Y_t - \sum_{s=1}^{S} p_s N_{ist} x_{ist},\ i = 1,\dots,n; t = 1,\dots,\tau,\\
& \lambda \ge \sum_{s=1}^{S} p_s \left\{k_{ist} a_{ist} + l_{ist} v_{ist}\right\} R_{ist}, i = 1,\dots,n; t = 1,\dots,\tau,\\
& 0 \le \sum_{i\in A} a_{ist} \le \sum_{j=1, j\ne i}^{n} v_{ist}, s = 1,\dots,S; t = 1,\dots,\tau,\\
& 1 = \sum_{s=1}^{S} x_{ist}, i = 1,\dots,n; t = 1,\dots,\tau\\
& 0 \le v_{it} \le x_{it},\ i=1,\dots, n;\ t=1,\dots,\tau,\\
& 0 \le x_{ist} \le U_{ist},\ i=1,\dots, n;\ s=1,\dots, S;\ t=1,\dots,\tau\\
& 0 \le a_{ist},\ i=1,\dots, n;\ s=1,\dots, S;\ t=1,\dots,\tau
\end{aligned}$$

The model (4) has a non-linear objective function, and the third and fourth constraints are also non-linear. In order to transform the model into a stochastic linear programming model, the following changes are made.

For each scenario s, let $K_{ist} \ge M_{ist} = |\min[0, R_{ist} - r_{pt}]|$, $s=1,2,\dots,S$. This gives the expected portfolio risk of period t as $\sum_{s=1}^{S} p_s K_{ist} x_{ist}$, with the expected portfolio risk for the period [0,τ] becoming $\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist}$. Then letting $J_t = \sum_{s=1}^{S} p_s K_{ist} x_{ist}$ gives the expected portfolio risk for the entire planning phase as

$$E = \frac{1}{\tau}\sum_{t=1}^{\tau} J_t$$

Similarly, for each scenario s, let $D_{ist} \ge N_{ist} = \max[0, R_{ist} - r_{pt}]$, $s=1,2,\dots,S$. This results in $\sum_{s=1}^{S} p_s D_{ist} x_{ist}$ as the expected upside deviation of the portfolio of period t. Therefore, the expected upside deviation (gain) of the portfolio for the period [0,τ] becomes $\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s D_{ist} x_{ist}$. By letting $Q_t = \sum_{s=1}^{S} p_s D_{ist} x_{ist}$, the expected portfolio gain for the entire planning phase becomes

$$F = \frac{1}{\tau}\sum_{t=1}^{\tau} Q_t$$

The model (4) becomes equivalent to the following stochastic linear programming model

$$\begin{aligned}
\text{Maximize} \quad & \frac{1}{\tau}\left[\sum_{t=1}^{\tau} Q_t - \sum_{t=1}^{\tau} J_t\right] \qquad (5)\\
\text{Subject to} \quad & W_t = (1+N_{pt})\, W_{t-1},\ t=1,\dots,\tau,\\
& \theta \le N_{pt},\ t=1,\dots,\tau\\
& 0 = J_t - \sum_{s=1}^{S} p_s K_{ist} x_{ist}, i = 1,\dots,n; t = 1,\dots,\tau,\\
& 0 = Q_t - \sum_{s=1}^{S} p_s K_{ist} x_{ist}, i = 1,\dots,n; t = 1,\dots,\tau,\\
& 0 = J_t - \sum_{s=1}^{S} p_s K_{ist} x_{ist}, i = 1,\dots,n; t = 1,\dots,\tau,\\
& 0 = -Q_t - \sum_{s=1}^{S} p_s D_{ist} x_{ist},\ i = 1,\dots,n; t = 1,\dots,\tau,\\
& \lambda \ge \sum_{s=1}^{S} p_s \left\{k_{ist} a_{ist} + l_{ist} v_{ist}\right\} R_{ist}, i = 1,\dots,n; t = 1,\dots,\tau,\\
& 0 \le \sum_{i\in A} a_{ist} \le \sum_{j=1, j\ne i}^{n} v_{ist}, s = 1,\dots,S; t = 1,\dots,\tau,\\
& 1 = \sum_{s=1}^{S} x_{ist}, i = 1,\dots,n; t = 1,\dots,\tau,\\
& 0 \le v_{it} \le x_{it},\ i=1,\dots, n;\ t=1,\dots,\tau,\\
& 0 \le x_{ist} \le U_{ist},\ i=1,\dots, n;\ s=1,\dots, S;\ t=1,\dots,\tau\\
& 0 \le a_{ist},\ i=1,\dots, n;\ s=1,\dots, S;\ t=1,\dots,\tau
\end{aligned}$$

The following Theorem shows that models (4) and (5) are equivalent and yield the same optimal values.

3.6. Theorem 1

If x^* is an optimal solution to (4), then (x^*, E^*, F^*) is an optimal solution to (5). Conversely, if (x^*, E^*, F^*) is an optimal solution to (5), then x^* is an optimal solution to (4).

Proof

Without loss of generality, let $x^* = x^*_{ist}$. If x* is an optimal solution to (4), then (x^*, E^*, F^*) is a feasible solution to (5), where

And

$$E = \frac{1}{\tau}\sum_{t=1}^{\tau} J_t \quad = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist}.$$

$$\geq \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s M_{ist} x_{ist}$$

$$= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot \left|\min\left[0, R_{ist} - r_{pt}\right]\right|$$

$$F = \frac{1}{\tau}\sum_{t=1}^{\tau} Q_t \quad = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s D_{ist} x_{ist}$$

$$\geq \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s N_{ist} x_{ist}$$

$$= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot (\max\left[0, R_{ist} - r_{pt}\right])$$

If (x*, E*, F*) is not an optimal solution to (4), then there exists a feasible solution (x, E, F) such that E*<E and F*<F, where

$$E = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist}$$

$$= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\min[0, R_{ist} - r_{pt}]|$$

And

$$F = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s D_{ist} x_{ist}$$

$$= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\mathrm{mix}[0, R_{ist} - r_{pt}]|$$

It is observed that

$$\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist}^{*} \cdot |\min[0, R_{ist} - r_{pt}]| = E^{*} < E.$$

And that $E^{*} < E = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\min[0, R_{ist} - r_{pt}]|$.

Also $\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist}^{*} \cdot \max\left[0, R_{ist} - r_{pt}\right] = F^{*} < F$ and that

$$F^{*} < F = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot \max\left[0, R_{ist} - r_{pt}\right].$$

This is a contradiction since x* is an optimal solution of (4).

Conversely, if (x*,E*,F*) is an optimal solution of (5), then x* is an optimal solution of (4). Otherwise, there exists a feasible solution x to (4) such that

$$E^{*} = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist}^{*} \cdot |\min[0, R_{ist} - r_{pt}]|$$

$$< \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\min[0, R_{ist} - r_{pt}]|$$

And

$$F^{*} = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist}^{*} \cdot (\max\left[0, R_{ist} - r_{pt}\right])$$

$$< \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot (\max\left[0, R_{ist} - r_{pt}\right])$$

$$= F$$

Which contradicts that (x*, E*, F*) is an optimal solution to (5). This completes the proof.

3.7. Transaction Cost Measurement

Transaction cost analysis has become increasingly important in helping firms and individual investors measure how effectively both perceived and actual orders are executed. Transaction costs are either implicit or explicit. Market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties are all explicit costs. Implicit costs are invisible and are strongly related to the trading strategy. They can broadly be put into three categories which are market impact, opportunity costs and spread. When an investment is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only. In such a case, market impact is defined as the deviation of the transaction price from the `unperturbed' price that would have prevailed if the trade had not occurred. In the proposed model, immediate trade execution is assumed and market impact is taken to account for the total implicit transaction costs. The study applies the approach by Hau (2006) and considers the transaction price to be the asset's last price of the month. The spread mid-point benchmark is used and the effective spread is evaluated as twice the distance from the mid-price measured in basis points. Taking P^M as the mid-point of the bid-ask spread and P^T as the transaction price, the effective spread (implicit transaction cost) is calculated as

$$SPREAD^{Trade} = \frac{200 \times |P^T - P^M|}{P^M}$$

4. DATA, MODEL APPLICATION AND RESULTS

The study uses historical monthly data of securities traded on the Johannesburg Stock Market from January 2008 to September 2012. The following criteria are used to select securities available for portfolio selection:

a. Stocks with negative mean returns for the entire period of study are excluded from the sample,
b. Companies which were not on the list by January 2008 and only entered the Johannesburg Stock Exchange (JSE) afterwards are excluded, and
c. Assets having the highest positive mean returns calculated for the entire period are taken to become our initial portfolio.

The study uses historical simulation and takes empirical distributions computed from monthly returns as equi-probable scenarios. Taking $P_{i,t}$ to be a monthly price of asset i considered in period t, a return scenario, R_{ist}, for the asset i of period t is calculated as $R_{ist} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}}$. Five scenarios for each asset in each period t are considered, giving a total of $5n$ scenarios in each period where n is the number of assets for selection. A scenario consists of an asset return and the corresponding implicit transaction cost. It is assumed that both asset return and the corresponding implicit cost are random since it is in the buying or selling of securities, whose prices are random, that implicit transaction costs are incurred by an investor. Implicit costs are calculated from the effective bid-ask spread corresponding to each selected asset return. The initial portfolio is selected from 13 securities and empirical distributions of these 13 securities are considered. Each security has 54 historical monthly returns and random numbers are used to select an asset return and associated implicit transaction cost corresponding to a scenario of a security. This is done by numbering the months 1–54. The transaction cost is given as a rate and each scenario is considered as equally likely to occur, giving a probability of $\frac{1}{5n}$ for each scenario. It should be noted that scenarios are only considered for the proposed SMUDTC model and other models use mean returns, mean implicit transaction costs and risks calculated for the entire period in the study. The period under study is divided into two, giving in-sample and out-of-sample data. The in-sample period starts from January 2008 and ends March 2012. The second period starts from April 2010 to September 2012. In each period, the performance of each model is analysed. First-stage optimal portfolios generated by the SMUDTC model are compared with optimal portfolios from the MAD, MV and minimax models (Appendix). In a real-life environment, comparison of models is usually done by means of ex-post analysis. It is also noted that portfolio performances are usually affected by market trend, hence the need to subject the proposed SMUDTC model to these two periods and evaluate its performance. An investor who has R10,000 to spend on his initial portfolio is considered in the study.

Table 1: Summary statistics of in-sample optimal portfolios

Div. Lim	MAD	MV	MM	SMUDTC
0.10				
Mean return	0.027	0.030	0.034	0.003
Risk	0.024	0.001	0.034	0.001
Gross wealth	10265.76	10303.55	10339.30	10066.72
Implicit cost	710.10	835.42	901.70	40.22
Net wealth	9555.66	9468.12	9437.60	10026.50
0.15				
Mean return	0.024	0.032	0.038	0.004
Risk	0.027	0.001	0.038	0.001
Gross wealth	10236.14	10316.92	10380.50	10065.00
Implicit cost	855.25	929.73	721.45	29.33
Net wealth	9380.89	9387.18	9659.05	10035.67
0.20				
Mean return	0.023	0.033	0.041	0.003
Risk	0.028	0.001	0.041	0.001
Gross wealth	10225.12	10330.33	10409.00	10056.44
Implicit cost	1092.00	818.23	746.40	25.36
Net wealth	9133.12	9512.10	9662.60	10031.08
0.25				
Mean return	0.022	0.033	0.043	0.003
Risk	0.029	0.001	0.043	0.001
Gross wealth	10216.15	10330.92	10428.00	10050.86
Implicit cost	396.25	760.18	883.25	23.20
Net wealth	9819.90	9570.74	9544.75	10027.66
0.30				
Mean return	0.021	0.033	0.044	0.003
Risk	0.029	0.001	0.044	0.001
Gross wealth	10211.78	10331.02	10443.60	10045.83
Implicit cost	445.30	757.81	1023.30	53.84
Net wealth	9766.48	9573.21	9420.30	10028.63
0.35				
Mean return	0.021	0.033	0.046	0.003
Risk	0.029	0.001	0.046	0.001
Gross wealth	10207.83	10331.02	10457.80	10051.63
Implicit cost	444.45	757.80	1083.60	18.79
Net wealth	9763.38	9573.22	9374.20	10032.84
0.40				
Mean return	0.020	0.033	0.047	0.004
Risk	0.030	0.001	0.047	0.001
Gross wealth	10204.72	10331.02	10469.20	10047.63
Implicit cost	343.80	757.79	984.40	5.52
Net wealth	9860.92	9573.23	9484.80	10042.11

4.1. In-sample Analysis

A GAMS software is used to solve the four models. In analysing the performance of each model, the gross portfolio mean return, portfolio risk, total implicit transaction costs incurred and the gross and net portfolio wealth are evaluated. Neither portfolio mean return nor portfolio risk is constrained, and diversification limits from 0.1 to 0.4 are used. The optimal portfolios generated by the four models are shown in Table 1. The phrase `Div. Lim' stands for diversification limit.

It is observed that, for each given diversification limit, the SMUDTC optimal portfolios have the least gross mean portfolio return and the investor incurs the least implicit transaction costs. The MM optimal portfolios have the highest gross expected portfolio returns followed by the MV-generated optimal portfolios. However, the MM-investor incurs the greatest implicit trading cost followed by the MV-investor. It is clearly evident, that despite having the least expected portfolio returns, SMUDTC optimal portfolios have the greatest portfolio wealth for every diversification limit considered. The MAD-generated optimal portfolios have greater wealth than those of MV and MM optimal portfolios. The trend remains the same as portfolios become less diversified.

4.2. Out-of-sample Analysis

In a real-life environment, validation of models is usually done by ex-post analysis. This is achieved by making a comparative evaluation of model performance on in-sample and out-of-sample data sets. The period from April 2010 to September 2012 provides the out-of-sample data set for the SMUDTC model. Table 2 shows the performance of the SMUDTC model together with MV, MAD and minimax models (Appendix). The information in Table 3 shows that MM optimal portfolios have the highest gross expected portfolio returns followed by MV and MAD optimal portfolios respectively.

The information in the table shows that MM optimal portfolios have the highest gross expected portfolio returns followed by

Table 2: In-sample SMUDTC model sensitivity analysis

Div. Lim	Parameter	Max. value	Cost SI	Risk SI	Wealth SI
0.1	X_1	0.1	0.0000	0.2297	−0.00045
	X_2	0.1	0.1000	0.1228	−0.00024
	X_4	0.1	0.0000	0.0397	0.00002
	X_5	0.1	0.0000	0.1228	−0.00024
	X_6	0.1	0.0000	0.1561	−0.00017
	X_7	0.1	0.1000	0.1367	−0.00034
	X_8	0.1	0.0000	0.3262	−0.00033
	X_9	0.1	0.0000	0.0552	−0.00012
	X_{10}	0.1	0.0000	0.0776	−0.00032
	X_{11}	0.1	0.0000	0.0756	−0.00014
0.2	X_1	0.2	0.0000	0.1356	−0.000073
	X_2	0.2	0.1000	−0.0410	−0.000047
	X_4	0.2	0.0000	−0.1798	0.000495
	X_6	0.2	0.0000	0.1893	0.00012
	X_9	0.2	0.0000	−01309	0.00022
0.3	X_1	0.3	0.1000	−0.7836	−0.000509
	X_2	0.3	0.2000	−0.5172	−0.000867
	X_3	0.3	0.0000	0.0164	−0.000662
	X_4	0.3	0.0000	-0.4888	−0.000078

Table 3: Summary statistics of out-of-sample optimal portfolios

Div. Lim	MAD	MV	MM	SMUDTC
0.10				
Mean return	0.027	0.029	0.031	0.005
Risk	0.002	0.002	0.1	0.0009
Gross wealth	10269.80	10289.02	10314.80	10064.57
Cost	432.00	426.34	502.10	15.36
Net wealth	9837.80	9862.68	9812.70	10049.21
0.15				
Mean return	0.025	0.028	0.033	0.006
Risk	0.022	0.002	0.143	0.0009
Gross wealth	10251.75	10277.97	10332.30	10070.17
Cost	479.10	397.32	411.15	17.27
Net wealth	9772.65	9880.65	9921.15	10052.90
0.20				
Mean return	0.024	0.027	0.035	0.007
Risk	0.023	0.002	0.2	0.0007
Gross wealth	10240.20	10268.25	10346.80	10073.55
Cost	223.20	378.64	503.60	8.00
Net wealth	10017.00	9889.61	9843.20	10065.55
0.25				
Mean return	0.023	0.026	0.036	0.007
Risk	0.025	0.002	0.25	0.0006
Gross wealth	10229.25	10259.68	10356.25	10080.73
Cost	248.50	356.71	591.00	7.40
Net wealth	9980.75	9902.97	9765.25	10073.33
0.30				
Mean return	0.023	0.026	0.036	0.008
Risk	0.025	0.002	0.25	0.0006
Gross wealth	10221.70	10255.02	10362.50	10081.56
Cost	272.80	340.06	550.00	6.64
Net wealth	9948.90	9914.96	9812.50	10074.92
0.35				
Mean return	0.021	0.025	0.037	0.008
Risk	0.026	0.002	0.333	0.0006
Gross wealth	10214.90	10250.44	10368.30	10082.44
Cost	270.55	318.87	502.10	6.36
Net wealth	9944.35	9931.57	9866.20	10076.08
0.40				
Mean return	0.021	0.025	0.037	0.007
Risk	0.027	0.002	0.333	0.0007
Gross wealth	10209.60	10247.81	10373.20	10075.40
Cost	215.20	306.70	440.40	6.80
Net wealth	9994.40	9941.12	9932.80	10068.60

MV and MAD optimal portfolios respectively. It is also evident that the MM optimal portfolios have the highest portfolio risk with the SMUDTC optimal portfolios' risks being the least for all diversification levels considered in the study. The SMUDTC optimal portfolios bear the least implicit transaction costs with the MM optimal portfolios having the highest transaction costs. It is also clear that the SMUDTC optimal portfolio has the greatest wealth for every diversification limit considered. All these features are evident in the in-sample analysis above. Thus, the SMUDTC investor has advantages of having high net expected portfolio returns and incurring the least implicit transaction costs during trading over his MM-, MV- and MAD- counterparts.

4.3. Sensitivity Analysis

Model sensitivity analysis is carried out using both in-sample and out-of-sample data. This is done by finding the sensitivity index (SI) for each parameter where the output percentage difference is calculated by varying one input parameter from its minimum value (zero in this case) to its maximum value (Hoffman and Gardner [1983]; Bauer and Hamby [1991]). The SI is calculated using the formula $Sensitivity\,index = \frac{D_{max} - D_{min}}{D_{max}}$, where D_{min} and D_{max} are the minimum and maximum output values respectively resulting from varying the input parameter over its entire range. Zero is taken to be the minimum input value for each parameter with the maximum value constrained by the diversification limit considered and is given as the best weight of the chosen asset in the optimal portfolio. Diversification limits of 0.1, 0.2 and 0.3 are considered. The SMUDTC model is stochastic and works by replacing one parameter (of the optimal portfolio) by a `less' profitable one when the weight of the chosen asset is zero. The condition that the total weight of assets in the optimal portfolio must be unity provides for the inclusion of a less profitable asset in the `new' optimal portfolio. This results in a relative sensitivity value. Hence, the D_{min} value of model output is a relative value. Therefore, by applying the method above we are finding relative sensitivity analysis of the model parameters.

Table 2 shows in-sample model sensitivity analysis where there are very small variations in portfolio wealth resulting from each asset being removed from the optimal portfolio. The proxy optimal portfolio has wealth increase of at most 0.05% and decrease of at most 0.049%. This shows that the SMUDTC model is not significantly influenced by individual parameter choices. The risk SI varies from 0 to 0.2, with zero being the modal SI. This again may imply that model output values are not strongly dependent on certain individual parameters. The cost SI range from −78.4% to 32.6%. These large values (numerically) are a result of very small implicit transaction costs incurred which have been used in the calculation. Table 1 shows the implicit transaction costs associated with SMUDTC optimal portfolios.

5. CONCLUSION

Portfolio selection that incorporates transaction costs in the literature is mostly devoted to proportional transaction costs. There is extensive use of the MV and MAD models in financial optimization yet both models penalise upside deviations (gains) and downside deviations (losses) in the same way. It is important to differentiate upside deviations from downside deviations since positive deviation is desirable to any investor while negative deviation is undesirable. Hence this paper proposes a new model that accounts for better investment opportunities (upside deviations), risk (downside deviations) and implicit transaction costs in a dynamic uncertain environment.

The study considers uncertainty in asset returns, portfolio risk and implicit trading costs incurred during initial trading and in subsequent rebalancing of the portfolio. The proposed model is tested using real data from an emerging market and its performance is compared with those of MV, MAD and minimax models. The results show that the proposed model generates optimal portfolios with least risk, highest portfolio wealth and minimum implicit transaction costs. The model is suitable for a risk-averse and conservative investor.

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APPENDIX

6. A REVIEW OF SOME VALIDATION MODELS

There are a number of models on portfolio selection presented in the literature to date. However, comparison in this study is restricted to the widely used mean-variance, MAD and minimax models. The following is a review of these models.

6.1. MV Model

The MV model is proposed by Markowitz (1952). This model minimizes portfolio variance (risk) subject to expected portfolio return achieving a prescribed level. The mathematical formulation of the model is as follows:

Maximize
$$\varphi = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

Subject to
$$\rho \le \sum_{i=1}^{n} r_i x_i,$$
$$1 = \sum_{i=1}^{n} x_i,$$
$$0 \le xi \le ui,\ i = 1,\ldots,n$$

Where σ_{ij} is the covariance between assets i and j, x_i is the proportion of wealth invested in asset i, r_i is the expected return of asset i in each period, ρ is the minimum rate of return desired by an investor, and u_i is the maximum proportion of wealth which can be invested in asset i.

6.2. Minimax Model

The minimax model is proposed by Young (1998) in which minimum return is used as a measure of risk. It is a linear programming model which is formulated as follows:

Maximize M_p

Subject to
$$0 \le \sum_{i=1}^{n} w_i y_{it} - M_p, t = 1,\ldots,T,$$
$$G \le \sum_{i=1}^{n} w_i \overline{y}_i,$$
$$W \ge \sum_{i=1}^{n} w_i,$$
$$0 \le wi,\ i=1,\ldots,n.$$

Where y_{it} is the return of one dollar invested in security i in period t, $\overline{y}_i$ is the mean return of security i, w_i is the portfolio allocation to security i, M_p is the minimum return on the portfolio, G is the minimum level of return, and W is the total allocation.

6.3. MAD

Konno and Yamazaki (1991) propose the MAD model and show that it behaves in the same manner as the MV model when the assets' returns are multi-variate normally distributed. The MAD model is formulated as given below:

Maximize
$$\beta = \frac{1}{T} \sum_{t=1}^{T} y_t$$

Subject to
$$0 \le y_t + \sum_{i=1}^{n} a_{it} x_i, t = 1,\ldots,T,$$
$$0 \le y_t - \sum_{i=1}^{n} a_{it} x_i, t = 1,\ldots,T,$$
$$\rho \le \sum_{i=1}^{n} r_i x_i,$$
$$1 = \sum_{i=1}^{n} x_i,$$
$$0 \le xi \le u_i,\ i = 1,\ldots,n,$$

Where $a_{it} = r_{it} - r_i$, $y_t = |\sum_{i=1}^{n} a_{it} x_i|$, r_i is the expected portfolio return in period t, r_{it} is the return of security i of period t, ρ is the minimum rate of return desired by the investor, and u_i is the maximum proportion of wealth that can be invested in asset i.